On maps of tori

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Abstract

Given a field K, all polynomial maps $\mathbb{S}^{m_1}(K) \times \cdots \times \mathbb{S}^{m_k}(K) \to \mathbb{T}^n(K)$ of generalized tori over K are studied. Furthermore, a full description of holomorphic maps $\mathbb{C}^m \to \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k}$ and $\mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_k} \to \mathbb{C}^n$ which restrict to maps $\mathbb{T}^m \to \mathbb{S}^{2n_1-1} \times \cdots \times \mathbb{S}^{2n_k-1}$ and $\mathbb{S}^{2m_1-1} \times \cdots \times \mathbb{S}^{2m_k-1} \to \mathbb{T}^n$ respectively, is presented.

Introduction

In virtue of Wood [7] (see also [4, Chapter 13]) a necessary condition for the existence of a non-constant polynomial map $\mathbb{S}^m \to \mathbb{S}^n$ of spheres for $m \geq n$ is that $2^{k+1} > m \geq n \geq 2^k$ for some $k \geq 0$. It follows that all polynomial maps $\mathbb{S}^m \to \mathbb{S}^n$ are constant if $m \geq 2n$. In particular, any polynomial map $\mathbb{S}^m \to \mathbb{S}^1$ is constant for $m > 1$. If we replace the reals $\mathbb{R}$ by the complex numbers $\mathbb{C}$ the situation is different. Write $\mathbb{S}^m(\mathbb{C})$ for the complex sphere defined as the locus of complex zeros in $\mathbb{C}^{m+1}$ of the polynomial $\sum_{k=0}^{m} X_k^2 - 1$. Given a complex polynomial map $f : \mathbb{S}^m(\mathbb{C}) \to \mathbb{C}$ we get a non-constant one $(f, if, 1, 0\ldots, 0) : \mathbb{S}^m(\mathbb{C}) \to \mathbb{S}^n(\mathbb{C})$ for $n \geq 2$. Whence the only remaining cases for complex spheres to be considered are polynomial maps $\mathbb{S}^m(\mathbb{C}) \to \mathbb{S}^1(\mathbb{C})$. It was shown in [8] that from the homotopy point of view nothing is lost by complexifying the problem of which homotopy classes of maps of spheres contain a polynomial representative. Furthermore, in virtue of [5], any complex polynomial self-map of $\mathbb{S}^2(\mathbb{C})$ yields a regular self-map of the sphere $\mathbb{S}^2$ in a canonical way. Then Loday [6], using algebraic and topological K-theory,

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proved some results on polynomials maps into $\mathbb{S}^n$. For instance, every polynomial map from the torus $\mathbb{T}^n$ (as the n-th cartesian power of $\mathbb{S}^1$) to $\mathbb{S}^n$ is null-homotopic if $n > 1$. For n even those results were extended in [1, 2] to regular and then in [4] to polynomial maps $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k} \to \mathbb{S}^n$ with $n = n_1 + \cdots + n_k$ odd.

This paper grew out of our attempt to describe all polynomial self-maps of the real and complex circle as well. This was accomplished by some analytic methods and then some purely algebraic ideas came to simplify our investigations. We tend to transfer some results from [6, 7, 8] on spheres and their polynomial maps into spheres over any field. Given a field K, the *m-sphere* $\mathbb{S}^m(K) \subseteq K^{n+1}$ over K is defined as the locus of zeros in K^{m+1} of the polynomial $\sum_{k=0}^m X_k^2 - 1$. Write $\mathbb{T}^n(K)$ for the n-th cartesian power of $\mathbb{S}^1(K)$ for any $n \geq 1$ called the *n-torus* over K.

If K is the field of real or complex numbers then any continuous map $\mathbb{S}^{m_1}(K) \times \cdots \times \mathbb{S}^{m_k}(K) \to \mathbb{T}^n(K)$ is obviously null-homotopic provided $m_1, \ldots, m_t > 1$. On the other hand, those polynomial maps are worth to be studied from the algebraic point of view. Section 1 takes up the systematic study of polynomial maps $\mathbb{S}^{m_1}(K) \times \cdots \times \mathbb{S}^{m_k}(K) \to \mathbb{T}^n(K)$ for any field K. We make use of the abelian group structure on the sphere $\mathbb{S}^1(K)$ to show

Theorem 1.4. *Let K be a field and $f : \mathbb{S}^1(K) \to \mathbb{S}^1(K)$ a polynomial self-map. Then:*

