On the existence of basic sequences in non-archimedean locally convex spaces

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Abstract

We prove that there exist non-archimedean (n.a.) locally convex spaces without basic orthogonal sequences, and even without Schauder basic sequences. Among other things any n.a. Köthe space with the weak topology has no basic orthogonal sequence. On the other hand, we show that the strong dual of any infinite-dimensional n.a. polar Fréchet space and any n.a. LF-space have basic orthogonal sequences.

1 Introduction

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0, \infty)$. For fundamentals on locally convex spaces and normed spaces we refer to [10], [11] and [12]. Basic orthogonal sequences and Schauder bases in locally convex spaces are studied in [4], [5] and [14] – [18].

Any infinite-dimensional (i.d.) Banach space of countable type is isomorphic to the Banach space c_0 of all sequences in $\mathbb{K}$ converging to zero with the sup-norm ([11], Theorem 3.16), so it has an orthogonal basis. Thus any i.d. Banach space has a basic orthogonal sequence.

In [4] it is shown that any locally convex space (lcs) in which not every bounded set is compactoid has a basic orthogonal sequence ([4], Corollary 3.1) and that any i.d. metrizable lcs of finite type has an orthogonal basis ([4], Theorem 3.5).

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In [14] we proved that any i.d. metrizable lcs has a basic orthogonal sequence. Thus we solved the problem stated in [4] whether any i.d. Fréchet space has a basic orthogonal sequence.

In [15] we showed that there exist i.d. Fréchet spaces of countable type without orthogonal bases.

In this paper we prove that there exist i.d. locally convex spaces without basic orthogonal sequences: For any strongly polar Fréchet space E with a continuous norm the spaces $(E, \sigma(E, E'))$ and $(E', \sigma(E', E))$ have no basic orthogonal sequences (Propositions 2 and 4). We also show that a lcs E of finite type has a basic orthogonal sequence if and only if E has an i.d. closed metrizable subspace (Proposition 5).

It is known that $(l'_\infty, \sigma(l'_\infty, l_\infty))$ has no Schauder basis, if $\mathbb{K}$ is spherically complete ([5], Remark 2.14(ii)). We improve this result by proving that $(l'_\infty, \sigma(l'_\infty, l_\infty))$ has no Schauder basic sequence, if $\mathbb{K}$ is spherically complete and the valuation of $\mathbb{K}$ is dense (Proposition 7). Thus there exist i.d. locally convex spaces (at least over spherically complete fields with a dense valuation) without Schauder basic sequences.

On the other hand, we prove that the strong dual of any i.d. polar Fréchet space has a basic orthogonal sequence (Proposition 12).

We also note that any LF-space has a basic orthogonal sequence (Corollary 15).

2 Preliminaries

The linear span of a subset A of a linear space E is denoted by $\operatorname{lin} A$.

A *seminorm* on a linear space E is a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x + y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if $\ker p = \{0\}$.

We assume that a locally convex space (lcs) is a Hausdorff space.

A *Fréchet space* is a complete metrizable lcs.

Let E be a locally convex space.

Denote by $\mathcal{P}(E)$ the set of all continuous seminorms on E. A family $\mathcal{B} \subset \mathcal{P}(E)$ is a *base* in $\mathcal{P}(E)$ if for any $p \in \mathcal{P}(E)$ there is $q \in \mathcal{B}$ with $q \geq p$.

A sequence (x_n) in E is *orthogonal with respect to* a family $\mathcal{B}$ in $\mathcal{P}(E)$ if $p(\sum_{i=1}^n \alpha_i x_i) = \max_{1 \leq i \leq n} p(\alpha_i x_i)$ for all $p \in \mathcal{B}, n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$.

A sequence (x_n) of non-zero elements of E is a *basic orthogonal sequence* in E if it is orthogonal with respect to some base $\mathcal{B}$ in $\mathcal{P}(E)$. A linearly dense basic orthogonal sequence in E is called an *orthogonal basis* in E.

