Complete Hypersurfaces with Bounded Mean Curvature in $\mathbb{R}^{n+1}$

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Abstract

Let M be an n-dimensional complete non-compact hypersurface in $\mathbb{R}^{n+1}$ and assume that its mean curvature lies between two positive numbers. Denote by Δ and A the Laplacian operator and the second fundamental form of M, respectively. In this paper, we show that if $3 \leq n \leq 5$ and if $\mathrm{Ind}(\Delta + |A|^2)$ is finite, then M has finitely many ends. We also show that if $2 \leq n \leq 5$ and if $\mathrm{Ind}(\Delta + |A|^2) = 0$, then M has only one end.

1. Introduction

The well-known Bernstein theorem [1] states that the only complete minimal graphs in $\mathbb{R}^3$ are planes. As a natural generalization, it was shown independently by Do Carmo-Peng [3], Fischer Colbrie-Schoen [5], and Pogorelov [12] that a complete stable minimal surface in $\mathbb{R}^3$ must be a plane. Later Gulliver [6] and Fischer-Colbrie [4] proved independently that a complete immersed minimal surface in $\mathbb{R}^3$ with finite index is conformally equivalent to a compact Riemann surface with finitely punctures and in particular, it must have finitely many ends. In the higher dimensional case, Cao-Shen-Zhu [2] proved that a complete, immersed, stable minimal hypersurface M^n of $\mathbb{R}^{n+1}$ with $n \geq 3$ has only one end. Recall that a minimal submanifold is stable if the second variation of its volume is always nonnegative for any normal variation with compact support. Recently, Li-Wang generalized the result of Gulliver and Fischer-Coolbrie for finitely many ends to higher dimensional minimal hypersurfaces in Euclidean space. They proved in [10] that a complete, immersed

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minimal hypersurface in $\mathbb{R}^{n+1}(n \geq 3)$ with finite index must have finitely many ends.

In this paper, we study the similar topological properties for hypersurfaces with mean curvature bounded between two positive numbers. Before stating our results, let us fix some notation.

Let M be a complete non-compact hypersurface in $\mathbb{R}^{n+1}$. Denote by Δ Laplacian operator acting on functions on M. Let $|A|^2$ be the square of the length of the second fundamental form A of M. Let $D_1 \subset D_2 \subset \cdots$ be an increasing sequence of exhausting compact domains in M. Denote by $\mathrm{Ind}_L(D_i)$ the index of $L \equiv \Delta + |A|^2$ on D_i which is the number of negative eigenvalues of the eigenvalue problem

$$\begin{cases} (\Delta + |A|^2)f + \lambda f = 0 \text{ on } D_i, \\ f|_{\partial D_i} = 0. \end{cases}$$

Set

$$\mathrm{Ind}(L) = \lim_{i\to\infty} \mathrm{Ind}_L(D_i).$$

Our first result is the following finiteness theorem for ends of complete hypersurfaces in $\mathbb{R}^{n+1}$.

Theorem 1.2. *Let $M^n(3 \leq n \leq 5)$ be a complete, non-compact, immersed hypersurface in $\mathbb{R}^{n+1}$. Assume that there are two positive constants $H_i, i = 1, 2$, such that the mean curvature H of M satisfies $H_1 \leq |H| \leq H_2$. If* $\mathrm{Ind}(L)$ *is finite, then M has only finitely many ends.*

We then prove the following

Theorem 1.3. *Let $M^n(2 \leq n \leq 5)$ be a complete, non-compact, immersed hypersurface in $\mathbb{R}^{n+1}$ and denote by H the mean curvature of M. Assume that* $\mathrm{Ind}(L) = 0$. *If there are two positive constants $H_i, i = 1, 2$, such that $H_1 \leq |H| \leq H_2$, then M has only one end.*

2. Proofs of the Results

Before proving the Theorems, we list some facts we need.

Lemma 2.1. ([9], [10]) *Let M be a complete Riemannian manifold. Let $\mathcal{H}^0_D(M)$ be the space of bounded harmonic functions with finite energy and denote by $H^1(L^2(M))$ the first L^2-cohomology group of M. Then the number of non-parabolic ends of M is bounded from above by* $\dim \mathcal{H}^0_D(M) \leq \dim\ H^1(L^2(M)) + 1$.

