Expanding graphs, Ramanujan graphs, and 1-factor perturbations

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Abstract

We construct $(k \pm 1)$-regular graphs which provide sequences of expanders by adding or substracting appropriate 1-factors from given sequences of k-regular graphs. We compute numerical examples in a few cases for which the given sequences are from the work of Lubotzky, Phillips, and Sarnak (with $k - 1$ the order of a finite field). If $k + 1 = 7$, our construction results in a sequence of 7-regular expanders with all spectral gaps at least $6 - 2\sqrt{5} \approx 1.52$; the corresponding minoration for a sequence of Ramanujan 7-regular graphs (which is not known to exist) would be $7 - 2\sqrt{6} \approx 2.10$.

1 Introduction

Let $X = (V, E)$ be a simple finite graph with n vertices, where V denotes the vertex set and E the set of geometrical edges of X. The adjacency matrix A of X, with rows and columns indexed by V, is defined by $A_{v,w} = 1$ if there exists an edge connecting v and w, and $A_{v,w} = 0$ otherwise (in particular $A_{v,v} = 0$). The eigenvalues of X, which are those of A, constitute a decreasing sequence $\lambda_0(X) \geq \lambda_1(X) \geq \ldots \geq \lambda_{n-1}(X)$. The *spectral gap* $\lambda_0(X) - \lambda_1(X)$ of X is positive if and only if X is connected. Let us assume from now on that X is k-regular for some $k \geq 3$, namely that $\sum_w A_{v,w} = k$ for all $v \in V$, so that $\lambda_0(X) = k$.

Recall that, for any infinite sequence $(X_i)_{i \in I}$ of connected k-regular simple finite graphs with increasing vertex sizes, we have the Alon-Boppana inequality $\liminf_{i \to \infty} \lambda_1(X_i) \geq 2\sqrt{k-1}$. A graph X is said to be a *Ramanujan graph* if it

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is connected and if $|\mu| \leq 2\sqrt{k-1}$ for any eigenvalue $\mu \neq \pm k$ of X. From elaborate arithmetic constructions, we know explicit infinite sequences of Ramanujan graphs for degree k when $k-1$ is the order of a finite field; but the existence of such sequences is an open problem for other degrees, for example when $k = 7$. It is thus interesting to find *sequences of expanders of degree* k, namely infinite sequences $(X_i)_{i \in I}$ of k-regular connected simple finite graphs with increasing vertex sizes such that $\inf_{i \in I}(k - \lambda_1(X_i))$ is strictly positive, and indeed as large as possible (short of being equal to $k - 2\sqrt{k-1}$).

For all this, see for example [Lubot–94], [Valet–97], [Colin–98], and [DaSaV–03].

The object of the present Note is to examine a procedure of construction of sequences of expanders $(X_i)_{i \in I}$ of degree k by perturbation of sequences of Ramanujan graphs. When $k - l - 1$ is the order of a finite field, we obtain estimates $\lambda_1(X_i) \leq l + 2\sqrt{k-l-1}$; for example, for $k = 7$ and $l = 1$, this corresponds to a spectral gap

$$7 - \lambda_1(X_i) \geq 6 - 2\sqrt{5} \approx 1.52 \qquad \text{for all } i \in I,$$

to be compared with the Alon-Boppana lower bound for the spectral gap:

$$7 - \liminf_{i \in I} \lambda_1(X_i) \leq 7 - 2\sqrt{6} \approx 2.10.$$

We insist on finding explicit constructions, but we record however the following results of J. Friedman based on random techniques: for all $k \geq 3$ and all $\epsilon > 0$, there exist sequences $(X_i)_{i \in I}$ of connected k-regular simple finite graphs with increasing vertex sizes and with $\lambda_1(X_i) \leq 2\sqrt{k-1} + \epsilon$ for all $i \in I$. See [Fried–04], and also [Fried–94].