A sequence (x_n) in a lcs E is a *Schauder basis* of E if each element x in E can be written uniquely as $x = \sum_{n=1}^\infty \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$ and the coefficient functionals $f_n : E \to \mathbb{K}, x \to \alpha_n \, (n \in \mathbb{N})$ are continuous. A sequence (x_n) in a lcs E is a *Schauder basic sequence* if it is a Schauder basis of the closed linear span of (x_n).

It is known that any basic orthogonal sequence in a lcs E is a Schauder basic sequence in E and that every Schauder basic sequence in a Fréchet space F is a basic orthogonal sequence in F ([4], Propositions 1.4 and 1.7).

For any seminorm p on a linear space E the map $\overline{p} : E_p \to [0, \infty), x + \ker p \to p(x)$ is a norm on $E_p = (E / \ker p)$.

A lcs E is *of finite type* if $\dim E_p < \infty$ for any $p \in \mathcal{P}(E)$, and *of countable type* if the normed space $(E_p, \overline{p})$ contains a linearly dense countable set for any $p \in \mathcal{P}(E)$.

Any i.d. Fréchet space of finite type is isomorphic to the Fréchet space $\mathbb{K}^{\mathbb{N}}$ of all sequences in $\mathbb{K}$ with the topology of pointwise convergence (see [4], Theorem 3.5).

The strong dual of $\mathbb{K}^{\mathbb{N}}$ we denote by ϕ. The topology of ϕ is the finest locally convex topology on ϕ.

By a *Köthe space* we mean a Fréchet space E with a Schauder basis (x_n) and with a continuous norm.

An *LM-space* (respectively, *LF-space*) is a lcs E which is the inductive limit of an inductive sequence (E_n) of metrizable locally convex spaces (respectively, of Fréchet spaces).

Put $B_{\mathbb{K}} = \{\alpha \in \mathbb{K} : |\alpha| \leq 1\}$. By the *absolutely convex hull* of a subset A of a lcs E we mean the set $\operatorname{co} A = \{\sum_{i=1}^{n} \alpha_i a_i : n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in B_{\mathbb{K}}, a_1, \ldots, a_n \in A\}$.

A subset B of a lcs E is *absolutely convex* if $\operatorname{co} B = B$.

A lcs E is *Baire-like* if for any increasing sequence (A_n) of absolutely convex closed subsets of E covering E there is $n \in \mathbb{N}$ such that A_n is a neighbourhood of zero in E.

For a subset A of a lcs E we put $A^{\circ} = \{f \in E' : |f(x)| \leq 1 \text{ for all } x \in A\}$ and $A^{\circ\circ} = \{x \in E : |f(x)| \leq 1 \text{ for all } f \in A^{\circ}\}$; A is a *polar* set if $A = A^{\circ\circ}$.

For an absolutely convex subset A of a lcs E we put $A^e = A$ if the valuation of $\mathbb{K}$ is discrete, and $A^e = \bigcap\{\alpha A : \alpha \in \mathbb{K} \wedge |\alpha| > 1\}$ if the valuation of $\mathbb{K}$ is dense.

A lcs E is *polar* if any neighbourhood of zero in E contains a polar one, and *strongly polar* if every absolutely convex neighbourhood U of zero in E with $U = U^e$ is a polar set. Any lcs of countable type is strongly polar ([12], Theorem 4.4); if $\mathbb{K}$ is spherically complete, then any lcs over $\mathbb{K}$ is strongly polar ([12]).

A subset B of a lcs E is *compactoid* if for each neighbourhood U of zero in E there exists a finite subset A of E such that $B \subset U + \operatorname{co} A$. By a *Fréchet-Montel space* we mean a Fréchet space in which any bounded subset is compactoid.

3 Results

First we show the following.

