On discontinuous transitive maps and dense orbits

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Abstract

It is well-known that many transitive continuous self-maps have a dense orbit. On the other hand several interesting transitive dynamical systems present some kind of discontinuity. Motivated by a result of A. Peris, we investigate under which conditions a discontinuous transitive map has a dense orbit.

A discrete dynamical system is a pair (X, f) where X is a topological space (called phase space) and $f : X \to X$ is a map (called transition function).

One of the most important property in the theory of dynamical systems is (topological) transitivity : a map $f : X \to X$ is (topologically) transitive if for every pair U and V of non-empty open subsets of X there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$ (for more informations on topological transitivity see, e.g., [7])

It is well-known that for a wide class of phase spaces, including all Baire separable metric spaces, every continuous transitive map $f : X \to X$ has a dense orbit.

On the other hand, there are noteworthy discrete dynamical systems (e.g., the Baker's map) whose transition function is discontinuous.

This considerations led A. Peris to ask when a discontinuous transitive map has a dense orbit.

He showed in [8] that every transitive function $f : X \to X$ with one point of discontinuity has a dense orbit when X is a Baire separable metric space. He also constructed an example of a transitive self-map on the closed interval $[0, 2]$ with two points of discontinuity and without dense orbits.

Therefore it is natural to study the following problem: under which con-

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ditions a transitive map with more than one point of discontinuity has a dense orbit ?

The aim of this note is to give an answer to this question.

In particular we will see, as an application of the main result, a condition ensuring the existence of a dense orbit for a wide class of transitive self-maps with finitely many discontinuity points.

Let $f : X \to X$ be a map, $x \in X$ and $A \subset X$. Let us recall that :

i) the orbit $O(f,x)$ of x under f is the set $\{x, f(x), f^2(x), ...\}$;

ii) A is said to be f-invariant if $f(A) \subset A$.

For a self-map of a topological space X let $D(f)$ be the set of its discontinuity points, moreover let us set $\mathcal{C}(D(f)) = \{\overline{O(f,x)} : x \in D(f)\}$.

A topological space X is second category (a Baire space) if the intersection of every countable family of open dense sets in X is non-empty (dense).

A set A contained in a space X is said to be nowhere dense in X if $\mathrm{Int}(\overline{A}) = \emptyset$.

A family $\mathcal{F}$ of subsets of a space X is called locally finite if every point of X has an open neighbourhood which meets finitely many members of $\mathcal{F}$.

A π-base for a topological space X is a family $\mathcal{B}$ of non-empty open subsets of X such that every non-empty open subset of X contains a member of $\mathcal{B}$. The π-weight of a space X is the smallest infinite cardinality of a π-base for X (see, e.g., [5]). It is clear that every first countable separable space (and, a fortiori, every separable metric space) has countable π-weight, and every space with countable π-weight is separable. Observe that the product 2^{ω_1} of ω_1 copies of the discrete space with two points is a separable space with uncountable π-weight, whereas the Stone-Čech compactification βN of the discrete space N of positive integers has countable π-weight but it is not first countable.

Let us also recall that a map f between topological spaces is called sequentially continuous if $f(x_n) \to f(x)$ whenever $x_n \to x$ (see, e.g., [9]). Clearly every continuous map is sequentially continuous.

Now let us introduce some concepts which will play a significant role to show the existence of a dense orbit.

A map $f : X \to X$ is said to be π-continuous if there is a π-base $\mathcal{B}$ for X such that $\bigcup\{f^{-k}(B) : k \in N\}$ is open in X for every $B \in \mathcal{B}$. Observe that every continuous self-map is π-continuous.

A map $f : X \to X$ is said to be orbit-continuous at a point x of X if $\overline{O(f,x)}$ is f-invariant.

It is clear that every continuous self-map is orbit-continuous at each point (in fact for a continuous map $f : X \to X$ we have $f(\overline{O(f,x)}) \subset \overline{f(O(f,x))} \subset \overline{O(f,x)}$ for every $x \in X$).

Note also that every self-map is orbit-continuous at each point whose orbit is either closed or dense.