(1) *if K is of characteristic two, $f = (f_0, 1 + f_0)$;*
(2) *if K is finite, any set self-map of $\mathbb{S}^1(K)$ is a polynomial one;*
(3) *if K is an infinite field of characteristic different from two, $f(z) = \alpha z^k$ for any $z \in \mathbb{S}^1(K)$ with some $\alpha \in \mathbb{S}^1(K)$ and an integer k.*

However, any self-map $\mathbb{S}^1(K) \to \mathbb{S}^1(K)$ is polynomial for a finite field K. Then, we observe that $\mathbb{S}^n(K)$ and K^n are birationally equivalent to assert

Theorem 1.7. *Let K be a field, $f : \mathbb{S}^n(K) \to \mathbb{S}^1(K)$ a polynomial map and $n \geq 2$. Then:*

(1) *if K is of characteristic two, $f = (f_0, 1 + f_0)$;*
(2) *if K is finite, any set map $\mathbb{S}^n(K) \to \mathbb{S}^1(K)$ is a polynomial one;*
(3) *if K is an infinite field of characteristic different from two, f is constant.*

A full description of all polynomial maps $\mathbb{S}^{m_1}(K) \times \cdots \times \mathbb{S}^{m_k}(K) \to \mathbb{T}^n(K)$ is derived in

Corollary 1.8. *Let K be an infinite field and $f = (f_1, \ldots, f_n) : \mathbb{S}^{m_1}(K) \times \cdots \times \mathbb{S}^{m_k}(K) \to \mathbb{T}^n(K)$ a polynomial map with $m_1, \ldots, m_k \geq 1$. Then:*

(1) *$f_j = (f'_j, f'_j + 1)$, where f'_j is a polynomial function on the sphere $\mathbb{S}^{m_j}(K)$ for $j = 1, \ldots, n$ provided K is of characteristic two;*

(2)

$$f_j(x_1, \ldots, x_k) = \alpha_j x_1^{l_{1j}} \cdots x_k^{l_{kj}}$$

for any point $(x_1, \ldots, x_k) \in \mathbb{S}^{m_1}(K) \times \cdots \times \mathbb{S}^{m_k}(K)$ and $j = 1, \ldots, n$ with some $\alpha_1, \ldots, \alpha_n \in \mathbb{S}^1(K)$ and integers l_{uv} such that $l_{uv} = 0$ provided $m_u > 1$ for $u = 1, \ldots, k$, $v = 1, \ldots, n$ and K is of characteristic different from two. We put $x^0 = 1$ whenever $x \in \mathbb{S}^d(K)$ with $d \geq 1$ and identify points of $\mathbb{S}^1(K)$ with elements of the group $U(\widetilde{K})$.

Section 2 ends the paper with purely topological investigations. Holomorphic maps $\mathbb{C}^m \to \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k}$ and $\mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_k} \to \mathbb{C}^n$ which restrict to $\mathbb{T}^m \to$

$\mathbb{S}^{2n_1-1}\times\cdots\times\mathbb{S}^{2n_k-1}$ and $\mathbb{S}^{2m_1-1}\times\cdots\times\mathbb{S}^{2m_k-1}\to\mathbb{T}^n$ respectively, are then considered. It follows that any map $\mathbb{T}^m\to\mathbb{S}^{2n_1-1}\times\cdots\times\mathbb{S}^{2n_k-1}$ is null-homotopic through a complex polynomial homotopy provided $n_1,\ldots,n_k>1$ (see [1, 3, 6] for the real case).

1 Tori over fields and their maps

Let K be a field. The affine set

$$\mathbb{S}^n(K)=V_K\Big(\sum_{k=0}^{n}X_k^2-1\Big)$$

in K^{n+1}, given by the zeros of the polynomial $\sum_{k=0}^{n}X_k^2-1$, is called the *n-sphere* over the field K for $n\geq 0$. We say that a map $f=(f_0,\ldots,f_n):\mathbb{S}^m(K)\to\mathbb{S}^n(K)$ is *polynomial* if there exist polynomials $F_0,\ldots,F_n\in K[X_0,\ldots,X_m]$ such that $f_k(x)=F_k(x)$ and $\sum_{k=0}^{n}f_k(x)^2=1$ for all $x\in\mathbb{S}^m(K)$. If characteristic of K is two then $\mathbb{S}^n(K)=V_K(\sum_{k=0}^{n}X_k+1)$. Whence any polynomials $F_1,\ldots,F_n\in K[X_0,\ldots,X_m]$ with $F_0=1+\sum_{k=1}^{n}F_k$ determine a polynomial map $f=(f_0,\ldots,f_n):\mathbb{S}^m(K)\to\mathbb{S}^n(K)$.