Proposition 1. *Let E^* be the algebraic dual of a linear space E. Then the lcs $E_\sigma = (E, \sigma(E, E^*))$ has no basic orthogonal sequence.*

Proof. Suppose, by contradiction, that (x_n) is a basic orthogonal sequence in E_σ. Let B be a Hamel basis of E such that $(x_n) \subset B$. Let $(f_b)_{b \in B} \subset E^*$ be the family of coefficient functionals associated with the basis B. Then for any $x \in E$ the set $\{b \in B : f_b(x) \neq 0\}$ is finite and $x = \sum_{b \in B} f_b(x) b$. Put $f(x) = \sum_{b \in B} f_b(x), x \in E$; clearly $f \in E^*$ and $f(b) = 1, b \in B$. Since $|f| \in \mathcal{P}(E_\sigma)$, then there exists $q \in \mathcal{P}(E_\sigma)$ with $q \geq |f|$ such that (x_n) is orthogonal with respect to q. Because $\dim(E/\ker q) < \infty$, the set $M = \{n \in \mathbb{N} : q(x_n) \neq 0\}$ is finite. Indeed, let $m \in \mathbb{N}, \alpha_1, \ldots, \alpha_m \in \mathbb{K}, i_1, \ldots, i_m \in M$ with $i_1 < \cdots < i_m$ such that $\sum_{k=1}^{m} \alpha_k (x_{i_k} + \ker q) = \ker q$. Then $0 = q(\sum_{k=1}^{m} \alpha_k x_{i_k}) = \max_{1 \leq k \leq m} |\alpha_k| q(x_{i_k})$, so $\alpha_1 = \cdots = \alpha_m = 0$. Thus $(x_m + \ker q)_{m \in M}$ is linearly independent in $E/\ker q$, so M is finite. Hence the set $\{n \in \mathbb{N} : |f(x_n)| \neq 0\}$ is finite; a contradiction. ■

For Fréchet spaces we have the following.

Proposition 2. *Let E be a strongly polar Fréchet space. Then the lcs $E_\sigma = (E, \sigma(E, E'))$ has a basic orthogonal basic sequence if and only if E has no continuous norm.*

Proof. If E has no continuous norm, then E contains an isomorphic copy F of $\mathbb{K}^{\mathbb{N}}$ ([1], Proposition 2.6). Hence $(F, \sigma(F, F'))$ has an orthogonal basis. Clearly, $\sigma(F, F') = \sigma(E, E')|F$; so E_σ has a basic orthogonal sequence.

Now we assume that E has a continuous norm p. Suppose, by contradiction, that E_σ has a basic orthogonal sequence (x_n). Let D be the closed linear span of (x_n) in E_σ. By [9], Theorem 1.3, (x_n) is a Schauder basis in the strongly polar Fréchet space D. Let (f_n) be the sequence of coefficient functionals associated with the basis (x_n). Without loss of generality we can assume that $p(x_n) \geq 1, n \in \mathbb{N}$. Clearly, $f_n(x)x_n \to_n 0$ in D for any $x \in D$. Hence $|f_n(x)| \to_n 0, x \in D$. Thus the series $\sum_{n=1}^{\infty} f_n(x)$ is convergent for any $x \in D$. By the Banach-Steinhaus theorem the linear functional $f(x) = \sum_{n=1}^{\infty} f_n(x), x \in D$, is continuous on D; clearly $f(x_n) = 1, n \in \mathbb{N}$. Let $g \in E'$ with $g|D = f$ ([10], Theorem 4.2). Then there is $q \in \mathcal{P}(E_\sigma)$ with $q \geq |g|$ such that (x_n) is orthogonal with respect to q. Since $\dim(E/\ker q) < \infty$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we have $q(x_n) = 0$. Hence, $f(x_n) = 0$ for any $n \geq n_0$; a contradiction. ∎