We can estimate by using Lemma 2.1 the number of ends of a non-compact hypersurface in $\mathbb{R}^{n+1}$ if we can prove that all its ends are non-parabolic. It has been shown by Cao-Shen-Zhu [2] that this is indeed the case if the hypersurface is minimal. Now we want to show that this is also the case for non-compact hypersurfaces M with mean curvature bounded between two positive numbers and finite $\mathrm{Ind}(L)$. In order to see this, let us first prove the following

Lemma 2.2. *Let $M^n \subset \mathbb{R}^m$ be an oriented complete non-compact submanifold. Assume that the mean curvature vector $\mathbf{H}$ of M satisfies $|\mathbf{H}| \leq H_0 < +\infty$. Then each end of M^n has infinite volume.*

Proof. We will use the methods in [2] and [14]. Take an arbitrary point $p \in M$, without loss of generality, we can assume that $p = 0$. Let $\mathbf{X}$ be the position vector in $\mathbb{R}^{n+1}$ and let Δ be the Laplacian operator acting on functions on M. We have from $|\mathbf{H}| \leq H_0$ that

$$\begin{aligned} \Delta\left(|\mathbf{X}|_M^2\right) &= 2n(1+ <\mathbf{H}, \mathbf{X}>) \\ &\geq 2n(1 - |\mathbf{H}||\mathbf{X}|) \\ &\geq 2n(1 - H_0|\mathbf{X}|). \end{aligned} \tag{2.1}$$

Let d be the distance function on M from 0 ; then we have

$$d \geq |\mathbf{X}|_M. \tag{2.2}$$

For $s > 0$, let $B(s)$ be the geodesic ball of M of radius s with center 0. If we denote by $\mathbf{n}$ the outward unit normal of $\partial B(s)$, then we have

$$\begin{aligned} \frac{\partial |\mathbf{X}|^2}{\partial \mathbf{n}} &= 2 <\mathbf{X}, \mathbf{n}> \\ &\leq 2|\mathbf{X}|_M \\ &\leq 2s. \end{aligned} \tag{2.3}$$

Thus, we obtain by integrating (2.1) over $B(s)$ and using the divergence theorem that

$$\begin{aligned} 2n(1 - H_0 s)V(B(s)) &\leq 2n\int_{B(s)}(1 - H_0 d) \\ &\leq 2n\int_{B(s)}(1 - H_0|\mathbf{X}|) \\ &\leq \int_{B(s)}\Delta\left(|\mathbf{X}|_M^2\right) \\ &= \int_{\partial B(s)}\frac{\partial|\mathbf{X}|^2}{\partial \mathbf{n}} \\ &\leq 2\int_{\partial B(s)} s \\ &= 2sA(\partial B(s)), \end{aligned} \tag{2.4}$$

where $A(\partial B(s))$ and $V(B(s))$ denote the area of $\partial B(s)$ and the volume of $B(s)$, respectively.

Set $V(s) = V(B(s))$. Since

$$A(\partial B(s)) = \frac{\partial}{\partial r}V(r)\,|_{r=s}\ ,$$

we have from (2.4) that

$$s\frac{\partial}{\partial r}V(r) \geq n(1 - H_0 s)V(s)\,|_{r=s}\ ,$$

that is

$$\frac{V'(s)}{V(s)} \geq \frac{n}{s} - nH_0. \tag{2.5}$$

Fix an arbitrary $t > 0$. For any $s > t$, we get by integrating the above inequality from t to s that

$$\log \frac{V(s)}{V(t)} \geq n \log \frac{s}{t} - nH_0(s-t),$$

which gives

$$V(s) \geq V(t) \left(\frac{s}{t}\right)^n e^{-nH_0(s-t)}.$$

Therefore for any $s > 0$, we have

$$\begin{aligned} V(s) &\geq \lim_{t\to 0} \left\{ V(t) \left(\frac{s}{t}\right)^n e^{-nH_0(s-t)} \right\} \\ &= \omega_n s^n e^{-nH_0 s}, \end{aligned} \tag{2.6}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. In particular, we have