Write now $U(K)$ for the multiplicative group of K.

Proposition 1.1. *Let K be an algebraically closed field and $f:U(K)\to U(K)$ a self-map given by a Laurent polynomial $F\in K[X,X^{-1}]$. Then, there exist $\alpha\in U(K)$ and an integer k such that $f(r)=\alpha r^k$ for all $r\in U(K)$.*

Proof. Let $F(X)=X^{-n}G(X)$ for some natural number n, where $G(X)\in K[X]$. Since $F(r)\neq 0$ for all $r\in U(K)$ and K is an algebraically closed so we have $G(X)=\alpha X^m$ with $\alpha\neq 0$ and some $m\geq 0$. Thus $f(r)=\alpha r^{m-n}$ for any $r\in U(K)$. ∎

Observe that on the 1-sphere $\mathbb{S}^1(K)$ called also a *circle* there is an abelian group structure defined by $(x_0,x_1)\circ(x_0',x_1')=(x_0x_0'-x_1x_1',x_0x_1'+x_1x_0')$ for any points $(x_0,x_1),(x_0',x_1')\in\mathbb{S}^1(K)$. On the other hand, consider for convenience the quotient ring $\widetilde{K}=K[X]/(X^2+1)$ and denote by I the class of X in $\widetilde{K}$. Whence any element of $\widetilde{K}$ is uniquely written as $r+sI$ with $r,s\in K$. Then, the circle $\mathbb{S}^1(K)$ might be identified with the subgroup of units $U(\widetilde{K})$ of the ring $\tilde{K}$ given by elements x_0+x_1I with $x_0,x_1\in K$ and $x_0^2+x_1^2=1$.

Suppose first that K is a field of characteristic different from two and there is an element $i\in K$ with $i^2=-1$. Be aware that $I\neq i$ and observe that $\widetilde{K}$ always has zero divisors because $(I+i)(I-i)=0$. It is clear that the map

$$\rho:\mathbb{S}^1(K)\to U(K)$$

given by $\rho(x_0+x_1I)=x_0+x_1i$ for $x_0+x_1I\in\mathbb{S}^1(K)$ is an isomorphism of groups with the inverse

$$\rho^{-1}:U(K)\to\mathbb{S}^1(K)$$

given by $\rho^{-1}(r) = \frac{1+r^2}{2r} + \frac{i(1-r^2)}{2r}I$ for $r \in U(K)$. Furthermore, any polynomial self-map $f = (f_0, f_1) : \mathbb{S}^1(K) \to \mathbb{S}^1(K)$ is given by $f(z) = f_0(\frac{1+z^2}{2z}, \frac{1-z^2}{2z}I) + f_1(\frac{1+z^2}{2z}, \frac{1-z^2}{2z}I)I$ for any $z \in \mathbb{S}^1(K)$ using the identification of $\mathbb{S}^1(K)$ as a subgroup of $U(\widetilde{K})$.

Corollary 1.2. *Let K be an algebraically closed field of characteristic different from two. Then for any polynomial self-map $f : \mathbb{S}^1(K) \to \mathbb{S}^1(K)$ there exist $\beta \in \mathbb{S}^1(K)$ and an integer k such that $f(z) = \beta z^k$ for any $z \in \mathbb{S}^1(K) \subseteq \widetilde{K}$.*

Proof. Given a polynomial self-map $f = (f_0, f_1) : \mathbb{S}^1(K) \to \mathbb{S}^1(K)$ consider the Laurent polynomial map $g : U(K) \to U(K)$ defined by $g(r) = \rho(f(\rho^{-1}(r)) = f_0(\frac{1+r^2}{2r}, \frac{1-r^2}{2r}i) + f_1(\frac{1+r^2}{2r}, \frac{1-r^2}{2r}i)i$ for $r \in U(K)$. Then Proposition 1.1 says that there are $\alpha \in U(K)$ and an integer k such that $g(r) = \alpha r^k$ for all $r \in U(K)$ and so taking $z = \rho^{-1}(r)$ and $\beta = \rho^{-1}(\alpha)$ we get $f(z) = \beta z^k$ for all $z \in \mathbb{S}^1(K)$. ∎