Corollary 3. *For any strongly polar Fréchet space E with a continuous norm the lcs E_σ has no basic orthogonal sequence. In particular, any Köthe space E with the weak topology has no basic orthogonal sequence.*

The conclusion of Proposition 2 (and Corollary 3) also holds if we assume that the Fréchet space E has the following property, usually called *property* (*): For every subspace of countable type D of E, each $f \in D'$ has a continuous linear extension $g \in E'$. The class of strongly polar Fréchet spaces is strictly contained in the class of Fréchet spaces with property (*). For example, a Banach space with a basis of a non-countable cardinality (in the sense of [11]) is not necessarily strongly polar, but such a space has property (*) (see the proof (i)–(iii) of Theorem 1.3 of [9], where it is also shown that each lcs with property (*) is an (O.P.)-space i.e. every weakly convergent sequence is convergent).

Clearly, for any i.d. strongly polar Fréchet space E, the lcs E_σ has a Schauder basic sequence, since E has a Schauder basic sequence.

It is easy to see that the weak dual E'_σ of a Fréchet space E has a Schauder basic sequence if E has a quotient F with a Schauder basis (x_n). Thus, by [19], Theorems 2 and 11, the weak dual of any i.d. Fréchet space of countable type has a Schauder basic sequence. For basic orthogonal sequences we have the following.

Proposition 4. *The weak dual $E'_\sigma = (E', \sigma(E', E))$ of any Fréchet space E has no basic orthogonal sequence.*

Proof. Suppose, by contradiction, that E'_σ has a basic orthogonal sequence (f_n). Let F be the closed linear span of (f_n) in E'_σ and let $(f_n^*) \subset F'$ be the sequence of coefficient functionals associated with the orthogonal basis (f_n) in F. Clearly, E'_σ is of countable type; so for any $n \in \mathbb{N}$ there is $x_n \in E$ such that $f_n^*(f) = f(x_n)$ for any

$f \in F$ ([12], Theorems 4.2 and 4.4). For some sequence (α_n) of non-zero scalars the series $\sum_{n=1}^{\infty} \alpha_n x_n$ is convergent in E to some element x_0. Clearly, $f_n(x_0) = \alpha_n \neq 0$ for any $n \in \mathbb{N}$. The seminorm $p(f) = |f(x_0)|, f \in E'$, is continuous on E'_σ, so there is $q \in \mathcal{P}(E'_\sigma)$ with $q \geq p$ such that (f_n) is orthogonal with respect to q. Since $\dim(E'/\ker q) < \infty$, then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we have $q(f_n) = 0$. Hence $p(f_n) = 0$ for $n \geq n_0$, so $f_n(x_0) = 0$ for $n \geq n_0$; a contradiction. ∎

The above results also can be obtained using the following.

Proposition 5. *A lcs E of finite type has a basic orthogonal sequence if and only if E contains an i.d. metrizable closed subspace.*

Proof. It is known that any i.d. metrizable lcs of finite type has an orthogonal basis ([4], Theorem 3.5). Thus it is enough to show that any lcs E of finite type with an orthogonal basis (x_n) is metrizable.

Let (f_n) be the sequence of coefficient functionals associated with the basis (x_n). Let $p \in \mathcal{P}(E)$. Then there is $q \in \mathcal{P}(E)$ with $q \geq p$ such that (x_n) is orthogonal with respect to q. Since $\dim(E/\ker q) < \infty$, there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$ we have $q(x_n) = 0$. Put $C = \max_n q(x_n)$ and $p_n(x) = \max_{1 \leq k \leq n} |f_k(x)|, x \in E, n \in \mathbb{N}$. Clearly $(p_k) \subset \mathcal{P}(E)$. It is easy to check that (x_n) is orthogonal with respect to (p_k). For any $x \in \operatorname{lin}(x_n)$ we have $q(x) \leq C p_{n_0}(x)$. Indeed, let $m \geq n_0$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$; then $q(\sum_{i=1}^m \alpha_i x_i) = \max_{1 \leq i \leq m} q(\alpha_i x_i) \leq C \max_{1 \leq i \leq m} p_{n_0}(\alpha_i x_i) = C p_{n_0}(\sum_{i=1}^m \alpha_i x_i)$. Since (x_n) is linearly dense in E, we get $p(x) \leq q(x) \leq C p_{n_0}(x), x \in E$. Thus E is metrizable. ∎

By the proof of Proposition 5 we have the following.