$$V((B(q,1)) \geq \omega_n e^{-nH_0}, \quad \forall q \in M, \tag{2.7}$$

where $B(q,r)$ is the open geodesic ball of radius r with center q. Let E be an end of M and assume that E is a connected component of $M \setminus \Omega$ for some compact $\Omega \subset M$. If E has finite volume, choose a positive integer T such that

$$\omega_n e^{-nH_0} T > \mathrm{vol}(E). \tag{2.8}$$

Take a point $x \in E$ and a $y \in \partial E$ such that $d(x,y) = d(x, \partial E) \geq 2T$. Let γ be a minimizing geodesic from x to y; then $B(\gamma(0),1)$, $B(\gamma(2),1), \ldots, B(\gamma(2(T-1)),1)$ are disjoint and are contained in E. Thus we have

$$\begin{aligned} \mathrm{vol}(E) &\geq \sum_{i=0}^{T-1} V(B(\gamma(2i),1)) \\ &\geq T\omega_n e^{-nH_0}, \end{aligned}$$

which contradicts to (2.8). This proves the infinity of $\mathrm{vol}(E)$.

Now we can prove the following

Lemma 2.3. *Let $M^n (n \geq 2)$ be a complete immersed non-compact hypersurface in $\mathbb{R}^{n+1}$. Assume that* $\mathrm{Ind}(L) < +\infty$. *If the mean curvature H of M satisfies $0 < H_1 \leq |H| \leq H_2 < +\infty$, then each end of M is non-parabolic.*

Proof. Let A denote the second fundamental form of M. Since $\mathrm{Ind}(L)$ is finite, one can use the same arguments in [4] to prove that there exists a compact set $\Omega \subset M$ such that $\mathrm{Ind}_L(M \setminus \Omega) = 0$. We can assume that $\Omega \subset B(p, R_0)$ for some $p \in M$ and

$R_0 > 0$. The monotonicity of eigenvalues [4] implies that $\mathrm{Ind}_L(M \setminus B(p, R_0))$ is also zero. Thus for all compactly supported function $\phi \in H_{1,2}(M \setminus B(p, R_0))$, we have

$$\int_{M\setminus B(p,R_0)} \phi^2 |A|^2 \leq \int_{M\setminus B(p,R_0)} |\nabla \phi|^2. \tag{2.9}$$

Take an end E of M. By choosing the above R_0 properly, we can assume without lose of generality that E is a connected component of $M \setminus B(p, R_0)$. For $x \in M$, let $d(x) = d(x, p)$ and for R sufficiently large, set $E_R = E \cap B(p, R)$. Assume that f_R is the solution of the equation

$$\begin{cases} \Delta f_R = 0, & \text{on } E_R, \\ f_R = 1, & \text{on } \partial E, \\ f_R = 0, & \text{on } E \cap \partial B(p, R). \end{cases}$$

The maximum principle implies that f_R is uniformly bounded between 0 and 1. Since $\int_{E_R} |\nabla f_R|^2 \leq \int_{E'_R} |\nabla f'_R|^2$ for $R' > R$, there is a universal constant C such that

$$\int_{E_R} |\nabla f_R|^2 \leq C.$$

Therefore by passing to a subsequence, still denoted by $\{f_R\}$, we can find a harmonic function f on E such that

$$\lim_{R\to+\infty} f_R(x) = f(x), \;\; x \in E.$$

From the construction we have

$$f|_{\partial E} = 1. \tag{2.10}$$

Furthermore, f satisfies the bounds

$$0 \leq f \leq 1.$$

Let us prove that f is non-constant, which will imply the non-parabolicity of E (Cf. [7], [8], [9]). For fixed $R > R_1 > R_0$, let ψ be a non-negative cut-off function satisfying the following conditions

$$\psi = \begin{cases} 1 & \text{on } E_R \setminus E_{R_1}, \\ 0 & \text{on } \partial E, \end{cases}$$

and

$$|\nabla \psi|^2 \leq C_1,$$

for some positive constant C_1. We change the definition of ψf_R to be zero at the other ends of M and also in $E \setminus E_R$. This will be continuous and, properly choosing f_R and ψ, can be even C^∞.

