Yet another application of the Gauss-Bonnet Theorem for the sphere*

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The Gauss-Bonnet Theorem, when applied to any Riemannian metric g on the sphere $\mathbb{S}^2$, claims that

$$\int_{\mathbb{S}^2} K dA_g = 4\pi,$$

where K and dA_g denote the Gaussian curvature and the area element of g, respectively. In particular, this implies the existence of at least an elliptic point of $(\mathbb{S}^2, g)$. In this short note we prove the Fundamental Theorem of Algebra as a consequence of this fact. As far as the authors know, this proof is new. Moreover, it does have the advantage of being directly based on the work of Gauss, who was the first mathematician giving an universally accepted proof of the Fundamental Theorem of Algebra (see [2], [3]). Hence our argument can also have some historical and pedagogical interest.

As it is well known, the stereographic projection allows to identify the sphere $\mathbb{S}^2$ with the one-point compactification $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ of the complex plane $\mathbb{C}$. Let us take $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ with $a_0\, a_n \neq 0$ and let us assume that $p(z) \neq 0$ for all $z \in \mathbb{C}$.

We set $p_*(z) = a_n + a_{n-1} z + \cdots a_0 z^n$, that is, the i-th coefficient of $p_*(z)$ is the $(n-i)$-th coefficient of $p(z)$. Clearly, $p_*(z) = z^n\, p(1/z)$ for all $z \in \mathbb{C} \setminus \{0\}$. Put

$$f(z) = p(z)\, p_*(z),$$

for all $z \in \mathbb{C}$. Then f satisfies the functional equation

$$\left| f\left(\frac{1}{z}\right) \right| = \frac{1}{|z|^{2n}}\, |f(z)|,$$

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for all $z \in \mathbb{C} \setminus \{0\}$, which allow us to introduce the Riemannian metric g on $\widehat{\mathbb{C}}$ such that

$$g = \frac{1}{|f(z)|^{\frac{2}{n}}}\,|dz|^2 \quad \text{for } z \in \mathbb{C} \quad \text{and}$$

$$g = \frac{1}{|f(1/z)|^{\frac{2}{n}}}\,|d(1/z)|^2 \quad \text{for } z \in \widehat{\mathbb{C}} \setminus \{0\}.$$

Now, a simple computation shows that the Gaussian curvature K of g satisfies

$$\frac{1}{|f(z)|^{\frac{2}{n}}}\,K = \frac{1}{n}\,\Delta\,(\log|f(z)|) = \frac{1}{n}\,\Delta\mathbf{Re}\,\log(f(z)) = 0,$$

for all $z \in \mathbb{C}$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denotes the standard Laplacian operator. Obviously, the last equality in the formula above holds true because real parts of analytic functions are harmonic. Therefore, it follows that $K = 0$ over all the sphere, which contradicts the existence of an elliptic point of g.

References

[1] **J. M. Almira, M. Jiménez, N. Del Toro,** Another topological proof of the Fundamental Theorem of Algebra, *Elem. Math.* **57** (2002) 32–37.

[2] **B. Fine, G. Rosenberger,** *The Fundamental Theorem of Algebra*, Undergraduate Texts in Mathematics, Springer-Verlag 1997.

[3] **R. Remmert,** *The Fundamental Theorem of Algebra*, in H. D. Ebinghaus et. al. *Numbers*, Graduate Texts in Mathematics, **123** Springer-Verlag (1995) 97–122.

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