Corollary 6. *A lcs E of finite type has an orthogonal basis (x_n) if and only if E is metrizable.*

Now we shall prove that there exist i.d. locally convex spaces (at least over some fields) without Schauder basic sequences. It is known that if $\mathbb{K}$ is spherically complete and E is an i.d. Banach space over $\mathbb{K}$, then $(E'', \sigma(E'', E'))$ does not have a Schauder basis ([5], Remark 2.16(iii)). We can improve this result by proving the following.

Proposition 7. *Assume that the field $\mathbb{K}$ is spherically complete and the valuation of $\mathbb{K}$ is dense. If E is an i.d. Banach space over $\mathbb{K}$, then $(E'', \sigma(E'', E'))$ has no Schauder basic sequence. In particular, the lcs $(l'_\infty, \sigma(l'_\infty, l_\infty))$ has no Schauder basic sequence.*

Proof. By [11], Corollary 4.5, E' is spherically complete. Using [11], Corollary 5.20, we infer that E' has no quotient with a Schauder basis (see also [8], Remark 2.6).

Suppose, by contradiction, that $(E'', \sigma(E'', E'))$ has a Schauder basic sequence (f_n); denote by F its closed linear span. Put $G = \{x \in E' : f(x) = 0 \text{ for all } f \in F\}$ and $H = (E'/G)$. Let $T : E' \to H$ be the quotient map and let $T' : H' \to E''$ be the adjoint map of T. As in the archimedean case one can show that $T'(H') = F$ and T' is an isomorphism between $(H', \sigma(H', H)))$ and $(F, \sigma(E'', E')|F)$. Thus $(H', \sigma(H', H))$ has a Schauder basis. Hence, by [5], Proposition 2.13, the Banach space H has a Schauder basis; a contradiction. ∎

One can check that the lcs $(l_\infty, \sigma(l_\infty, l'_\infty))$ possesses a Schauder basic sequence and has no basic orthogonal sequence (for any field $\mathbb{K}$).

If $\mathbb{K}$ has a discrete valuation, then by [8], Remark 2.6 (or [11], Corollary 4.14) any i.d. Banach space E over $\mathbb{K}$ has a quotient with a Schauder basis (x_n), so $(E', \sigma(E', E))$ has a Schauder basic sequence.

Next we will show that the spaces $C_p(X)$ and $C_c(X)$ of continuous functions have basic orthogonal sequences.

Let X be an infinite zero-dimensional Hausdorff space and let $\mathcal{F}$ be a family $(F_t)_{t\in T}$ of compact subsets of X with $\bigcup_{t\in T} F_t = X$ such that for all $t_1, t_2 \in T$ there exists $t_3 \in T$ with $F_{t_1} \cup F_{t_2} \subset F_{t_3}$. Denote by $C_{\mathcal{F}}(X)$ the space $C(X)$ of all continuous functions from X to $\mathbb{K}$, with the topology of uniform convergence on the elements of $\mathcal{F}$. The seminorms $p_{n,t}(f) = n \sup_{x\in F_t} |f(x)|, f \in C(X), n \in \mathbb{N}, t \in T$, form a base in $\mathcal{P}(C_{\mathcal{F}}(X))$.