Let $\lambda_1, ..., \lambda_n$ be the principal curvatures of M; then we have

$$\begin{aligned} |A|^2 &= \sum_{i=1}^{n} \lambda_i^2 \\ &\geq \frac{1}{n}\left(\sum_{i=1}^{n}\lambda_i\right)^2 \\ &= n|H|^2 \\ &\geq nH_1^2. \end{aligned} \tag{2.11}$$

Using (2.9) and (2.11), integration by parts and the fact that f_R is harmonic, we get

$$\begin{aligned} \int_{E_R} (f_R\psi)^2 &\leq \frac{1}{nH_1^2}\int_{E_R} |\nabla(\psi f_R)|^2 \\ &= \frac{1}{nH_1^2}\left(\int_{E_R} |\nabla\psi|^2 f_R^2 + \frac{1}{2}\int_{E_R} < \nabla(\psi^2), \nabla(f_R^2) > \right. \\ &\quad \left. + \int_{E_R} \psi^2|\nabla f_R|^2\right) \\ &= \frac{1}{nH_1^2}\int_{E_R} |\nabla\psi|^2 f_R^2 \\ &\leq \frac{C_1}{nH_1^2}\int_{E_{R_1}} f_R^2. \end{aligned}$$

Thus, for a fixed R_1 satisfying $R_0 < R_1 < R$ we have

$$\int_{E_R \setminus E_{R_1}} f_R^2 \leq \frac{C_1}{nH_1^2}\int_{E_{R_1}} f_R^2. \tag{2.12}$$

Assume now that f is a constant. Then f must be identically 1 because of its boundary condition. Taking $R \to \infty$ in (2.12), one gets

$$V(E_R) - V(E_{R_1}) \leq \frac{C_1}{nH_1^2} V(E_{R_1}).$$

Since $R > R_1$ is arbitrary, we conclude that E has finite volume. This contradicts to Lemma 2.2 and completes the proof of Lemma 2.3.

Now we are ready to prove the main results in this paper.

Proof of Theorem 1.2. We will prove that M has finite first L^2-Betti number, i.e. $\dim H^1(L^2(M)) < \infty$, which, combining with Lemma 2.1 and Lemma 2.3 will imply that M has finitely many ends. For any L^2 harmonic 1-form ω on M, let $h = |\omega|$ be the length of ω and denote by ω^* be the vector field dual to ω. It follows from the Bochner formula that

$$\frac{1}{2}\Delta h^2 = \mathrm{Ric}(\omega^*, \omega^*) + |\nabla\omega|^2, \tag{2.13}$$

where Ric is the Ricci curvature of M and $\nabla\omega$ denotes the covariant derivative of ω.

By using the same arguments as in the proof of Theorem 5 in [10], we have

$$|\nabla\omega|^2 \geq \frac{n|\nabla h|^2}{(n-1)}. \tag{2.14}$$

If we denote by $e_1, ..., e_n$ the orthonormal principal directions of M corresponding to the principal curvatures $\lambda_1, ..., \lambda_n$, then we have from the Gauss equation that

$$\mathrm{Ric}(e_i, e_j) = 0, \quad 1 \leq i \neq j \leq n,$$

and for any i,

$$\mathrm{Ric}(e_i, e_i) \;\; = \;\; \left(\sum_{k=1}^{n} \lambda_k\right)\lambda_i - \lambda_i^2 = \left(\sum_{k\neq i} \lambda_k\right)\lambda_i.$$

Observe that

$$\left(\lambda_i + \frac{1}{2}\sum_{k=1,k\neq i}^{n} \lambda_k\right)^2 = \lambda_i^2 + \lambda_i \sum_{k=1,k\neq i}^{n} \lambda_k + \frac{1}{4}\left(\sum_{k=1,k\neq i}^{n} \lambda_k\right)^2.$$