Let $X = (V, E)$ be a graph. If X is not bipartite, we denote by $\overline{X} = (V, \overline{E})$ the *complement* of X; two distinct vertices are adjacent in $\overline{X}$ if and only if they are not so in X. If X is bipartite, given with a bipartition $V = V_0 \sqcup V_1$, we denote by $\overline{X} = (V, \overline{E})$ the *bipartite complement* of X; two vertices $v \in V_0$, $w \in V_1$ are adjacent in $\overline{X}$ if and only if they are not in X. A *matching* of a graph X is a subset M of E such that any vertex $x \in V$ is incident with at most one edge of M, and a *perfect matching* (also called 1-*factor*) is a subset F of E such that any vertex $x \in V$ is incident with exactly one edge of F.

Let $X = (V, E)$ be a graph. If F is a perfect matching of X, we denote by $X - F$ the graph $(V, E \setminus F)$; if X is k-regular, then $X - F$ is $(k-1)$-regular. If F is a perfect matching of $\overline{X}$, we denote by $X + F$ the graph $\overline{\overline{X} - F}$; if X is k-regular, then $X + F$ is $(k+1)$-regular.

The basic observation for the present Note is the set of inequalities

$$|\lambda_j(X \pm F) - \lambda_j(X)| \leq 1$$

for any perfect matching F of X (for $X - F$) or of $\overline{X}$ (for $X + F$), and for all $j \in \{0, \ldots, n-1\}$, where $n = |V|$ (Proposition 2). We can apply this to the Ramanujan graphs $X^{p,q}$ and their complements (notation of [DaSaV-03], see below). In Section 3, we apply an algorithm for finding perfect matchings in regular bipartite graphs (thus concentrating on pairs (p, q) for which the graph $X^{p,q}$ is bipartite). In conclusion, we report some numerical computations.

2 Graphs of the form $X^{p,q} \pm F$

Let us recall the definition of the graphs $X^{p,q}$.

If R is a commutative ring with unit, the *Hamilton quaternion algebra* $\mathbb{H}(R)$ *over* R is the free module R^4 with basis $\{1, i, j, k\}$, where multiplication is defined by $i^2 = j^2 = k^2 = -1$, and $ij = -ji = k$, plus circular permutations of i, j, k. A quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$ has a *conjugate* $\overline{q} = a_0 - a_1 i - a_2 j - a_3 k$ and a *norm* $N(q) = \overline{q}q = a_0^2 + a_1^2 + a_2^2 + a_3^2$.

Let $p \in \mathbb{N}$ be an odd prime. If $p \equiv 1 \pmod 4$, a theorem of Jacobi shows that there are exactly $p+1$ quaternions in $\mathbb{H}(\mathbb{Z})$ of norm p of the form $a_0 + a_1 i + a_2 j + a_3 k$ with $a_0 \equiv 1 \pmod 2$, and $a_0 \geq 1$. These occur in pairs $(\alpha, \overline{\alpha})$; we select arbitrarily one, say α_l, from each pair, and we set

$$S_p = \{\alpha_1, \overline{\alpha_1}, \ldots, \alpha_s, \overline{\alpha_s}\} \qquad \text{with} \quad 2s = p + 1.$$

If $p \equiv 3 \pmod 4$, there are quaternions in $\mathbb{H}(\mathbb{Z})$ of norm p of the form $a_0 + a_1 i + a_2 j + a_3 k$ with $a_0 \equiv 0 \pmod 2$, and $a_0 \geq 0$. From those with $a_0 \geq 2$, say $2s$ of them, we obtain $\alpha_1, \ldots, \alpha_s$ as above. Those of the form $a_1 i + a_2 j + a_3 k$, say $2t$ of them [1], occur in pairs $(\beta, -\beta)$; we select arbitrarily one, say β_m, from each pair, and we set

$$S_p = \{\alpha_1, \overline{\alpha_1}, \ldots, \alpha_s, \overline{\alpha_s}, \beta_1, \ldots, \beta_t\}.$$

Observe that $t/4$ is the number of solutions in $\mathbb{N}$ of the equation $a_1^2 + a_2^2 + a_3^2 = p$, and that we have again $|S_p| = 2s + t = p + 1$ by Jacobi's theorem. Observe also that we can have $s = 0$ (case of $p = 3$), as well as $t = 0$ (case of $p \equiv 7 \pmod 8$), or both s and t positive (case of $p = 19$, with $s = 4$ and $t = 12$).