We refer the reader to [2], [3], [4] and [10] for notations and terminology not explicitly given.

Our main result shows that, under certain hypothesis, the absence of a discontinuity point with a dense orbit ensures the π-continuity of a map; moreover it exhibits a class of transitive π-continuous self-maps which must have a dense orbit.

Proposition 1. *Let X be a space with countable π-weight and let $f : X \to X$ be a transitive map.*

i) If f is orbit-continuous at each $x \in D(f)$, $\mathcal{C}(D(f))$ is locally finite and $\overline{O(f,x)} \neq X$ for every $x \in D(f)$, then f is π-continuous.

ii) If X is a second category space and f is π-continuous, then f has a dense orbit.

Proof. i) Let us first observe that the only closed f-invariant set with non-empty interior of a (not necessarily continuous) transitive map is the whole space.

Since f is transitive and orbit-continuous at each discontinuity point, it follows that either $\overline{O(f,x)} = X$ or $\mathrm{Int}(\overline{O(f,x)}) = \emptyset$ for every $x \in D(f)$.

So, by hypothesis, it follows that $\mathrm{Int}(\overline{O(f,x)}) = \emptyset$ for every $x \in D(f)$.

Set $O(D(f)) = \bigcup\{O(f,x) : x \in D(f)\}$. Since $\mathcal{C}(D(f))$ is locally finite, it follows that $\overline{O(D(f))} = \bigcup\{\overline{O(f,x)} : x \in D(f)\}$.

Now the union of a locally finite family of nowhere dense sets is nowhere dense, so $\mathrm{Int}(\overline{O(D(f))}) = \emptyset$.

Let us take a countable π-base $\mathcal{B}$ for X. Since $\mathrm{Int}(\overline{O(D(f))}) = \emptyset$, we may assume that $B \subset X \setminus \overline{O(D(f))}$ for every $B \in \mathcal{B}$.

We will show that $\bigcup\{f^{-k}(B) : k \in N\}$ is open in X for every $B \in \mathcal{B}$. So let $B \in \mathcal{B}$ and let $p \in \bigcup\{f^{-k}(B) : k \in N\}$. Then there is some $m \in N$ such that $p \in f^{-m}(B)$, therefore $f^m(p) \in B$. Since f^m is continuous at p, there is some open neighbourhood V of p such that $V \subset f^{-m}(B) \subset \bigcup\{f^{-k}(B) : k \in N\}$. Hence $\bigcup\{f^{-k}(B) : k \in N\}$ is open and f is π-continuous.

ii) Let us first observe that for any (not necessarily continuous) transitive map $f : X \to X$, it follows that $\bigcup\{f^{-k}(U) : k \in N\}$ is dense in X for every non-empty open subset U of X.

Since X has countable π-weight, we may take, without loss of generality, a countable π-base $\mathcal{B} = \{B_n : n \in N\}$ for X witnessing the π-continuity of f.

Then $H_n = \bigcup\{f^{-k}(B_n) : k \in N\}$ is open (and dense) in X. Since X is of second category, there is some $x \in \bigcap\{H_n : n \in N\}$.

We claim that $\overline{O(f,x)} = X$, in fact let V be a non-empty open set in X, then there is a $B_m \subset V$, so $x \in f^{-k}(B_m)$ for some $k \in N$. Therefore $f^k(x) \in B_m \subset V$, and $V \cap O(f,x) \neq \emptyset$.

By the proposition above one obtains the following

Corollary 2. *Let X be a second category space with countable π-weight and let $f : X \to X$ be a transitive map such that $\mathcal{C}(D(f))$ is locally finite. If f is orbit-continuous at each $x \in D(f)$, then f has a dense orbit.*

In particular we have

Corollary 3. *Let X be a second category space with countable π-weight and let $f : X \to X$ be a transitive map with finitely many discontinuity points. If f is orbit-continuous at each $x \in D(f)$, then f has a dense orbit.*

The next lemma gives a condition which ensures the orbit-continuity of a self-map at a discontinuity point.