Hence

$$\begin{aligned}
&|A|^2 + \lambda_i \sum_{k=1,k\neq i}^{n} \lambda_k = \\
&\sum_{k=1,k\neq i}^{n} \lambda_k^2 + \left(\lambda_i + \frac{1}{2}\sum_{k=1,k\neq i}^{n} \lambda_k\right)^2 - \frac{1}{4}\left(\sum_{k=1,k\neq i}^{n} \lambda_k\right)^2.
\end{aligned}$$

But

$$\sum_{k=1,k\neq i}^{n} \lambda_k^2 \geq \frac{1}{n-1}\left(\sum_{k=1,k\neq i}^{n} \lambda_k\right)^2.$$

It then follows from $n \leq 5$ that

$$\begin{aligned}
& |A|^2 + \lambda_i \sum_{k=1,k\neq i}^{n} \lambda_k \\
\geq \;& \sum_{k=1,k\neq i}^{n} \lambda_k^2 + \left(\lambda_i + \frac{1}{2}\sum_{k=1,k\neq i}^{n} \lambda_k\right)^2 + \left(\frac{1}{n-1} - \frac{1}{4}\right)\left(\sum_{k=1,k\neq i}^{n} \lambda_k\right)^2 \\
\geq \;& 0.
\end{aligned}$$

Thus the Ricci curvature of M is bounded from below by $-|A|^2$ and so we have

$$\begin{aligned}
\mathrm{Ric}(\omega^*, \omega^*) \;\; &\geq \;\; -|A|^2|\omega^*|^2 \\
&= \;\; -|A|^2 h^2.
\end{aligned} \tag{2.15}$$

Combining (2.13)-(2.15), we obtain

$$\Delta h \geq -|A|^2 h + \frac{|\nabla h|^2}{(n-1)h}. \tag{2.16}$$

Since M has finite index, we know from the proof of Lemma 2.3 that the inequality (2.9) holds. By choosing $\psi = \phi h$ in (2.9) with ϕ being a non-negative compactly supported function on $M \setminus B(p, R_0)$, we arrive at

$$\begin{aligned} & \int_{M\setminus B(p,R_0)} \phi^2 |A|^2 h^2 \\ \leq & \int_{M\setminus B(p,R_0)} |\nabla \phi|^2 h^2 + 2\int_{M\setminus B(p,R_0)} \phi h < \nabla\phi, \nabla h > \\ & + \int_{M\setminus B(p,R_0)} \phi^2 |\nabla h|^2 \\ = & \int_{M\setminus B(p,R_0)} |\nabla \phi|^2 h^2 - \int_{M\setminus B(p,R_0)} \phi^2 h \Delta h. \end{aligned} \tag{2.17}$$

Substituting (2.16) into (2.17), we get

$$\int_{M\setminus B(p,R_0)} \phi^2 |\nabla h|^2 \leq (n-1) \int_{M\setminus B(p,R_0)} |\nabla \phi|^2 h^2. \tag{2.18}$$

The inequality (2.9) also implies that

$$\begin{aligned} \int_{M\setminus B(p,R_0)} \phi^2 h^2 & \leq \frac{1}{nH_1^2} \int_{M\setminus B(p,R_0)} \phi^2 |A|^2 h^2 \\ & \leq \frac{1}{nH_1^2} \int_{M\setminus B(p,R_0)} |\nabla(\phi h)|^2 \\ & \leq \frac{2}{nH_1^2} \int_{M\setminus B(p,R_0)} (\phi^2 |\nabla h|^2 + |\nabla\phi|^2 h^2). \end{aligned} \tag{2.19}$$

Thus we deduce from (2.18) and (2.19) that

$$\int_{M\setminus B(p,R_0)} \phi^2 h^2 \leq \frac{2}{H_1^2} \int_{M\setminus B(p,R_0)} |\nabla \phi|^2 h^2. \tag{2.20}$$