If $\mathcal{F}$ is the family $\mathcal{F}_p$ of all finite subsets of X, then $C_{\mathcal{F}}(X)$ is the space $C_p(X)$ of all continuous functions from X to $\mathbb{K}$ with the topology of pointwise convergence. If $\mathcal{F}$ is the family $\mathcal{F}_c$ of all compact subsets of X, then $C_{\mathcal{F}}(X)$ is the space $C_c(X)$ of all continuous functions from X to $\mathbb{K}$ with the compact-open topology.

Any infinite clopen (i.e. closed and open) subset A of X contains a non-empty clopen subset B such that $(A \setminus B)$ is infinite. Thus X possesses a sequence (U_n) of nonempty clopen subsets pairwise disjoint. Let f_n be the characteristic function of U_n for $n \in \mathbb{N}$. Then (f_n) is an orthogonal sequence in $C_{\mathcal{F}}(X)$. Indeed, let $n \in \mathbb{N}, t \in T, m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$. We have $p_{n,t}(\sum_{i=1}^m \alpha_i f_i) = n \sup_{x\in F_t} |\sum_{i=1}^m \alpha_i f_i(x)| = \max_{i\in M_t} n|\alpha_i| = \max_{1\le i\le m} p_{n,t}(\alpha_i f_i)$, where $M_t = \{i \in \mathbb{N} : 1 \le i \le m \wedge U_i \cap F_t \neq \emptyset\}$ $(\max \emptyset = 0)$. Thus (f_n) is orthogonal with respect to the base $\{p_{n,t} : n \in \mathbb{N}, t \in T\}$. Similarly one can check that any sequence $(f_n) \subset C(X)$ of functions with pairwise disjoint nonempty supports is a basic orthogonal sequence in $C_{\mathcal{F}}(X)$.

Thus we have the following.

Proposition 8. *The lcs $C_{\mathcal{F}}(X)$ has a basic orthogonal sequence.*

Corollary 9. *The spaces $C_p(X)$ and $C_c(X)$ have basic orthogonal sequences.*

Note that for some spaces X the lcs $C_p(X)$ possesses i.d. closed subspaces without basic orthogonal sequences. Indeed, let E be an i.d. polar Fréchet space. Then $C_p(E)$ has an i.d. closed subspace isomorphic to $(E', \sigma(E', E))$ which has no basic orthogonal sequence.

Using Corollary 6 it is easy to obtain the following.

Corollary 10. *The lcs $C_p(X)$ has an orthogonal basis (f_n) if and only if the set X is countable.*

To prove our next proposition we need the following lemma (compare with [3], Lemma 2.2.3 and Remarks 2.2.5).

Lemma 11. *Every barrelled lcs E which is not Baire-like contains an isomorphic copy of ϕ.*

Proof. Let (A_n) be an increasing sequence of closed absolutely convex subsets of E covering E such that A_n is not a neighbourhood of zero in E for any $n \in \mathbb{N}$. Without loss of generality we can assume that the sequence $\operatorname{lin} A_n, n \in \mathbb{N}$, is strictly increasing. Let $x_n \in (A_{n+1} \setminus \operatorname{lin} A_n), n \in \mathbb{N}$. We shall prove that the linear span X of (x_n) is isomorphic to ϕ. Let p be a seminorm on X. By [13], Corollary 1.2, and [3], Lemma 0.1, the sets $B_n = (A_n + \operatorname{lin}\{x_k : k < n\})^e, n \in \mathbb{N}$, are closed; clearly $x_n \in (B_{n+1} \setminus \operatorname{lin} B_n), n \in \mathbb{N}$.

Using [3], Proposition 0.2, we can find inductively a sequence $(p_n) \subset \mathcal{P}(E)$ such that $p_n(x) < 1$ for $x \in B_n$ and $p_n(x_n) > \max(\{p_k(x_n) : k < n\} \cup \{p(x_n)\}), n \in \mathbb{N}$. Then $q(x) = \sup_k p_k(x) < \infty$ for any $x \in E$. Since E is barrelled, $q \in \mathcal{P}(E)$.