Let q be another odd prime, $q \neq p$, and let $\tau_q : \mathbb{H}(\mathbb{Z}) \longrightarrow \mathbb{H}(\mathbb{F}_q)$ denote reduction modulo q. The equation $x^2 + y^2 + 1 = 0$ has solutions in $\mathbb{F}_q$. We choose one solution; then the mapping $\psi_q : \mathbb{H}(\mathbb{F}_q) \longrightarrow M_2(\mathbb{F}_q)$ defined by

$$\psi_q(a_0 + a_1 i + a_2 j + a_3 k) = \begin{pmatrix} a_0 + a_1 x + a_3 y & -a_1 y + a_2 + a_3 x \\ -a_1 y - a_2 + a_3 x & a_0 - a_1 x - a_3 y \end{pmatrix}$$

is an algebra isomorphism and $\psi_q(\tau_q(S_p))$ is in the group $GL_2(q)$ of invertible elements of $M_2(\mathbb{F}_q)$. We denote by $\phi : GL_2(q) \longrightarrow PGL_2(q)$ the reduction modulo the centre, and we set

$$S_{p,q} = \phi\Big(\psi_q(\tau_q(S_p))\Big) \subset PGL_2(q).$$

It follows from the definitions that $S_{p,q}$ is symmetric: if $s \in S_{p,q}$ is the image of $\alpha_l \in S_p$ (notation as above), then s^{-1} is the image of $\overline{\alpha_l}$; if s is the image of $\beta_m \in S_p$, then $s^2 = 1$. Moreover, it is known that $|S_{p,q}| = p + 1$. There are now two cases to consider.

Either p is a square modulo q. Then $S_{p,q} \subset PSL_2(q)$ and indeed $S_{p,q}$ generates $PSL_2(q)$. By definition, $X^{p,q}$ is the Cayley graph of $PSL_2(q)$ with respect to $S_{p,q}$; more precisely, $X^{p,q} = (V, E)$ with $V = PSL_2(q)$ and $\{v, w\} \in E$ if $v^{-1}w \in S_{p,q}$. It

[1] Observe that $2t$ is a multiple of 8, since each of a_1, a_2, a_3 is odd, in particular not 0, so that each sign change provides another writing of p as a sum of three squares.

is a $(p+1)$-regular graph with $\frac{1}{2}q(q^2-1)$ vertices which is connected, non-bipartite, and which is a Ramanujan graph.

Or p is not a square modulo q. Then $S_{p,q} \cap PSL_2(q) = \emptyset$ and $S_{p,q}$ generates $PGL_2(q)$. By definition, $X^{p,q}$ is the Cayley graph of $PGL_2(q)$ with respect to $S_{p,q}$. It is a $(p+1)$-regular bipartite graph with $q(q^2-1)$ vertices which is connected and which is a Ramanujan graph.

See [DaSaV–03] for proofs of a large part of the facts stated above, including the connectedness of the graphs $X^{p,q}$ when $p \geq 5$ and $q > p^8$, and the expanding property of this family. For the proof that $(X^{p,q})_q$ is actually a family [2] of Ramanujan graphs, see the original papers ([LuPhS–88], with a large part obtained independently in [Margu–88]), as well as [Sarna–90].

Table I shows the spectrum of $X^{3,q}$ for $q \in \{5, 7, 11\}$ and Table II that of $X^{5,q}$ for $q \in \{7, 11\}$. Numerical computations of eigenvalues reported in this paper have been computed with Mathlab.