For $R > R_0 + 1$, we take a ϕ satisfying the following conditions

$$\phi = \begin{cases} 0, & \text{on } B(p, R_0) \\ 1, & \text{on } B(p, R) \setminus B(p, R_0 + 1) \\ 0, & \text{on } M \setminus B(p, 2R), \end{cases}$$

$$|\nabla \phi| \leq \begin{cases} C_3, & \text{on } B(p, R_0 + 1) \setminus B(p, R_0) \\ C_3 R^{-1}, & \text{on } B(p, 2R) \setminus B(p, R) \end{cases}$$

for some constant $C_3 > 0$. It follows by substituting this ϕ into (2.20) that

$$\begin{aligned} \int_{B(p,R)\setminus B(p,R_0+1)} h^2 \leq & \ C_4 \int_{B(p,R_0+1)\setminus B(p,R_0)} h^2 \\ & + C_4 R^{-2} \int_{B(p,2R)\setminus B(p,R)} h^2. \end{aligned}$$

Since $h \in L^2$, we obtain by taking $R \to \infty$ that

$$\int_{M\setminus B(p,R_0+1)} h^2 \leq C_4 \int_{B(p,R_0+1)\setminus B(p,R_0)} h^2. \tag{2.21}$$

Thus we have

$$\begin{aligned}\int_{B(p,R_0+2)} h^2 &= \int_{B(p,R_0+1)} h^2 + \int_{B(p,R_0+2)\setminus B(p,R_0+1)} h^2 \\ &\le C_5 \int_{B(p,R_0+1)} h^2.\end{aligned} \tag{2.22}$$

According to the Sobolev inequality in [11], we have for compactly supported functions g on M, with $\nabla g \in L^1(M)$,

$$\begin{aligned}\left(\int_M |g|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} &\le C_6(n) \int_M (|\nabla g| + |g||\mathbf{H}|) \\ &\le C_7(n, H_2) \int_M (|\nabla g| + |g|).\end{aligned}$$

Let f be a compactly supported function on M. By replacing g by $f^{\frac{2(n-1)}{n-2}}$, and finally squaring the inequality obtained, we arrive at

$$\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le C_8(n, H_2) \int_M (|\nabla f|^2 + |f|^2). \tag{2.23}$$

From (2.16), we have

$$\Delta h \ge -|A|^2 h.$$

Fix an arbitrary $x \in M$. For any constant $a \ge 1$ and any compactly supported Lipschitz function ϕ on $B(x,1)$, it follows that

$$\int_{B(x,1)} \phi^2 |A|^2 h^{2a} \ge -\int_{B(x,1)} \phi^2 h^{2a-1} \Delta h. \tag{2.24}$$

Integrating by parts, we have

$$\begin{aligned}&-\int_{B(x,1)} \phi^2 h^{2a-1} \Delta h \\ = &\ 2\int_{B(x,1)} \phi h^{2a-1} < \nabla\phi, \nabla h > +(2a-1) \int_{B(x,1)} \phi^2 h^{2a-2} |\nabla h|^2 \\ \ge &\ 2\int_{B(x,1)} \phi h^{2a-1} < \nabla\phi, \nabla h > +a \int_{B(x,1)} \phi^2 h^{2a-2} |\nabla h|^2.\end{aligned}$$

Thus, by using (2.23), (2.24) and the identity

$$\begin{aligned}&\int_{B(x,1)} |\nabla(\phi h^a)|^2 = a^2 \int_{B(x,1)} \phi^2 h^{2a-2} |\nabla h|^2 \\ &+ \int_{B(x,1)} |\nabla\phi|^2 h^{2a} + 2a \int_{B(x,1)} \phi h^{2a-1} < \nabla\phi, \nabla h >,\end{aligned}$$

we get

$$\begin{aligned}&a\int_{B(x,1)} \phi^2 |A|^2 h^{2a} + \int_{B(x,1)} |\nabla\phi|^2 h^{2a} \\ \ge &\ \int_{B(x,1)} |\nabla(\phi h^a)|^2 \\ \ge &\ C_8^{-1} \left(\int_{B(x,1)} (\phi^2 h^{2a})^{\mu}\right)^{\frac{1}{\mu}} - \int_{B(x,1)} (\phi h^a)^2,\end{aligned} \tag{2.25}$$

where $\mu = \frac{n}{n-2}$.