It is easy to see that $p_n(x_k) = 0$ for all $n, k \in \mathbb{N}$ with $n > k$. Clearly, $q(x_j) = p_j(x_j)$ for any $j \in \mathbb{N}$. Let $(\alpha_j) \subset \mathbb{K}$. For any $n \in \mathbb{N}$ we have $q(\sum_{j=1}^n \alpha_j x_j) \geq p_n(\sum_{j=1}^n \alpha_j x_j) = p_n(\alpha_n x_n) = q(\alpha_n x_n)$. Hence, by induction, we get $q(\sum_{j=1}^n \alpha_j x_j) \geq \max_{1\leq j\leq n} q(\alpha_j x_j), n \in \mathbb{N}$. Thus we obtain $p(\sum_{j=1}^n \alpha_j x_j) \leq \max_{1\leq j\leq n} p(\alpha_j x_j) \leq \max_{1\leq j\leq n} p_j(\alpha_j x_j) = \max_{1\leq j\leq n} q(\alpha_j x_j) \leq q(\sum_{j=1}^n \alpha_j x_j)$ for any $n \in \mathbb{N}$.

It follows that $p(x) \leq q(x)$ for any $x \in X$. We have shown that any seminorm on X is continuous, so X is isomorphic to ϕ. ■

Now we can prove the following.

Proposition 12. (a) *The strong dual $E'_b = (E', b(E', E))$ of any i.d. polar Fréchet space E has a basic orthogonal sequence.*

(b) *The strong dual E'_b of a strongly polar Fréchet space has an orthogonal basis (f_n) if and only if E is a Fréchet-Montel space with an orthogonal basis.*

Proof. (a) Let (U_n) be a decreasing base of polar neighbourhoods of zero in E. Clearly, the sets $U_n^\circ, n \in \mathbb{N}$, are bounded in E'_b. By [4], Corollary 3.1, it is enough to consider the case when for any $n \in \mathbb{N}$ the set U_n° is compactoid in E'_b.

Then, by [12], Proposition 6.5, every $\sigma(E', E)$-bounded subset of E' (i.e. every equicontinuous subset of E') is compactoid in E'_b. Applying [6], Theorem 3.3, we deduce that E is a Fréchet-Montel space; so E is of countable type ([6], Theorem 3.1). It follows from [12], Theorem 10.3, Lemma 9.4, Theorem 8.5 and Corollary 10.10, that E'_b is barrelled.

The space E'_b is not Baire-like. Indeed, (U_n°) is an increasing sequence of absolutely convex closed subsets of E'_b covering E'_b. Recall that we consider the case when for any $n \in \mathbb{N}$ the set U_n° is compactoid in E'_b; then U_n° is not a neighbourhood of zero in E'_b for any $n \in \mathbb{N}$. Using Lemma 11, we infer that E'_b contains an isomorphic copy of ϕ; so E'_b has a basic orthogonal sequence.

(b) If E'_b has an orthogonal basis (f_n), then E is a Fréchet-Montel space ([12], Theorem 8.5) and reflexive ([12], Theorem 10.3); so any bounded subset of E'_b is compactoid ([12], Theorem 10.7) and E'_b is complete ([12], Proposition 6.8). Using [19], Proposition 6, we infer that $(E'_b)'_b$ has an orthogonal basis.

If E is a Fréchet-Montel space with an orthogonal basis then, by [19], Proposition 6, E'_b has an orthogonal basis. ■

For Fréchet-Montel spaces we can show a stronger result.

Let E be a lcs. The topology $c(E', E)$ of uniform convergence on compactoid subsets of E is a locally convex topology on the dual space E'. By E'_c we denote the lcs $(E', c(E', E))$. E is said to be a *(dF)-space* ([7]) if it has a fundamental sequence of compactoid subsets and the canonical linear map $J_E : E \to (E'_c)'_c$ is an isomorphism. It is easy to see that the strong dual E'_b of any Fréchet-Montel space E is a (dF)-space and $E'_c = E'_b$.