Proposition 1. *If the graph $X^{p,q}$ is bipartite, $X^{p,q}$ and its bipartite complement $\overline{X^{p,q}}$ have perfect matchings.*

Proof More generally, any bipartite graph which is regular of degree at least 1 has a perfect matching, as it follows of P. Hall's marriage theorem; see for example Corollary 1.1.4 and Lemma 1.4.16 in [LovPl–86]. Here is another reason for $X^{p,q}$ (bipartite *or not*): any connected vertex-transitive graph of even order has a perfect matching (Section 3.5 in [GodRo–01]); this applies in particular to Cayley graphs of finite groups of even order, such as $PGL_2(q)$ and $PSL_2(q)$. ∎

Proposition 2. *Let $X = (V, E)$ be a finite graph with n vertices and with eigenvalues $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}$. Let F be a matching of X [respectively of the complement $\overline{X}$] and let $\mu_0 \geq \mu_1 \ldots \geq \mu_{n-1}$ be the eigenvalues of $X - F$ [respectively $X + F$]. Then $|\mu_j - \lambda_j| \leq 1$ for $j \in \{0, 1, \ldots, n-1\}$.*

Proof Outside diagonal entries, the adjacency matrix A_F of (V, F) is a matrix of permutation which is a nonempty product of transpositions with disjoint supports, one transposition for each edge in F. Thus $\|A_F\| \leq 1$. Here, the norm of a matrix acting on the Euclidean space $\mathbb{R}^V$ is the operator norm $\|A_F\| = \sup \left\{ \|Af\|_2 \mid f \in \mathbb{R}^V, \ \|f\|_2 \leq 1 \right\}$, where $\|f\|_2^2 = \sum_{v \in V} |f(v)|^2$.

Thus Proposition 1 follows from the classical Courant-Fischer-Weyl minimax principle, according to which eigenvalues of symmetric operators are norms of appropriate restrictions of these operators. See e.g. Chapter III in [Bhati–97]. ∎

[2] The family is indexed by the set of all odd primes q, and p is a fixed arbitrary odd prime.

3 Tables

There are several standard efficient algorithms to find a perfect matching F in a graph X; see [LovPl–86] and [West–01], among others. We will not describe here the details of the algorithm we have used. Eigenvalues of $X - F$ can then be computed with Mathlab.

The eigenvalues of a graph of the form $X^{p,q} - F$ depend on the choice of F. Table III gives for each of three pairs (p, q) the values of the spectral gaps $p - \lambda_1(X^{p,q} - F)$ corresponding to four different F. Table III shows that there are situations ($p = 5, q = 7$) with $\lambda_0(X - F) = k - 1 < \lambda_0(X) = k$ and $\lambda_1(X - F) > \lambda_1(X)$.

Table IV shows the full spectrum of $X^{3,5} - F$ for one specific F. Tables V to VII show the ten largest eigenvalues of three graphs of the form $X^{p,q} + F$. Observe that the multiplicities in Tables IV to VII are much less than those of the unperturbed graphs.

Table I: spectra of $X^{3,q}$

q=5		q=7		q=11	
eigenvalues	multiplicities	eigenvalues	multiplicities	eigenvalues	multiplicities
-4.0000	1	-4.0000	1	-3.2361	30
-3.0000	12	-3.0000	24	-3.0000	33
-2.0000	28	-2.8284	30	-2.7321	10
-1.0000	4	-2.0000	28	-2.6180	24
0.0000	30	-1.4142	24	-2.3723	10
1.0000	4	-1.0000	40	-2.0468	36
2.0000	28	0.0000	42	-2.0000	10
3.0000	12	1.0000	40	-1.6180	36
4.0000	1	1.4142	24	-1.5616	33
		2.0000	28	-0.9191	36
		2.8284	30	-0.7321	30
		3.0000	24	-0.3820	24
		4.0000	1	0.0000	30
				0.3820	12
				0.6180	36
				0.7321	10
				1.0000	52
				1.2361	30
				1.9191	36
				2.0000	20
				2.5616	33
				2.6180	12
				2.7321	30
				3.0468	36
				3.3723	10
				4.0000	1