For $\rho > 0,\ \sigma > 0$, with $\rho + \sigma \leq 1$, let us choose $\phi(r)$ to be the Lipschitz cut-off function depending only on r which is the distance function to x, given by

$$\phi = \begin{cases} 0 & \text{on } B(x,1) \setminus B(x,\rho+\sigma) \\ \frac{\rho+\sigma-r}{\sigma} & \text{on } B(x,\rho+\sigma) \setminus B(x,\rho) \\ 1 & \text{on } B(x,\rho). \end{cases}$$

Setting $S = \max_{r(x,y)\leq 1} |A|^2(y)$ and substituting the above ϕ into (2.25), we have

$$\begin{aligned} \left(\int_{B(x,\rho)} h^{2a\mu}\right)^{\frac{1}{\mu}} &\leq \left(\int_{B(x,1)} (\phi^2 h^{2a})^{\mu}\right)^{\frac{1}{\mu}} \\ &\leq C_8 \left(aS + 1 + \frac{1}{\sigma^2}\right) \int_{B(x,\rho+\sigma)} h^{2a}. \end{aligned}$$

Therefore,

$$\left(C_8 \left(aS + 1 + \frac{1}{\sigma^2}\right)\right)^{\frac{1}{2a}} ||h||_{2a,\rho+\sigma} \geq ||h||_{2a\mu,\rho}, \tag{2.26}$$

where for $b > 0, c > 0$

$$||h||_{b,c} = \left(\int_{B(x,c)} h^b\right)^{\frac{1}{b}}.$$

Let us now choose the sequences of a_i, ρ_i and σ_i such that

$$a_0 = 1,\ a_1 = \mu,\ \cdots,\ a_i = \mu^i, ...,$$

$$\sigma_0 = 2^{-2},\ \sigma_1 = 2^{-3},\ \cdots,\ \sigma_i = 2^{-(2+i)}, ...,$$

$$\rho_{-1} = 1,\ \rho_0 = 1 - \sigma_0,\ \rho_1 = 1 - \sigma_0 - \sigma_1,\ \cdots,\ \rho_i = 1 - \sum_{j=0}^{i} \sigma_j, \cdots.$$

Applying (2.26) to $a = a_i,\ \rho = \rho_i$ and $\sigma = \sigma_i$, and iterating the inequality, we deduce that

$$||h||_{2a_{i+1},\rho_i} \leq \prod_{j=0}^{i} \left(C_8 \left(a_j S + 1 + \frac{1}{\sigma_j^2}\right)\right)^{\frac{1}{2a_j}} ||h||_{2,1}$$

Set $V = \mathrm{vol}(B(x, \frac{1}{2}))$. Since

$$\begin{aligned} &\lim_{i\to\infty} V^{\frac{-1}{2a_{i+1}}} ||h||_{2a_{i+1},\rho_i} \\ \geq\ &\lim_{i\to\infty} V^{\frac{-1}{2a_{i+1}}} ||h||_{2a_{i+1},\frac{1}{2}} = \sup_{B(x,\frac{1}{2})} h, \end{aligned}$$

we conclude that

$$\sup_{B(x,\frac{1}{2})} h \leq \prod_{j=0}^{\infty} \left(C_8 \left(\mu^j S + 1 + 16 \cdot 4^j\right)\right)^{\frac{1}{2\mu^j}} ||h||_{2,1}. \tag{2.27}$$

Observe that

$$\begin{aligned} & \prod_{j=0}^{\infty}\left(C_8\left(\mu^j S+1+16\cdot 4^j\right)\right)^{\frac{1}{2\mu^j}} \\ \leq\ & \prod_{j=0}^{\infty}\left(C_8(S+17)4^j\right)^{\frac{1}{2\mu^j}} \\ =\ & \left(C_8(S+17)\right)^{\frac{\mu}{2(\mu-1)}}\cdot 2^{\frac{\mu}{(\mu-1)^2}} \\ \equiv\ & C_9^{\frac{1}{2}}. \end{aligned}$$