Lemma 4. *Let f be a self-map on a topological space X and let $x \in D(f)$. If $f(D(f)) \subset \overline{O(f,x)}$, then f is orbit-continuous at x.*

Proof. It is enough to check that for every $p \in \overline{O(f,x)} \setminus D(f)$, it follows that $f(p) \in \overline{O(f,x)}$. So let us take $p \in \overline{O(f,x)} \setminus D(f)$ and let V be an open neighbourhood of $f(p)$. Since f is continuous at p, it follows that that there is some open neighbourhood U of p such that $f(U) \subset V$. Now $U \cap O(f,x) \neq \emptyset$, therefore $V \cap O(f,x) \neq \emptyset$. Hence $f(p) \in \overline{O(f,x)}$.

It should be noted that, by the results above, we may also obtain the nice result of Peris quoted before.

Corollary 5. [8] *Let X be a Baire separable metric space and let $f : X \to X$ be a transitive map with (only) one point of discontinuity $a \in X$. Then there is a point whose orbit is dense.*

Proof. It is enough to observe that f is orbit-continuous at a.

Remarks 6. (i) Let $f : X \to X$ be a map and let $x \in X$. The ω-limit set $\omega(x)$ of x is the (possibly empty) set of limits of all convergent subsequences of $(f^n(x))_{n=0}^{\infty}$. Let us call x a recurrent point if $\omega(x) = \overline{O(f,x)}$ (this definition is equivalent to the standard one when f is a continuous self-map on a compact metric space X). It is worth noting that a sequentially continuous self-map f is orbit-continuous at any recurrent point (in fact every ω-limit set is f-invariant for such a map).

(ii) Proposition 1 (ii) is no longer true if we omit the second category condition, even if we assume the continuity of f. In fact, let f be the restriction of the tent map to the rationals, i.e., let $f : Q \to Q$ be the map given by $f(x) = 1 - |2x - 1|$ for every $x \in Q$. It turns out that f is a continuous transitive map without dense orbits.

(iii) Let f be a self-map on a space X and let us call $DO(f)$ the set of all points of X with dense orbit. If the space X in Prop. 1 ii) is a Baire space, we can conclude that $\overline{DO(f)} = X$. It is interesting to note that if f is a continuous self-map on a compact Hausdorff space X without isolated points then either $DO(f) = X$ or $\mathrm{Int}(DO(f)) = \emptyset$ ([1]).

(iv) Let f be a self-map on a T_1-space without isolated points. The transitivity of f allows us to obtain some information on the nature of f. In fact, in such a case, f cannot be constant on any subset with non-empty interior. Suppose not, and let C be a subset of X with non-empty interior such that $f(C) = \{q\}$ for some $q \in X$. Since X is an infinite T_1-space, it follows that $O(f,q)$ cannot be dense in X (otherwise $f^n(q) \in \mathrm{Int}(C)$ for some $n \in N$, so $f^{n+1}(q) = q$, a contradiction).

Therefore $V = X \setminus \overline{O(f,q)}$ is a non-empty open subset of X.

Since f is transitive, there are some positive integer m and $c \in \mathrm{Int}(C)$ such that $f^m(c) = f^{m-1}(f(c)) = f^{m-1}(q) \in V \cap O(f,q)$, this is a contradiction.

In particular, let J be an interval of the real line and let f be a transitive self-map on J. Then f cannot send a non-degenerate subinterval of J to a singleton.

(v) Since a finite orbit of a self-map on a T_1-space is closed, we have also the following particular result which should be mentioned : let X be a second category T_1-space with countable π-weight and let $f : X \to X$ be a transitive map with finitely many discontinuity points x_1,..., x_n. If x_1,..., x_n have finite orbits, then f has a dense orbit.

(vi) The condition in Corollary 3 regarding the orbit-continuity of f at the discontinuity points cannot be omitted, even if X is a compact metric space. In fact, as already noted, in [8] it is constructed a transitive function $f : [0,2] \to [0,2]$ without dense orbits which is discontinuous (only) at 1 and 2. This function is not orbit-continuous at its discontinuity points, in fact it turns out that $\overline{O(f,1)} = [1,2]$ and $\overline{O(f,2)} = [0,1]$, while $f(1) > 1$ and $f(2) < 1$.

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