Table II: spectra of $X^{5,q}$			
q=7		q=11	
eigenvalues	multiplicities	eigenvalues	multiplicities
-6.0000	1	-4.0243	36
-4.0000	21	-3.7321	30
-3.0000	16	-3.0000	65
-2.8284	42	-2.2361	30
-2.0000	21	-1.7321	10
-1.4142	12	-1.6180	60
-1.0000	48	-1.3723	10
0.0000	14	-1.2361	12
1.0000	48	-0.5616	33
1.4142	12	-0.2679	30
2.0000	21	-0.1638	36
2.8284	42	0.6180	60
3.0000	16	1.0000	30
4.0000	21	1.7321	10
6.0000	1	1.7818	36
		2.2361	30
		3.0000	50
		3.2361	12
		3.4063	36
		3.5616	33
		4.3723	10
		6.0000	1

Table III: spectral gaps for $X^{p,q} - F$		
p=3,q=5	p=3,q=7	p=5,q=7
0.4457	0.2499	0.7910
0.3025	0.1862	0.7732
0.2993	0.1785	0.7367
0.2702	0.0272	0.7152

Table IV: spectrum of $X^{3,5} - F$					
eigenvalues	multiplicities	eigenvalues	multiplicities	eigenvalues	multiplicities
-3.0000	1	-0.8302	4	1.2929	8
-2.5543	8	-0.5086	8	1.8829	8
-2.5450	4	-0.4394	4	2.0000	6
-2.1542	4	0.0000	4	2.1542	4
-2.0000	6	0.4394	4	2.5450	4
-1.8829	8	0.5086	8	2.5543	8
-1.2929	8	0.8302	4	3.0000	1
-1.0000	3	1.0000	3		

Table V: largest eigenvalues for $X^{3,5} + F$					
eigenvalues	multiplicities	eigenvalues	multiplicities	eigenvalues	multiplicities
3.2578	1	3.2163	1	3.1707	1
3.3225	1	3.3208	1	3.1998	1
3.3425	1	3.3431	1	3.2214	1
3.4295	1	3.4417	1	3.2418	1
3.4859	1	3.4992	1	3.3046	1
3.5140	1	3.5358	1	3.5525	1
3.5687	1	3.6211	1	3.5653	1
3.5950	1	3.6822	1	3.5935	1
3.6758	1	3.8466	1	3.6547	1
5.0000	1	5.0000	1	5.0000	1

Table VI: largest eigenvalues for $X^{3,7} + F$					
eigenvalues	multiplicities	eigenvalues	multiplicities	eigenvalues	multiplicities
3.6042	1	3.6199	1	3.6138	1
3.6130	1	3.6478	1	3.6431	1
3.6349	1	3.6594	1	3.6524	1
3.6728	1	3.6826	1	3.6726	1
3.6892	1	3.6996	1	3.6922	1
3.6971	1	3.7203	1	3.7131	1
3.7073	1	3.7468	1	3.7275	1
3.7505	1	3.7548	1	3.7461	1
3.7697	1	3.7752	1	3.7985	1
5.0000	1	5.0000	1	5.0000	1

Table VII: largest eigenvalues for $X^{5,7} + F$					
eigenvalues	multiplicities	eigenvalues	multiplicities	eigenvalues	multiplicities
4.3702	1	4.3388	1	4.3229	1
4.4015	1	4.3738	1	4.3405	1
4.4271	1	4.4326	1	4.3882	1
4.4625	1	4.4790	1	4.4117	1
4.4888	1	4.5124	1	4.4671	1
4.4971	1	4.5618	1	4.5585	1
4.5819	1	4.5925	1	4.5875	1
4.5976	1	4.6417	1	4.6341	1
4.6512	1	4.6892	1	4.7260	1
7.0000	1	7.0000	1	7.0000	1

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