Therefore, it holds that

$$h^2(x)\leq C_9\int_{B(x,1)} h^2.$$

Since x is arbitrary, we get

$$\sup_{B(p,R_0+1)} h^2 \leq C_9'\int_{B(p,R_0+2)} h^2. \tag{2.28}$$

It follows by combining (2.22) and (2.28) that

$$\sup_{B(p,R_0+1)} h^2 \leq C_{10}\int_{B(p,R_0+1)} h^2, \tag{2.29}$$

for some constant $C_{10}>0$ which depends only on n, H_1, H_2 and $\sup_{B(p,R_0+2)}|A|^2$. Since (2.29) holds, by using the same arguments as in the final part of the proof of Theorem 5 in [10], one can deduce that $H^1(L^2(M))$ is finite dimensional. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. According to Lemma 2.1 and Lemma 2.3, it suffices to show that there exists no non-trivial L^2 harmonic 1-form on M. We will prove this by contradiction. Thus suppose that θ is a non-trivial L^2 harmonic 1-form on M. Set $u=|\theta|$. Observe that (2.15) still holds if $n=2$. Thus, by using the same arguments as in the proof of Theorem 1.1, we have

$$u\Delta u\geq -|A|^2u^2+\frac{|\nabla u|^2}{(n-1)}. \tag{2.30}$$

Since $\mathrm{Ind}(L)=0$, it follows from the definition that

$$\int_M|\nabla\psi|^2\geq\int_M|A|^2\psi^2, \tag{2.31}$$

for all compactly supported $\psi\in H_{1,2}(M)$. Fix a point $p\in M$. Let us choose $\phi\in C_0^\infty(M)$ satisfying

(i) $0\leq\phi\leq 1$

(ii) $\phi\equiv 1$ on $B(p,\frac{r}{2}),\phi\equiv 0$ on $M\setminus B(p,r)$

(iii) $\|\nabla\phi\|^2\leq\frac{c}{r^2}$, $c=$ constant independent of r.

By choosing $\psi = \phi u$ in (2.31) and using (2.30) and the divergence theorem, we obtain

$$\begin{aligned}
0 &\leq \int_M (|\nabla(\phi u)|^2 - |A|^2\phi^2 u^2) \\
&= \int_M \left(-\phi u \Delta(\phi u) - |A|^2\phi^2 u^2\right) \\
&= \int_M \left(-\phi u (u\Delta\phi + \phi\Delta u + 2 < \nabla\phi, \nabla u > - |A|^2\phi^2 u^2\right) \\
&\leq \int_M \left(-\phi u^2 \Delta\phi - \frac{\phi^2|\nabla u|^2}{n-1} - 2\phi u < \nabla\phi, \nabla u >\right) \\
&= \int_M \left(-\phi u^2 \Delta\phi - \frac{\phi^2|\nabla u|^2}{n-1} + \frac{1}{2}u^2\Delta\phi^2\right) \\
&= \int_M u^2|\nabla\phi|^2 - \int_M \frac{\phi^2|\nabla u|^2}{n-1} \\
&\leq \frac{cr^2}{\int} {}_{B(p,r)} u^2 - \int_M \frac{\phi^2|\nabla u|^2}{n-1}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\frac{1}{n-1}\int_{B(p,\frac{r}{2})} |\nabla u|^2 &\leq \int_M \frac{\phi^2|\nabla u|^2}{n-1} \\
&\leq \frac{c}{r^2}\int_{B(p,r)} u^2.
\end{aligned}$$

Since $u \in L^2(M)$, we conclude by taking $r \to \infty$ that

$$|\nabla u| \equiv 0.$$

Thus u is constant. On the other hand, we have by substituting the above ϕ into (2.31) that

$$\int_M |A|^2\phi^2 u^2 \leq \int_M u^2|\nabla\phi|^2,$$

which implies

$$\int_{B(p,r)} |A|^2 u^2 \leq \frac{1}{r^2}\int_{B(p,\frac{r}{2})} u^2.$$

If $u \neq 0$, we get by letting $r \to \infty$ that $|A| \equiv 0$. Thus M is totally geodesic and so is an affine n-plane. This clearly contradicts the fact that there exists no non-trivial L^2 harmonic 1-form on $\mathbb{R}^n$. Thus $u \equiv 0$ and the proof of Theorem 1.3 is complete.

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