Proposition 13. *Let E be a Fréchet-Montel space. Then any i.d. closed subspace F of the strong dual E'_b of E has a basic orthogonal sequence.*

Proof. E'_b is a reflexive (dF)-space. Hence, by [7], Theorem 5.12, F is isomorphic to $(E/F^{\perp})'_c$, where $F^{\perp} = \{x \in E : f(x) = 0 \text{ for any } f \in F\}$.

If $G = (E/F^{\perp})$ is a Fréchet-Montel space then $G'_c = G'_b$; so F has a basic orthogonal sequence, by Proposition 12.

In the opposite case, G contains a complemented isomorphic copy of c_0 (see [5], Corollary 6.7, [6], Theorem 3.1, and [18], Proposition 3) so G'_c contains an isomorphic copy of $\psi = (c_0)'_c$ ([7], Proposition 2.8).

Thus it is enough to show that ψ has a basic orthogonal sequence. For any $t = (t_n) \in c_0$ the set $B_t = \{x = (x_n) \in c_0 : |x_n| \leq |t_n| \text{ for } n \in \mathbb{N}\}$ is compactoid in c_0 and any compactoid set B in c_0 is contained in B_t for some $t \in c_0$ ([2], Corollary 3.7). The seminorms $q_t(f) = \sup_{x \in B_t} |f(x)|, f \in (c_0)', t \in c_0$ form a base $\mathcal{B}$ in $\mathcal{P}(\psi)$. It is easy to check that the sequence (f_n) of coefficient functionals associated with the coordinate basis (e_n) in c_0 is orthogonal with respect to $\mathcal{B}$; so (f_n) is a basic orthogonal sequence in ψ. ■

By the proof of Proposition 13, for any i.d. Fréchet space E of countable type, E'_c has a basic orthogonal sequence. Hence, by [7], Theorem 5.12, any i.d. quotient of the strong dual E'_b of any Fréchet-Montel space E has a basic orthogonal sequence.

Finally we show the following.

Proposition 14. *Any i.d. barrelled LM-space E has a basic orthogonal sequence.*

Proof. Clearly, ϕ has an orthogonal basis. Thus, by Lemma 11, it is enough to consider the case when E is Baire-like. Using the idea of the proof of Theorem 2.2.2 of [3], we show that E is metrizable. Let (E_n) be an inductive sequence of metrizable locally convex spaces defining E. For any $n \in \mathbb{N}$, let $(U_{n,k})_{k=1}^{\infty}$ be a base of absolutely convex neighbourhoods of zero in E_n. Let U be an absolutely convex closed neighbourhood of zero in E. Let $k \in \mathbb{N}$. Then $U \cap E_k$ is a neighbourhood of zero in E_k, so $U_{k,i_k} \subset U \cap E_k$ for some $i_k \in \mathbb{N}$. Let A_n be the closure of $\sum_{k=1}^{n} U_{k,i_k}$ in E for any $n \in \mathbb{N}$. Clearly, (A_n) is an increasing sequence of absolutely convex closed subsets of E and $A_n \subset U$ for $n \in \mathbb{N}$. Let $\alpha \in \mathbb{K}$ with $|\alpha| > 1$. Then $E = \bigcup_{n=1}^{\infty} \alpha^n A_n$; hence $\alpha^n A_n$ is a neighborhood of zero in E for some $n \in \mathbb{N}$. We have shown that some subfamily of the countable family $\{\mathrm{cl}_E(\sum_{k=1}^{n} U_{k,j_k}) : n, j_1, \ldots, j_n \in \mathbb{N}\}$ is a base of neighbourhoods of zero in E; so E is metrizable.

By [14], Theorem 2, E has a basic orthogonal sequence. ■

Corollary 15. *Any i.d. LF-space possesses a basic orthogonal sequence.*

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