Now, for a field K of characteristic different from two, let $V_K = V_K(X_0 - 1, \sum_{k=1}^n X_k^2)$ and $\mathbb{S}_i^{n-1}(K) = V_K(\sum_{k=1}^n X_k^2 + 1)$ be affine sets in K^{n+1} and K^n, respectively. Then, the rational function $\Phi_n(K) = (\frac{X_1}{1-X_0}, \cdots, \frac{X_n}{1-X_0})$ (determined by the familiar stereographic projection) yields the bijection

$$\phi_n(K) : \mathbb{S}^n(K)\backslash V_K \longrightarrow K^n\backslash\mathbb{S}_i^{n-1}(K)$$

with the inverse

$$\psi_n(K) : K^n\backslash\mathbb{S}_i^{n-1}(K) \longrightarrow \mathbb{S}^n(K)\backslash V_K$$

determined by the inverse $\Psi_n(K) = \frac{1}{1+\sum_{k=1}^n X_k^2}(\sum_{k=1}^n X_k^2 - 1, 2X_1, \ldots, 2X_n)$ to the stereographic projection. Whence, $\mathbb{S}^n(K)$ and K^n are birationally equivalent for any $n \geq 0$ and we are in a position to state

Proposition 1.3. *Let K be an infinite field of characteristic different from two. Then $\overline{\mathbb{S}^n(K)} = \mathbb{S}^n(\bar{K})$ for $n \geq 0$, where $\bar{K}$ denotes the algebraic closure of K and $\overline{\mathbb{S}^n(K)}$ the Zariski closure of $\mathbb{S}^n(K)$ in $\bar{K}^{n+1}$.*

Proof. Obviously, $\overline{\mathbb{S}^0(K)} = \mathbb{S}^0(\bar{K})$. If now $n \geq 1$ and a polynomial $F \in \bar{K}[X_0, \ldots, X_n]$ vanishes on $\mathbb{S}^n(K)$ then $F\Psi_n(K)$ vanishes on $K^{n+1}\backslash\mathbb{S}_i^n(K)$. Because the field K is infinite, K^n is dense in $\bar{K}^n$ (with respect to the Zariski topology on $\bar{K}^n$) and consequently $K^n\backslash\mathbb{S}_i^{n-1}(K)$ is dense in $\bar{K}^n\backslash\mathbb{S}_i^{n-1}(\bar{K})$ as well. The regular function $F\Psi_n(K)$ vanishes on $K^n\backslash\mathbb{S}_i^{n-1}(K)$ whence $F\Psi_n(\bar{K})$ vanishes on $\bar{K}^n\backslash\mathbb{S}_i^{n-1}(\bar{K})$. Consequently, the polynomial F vanishes on the non-empty open subset $\mathbb{S}^n(\bar{K})\backslash V_{\bar{K}}$. Because the principal ideal $(\sum_{k=0}^n X_k^2 - 1)$ is prime so the sphere $\mathbb{S}^n(\bar{K})$ is irreducible and hence the open set $\mathbb{S}^n(\bar{K})\backslash V_{\bar{K}}$ is dense in $\mathbb{S}^n(\bar{K})$. Thus, F vanishes on the whole sphere $\mathbb{S}^n(\bar{K})$, and the proof follows. ∎

Now, we can state our first result.

Theorem 1.4. *Let K be a field and $f : \mathbb{S}^1(K) \to \mathbb{S}^1(K)$ a polynomial self-map. Then:*

(1) *if K is of characteristic two, $f = (f_0, 1 + f_0)$;*
(2) *if K is finite, any set self-map of $\mathbb{S}^1(K)$ is a polynomial one;*
(3) *if K is an infinite field of characteristic different from two, $f(z) = \alpha z^k$ for any $z \in \mathbb{S}^1(K)$ with some $\alpha \in \mathbb{S}^1(K)$ and an integer k.*

Proof. The cases (1) and (2) are straightforward.

(3) By Proposition 1.3, the map $f : \mathbb{S}^1(K) \to \mathbb{S}^1(K)$ can be extended to a polynomial map $f' : \mathbb{S}^1(\bar{K}) \to \mathbb{S}^1(\bar{K})$, where $\bar{K}$ is the algebraic closure of K. Corollary 1.2 says that there is $\alpha \in \mathbb{S}^1(\bar{K})$ and an integer k such that $f'(z) = \alpha z^k$ for all $z \in \mathbb{S}^1(\bar{K})$. Taking $z \in \mathbb{S}^1(K)$ we deduce that $\alpha \in \mathbb{S}^1(K)$ and the proof follows. ∎

Now, we move to the study of polynomial maps from n-spheres over any field with $n > 1$. But before we need

Lemma 1.5. *Let K be an algebraically closed field of characteristic different from two. Then any polynomial map $f : \mathbb{S}^2(K) \to \mathbb{S}^1(K)$ with a finite image is necessarily constant.*

Proof. Let $\alpha_1, \dots, \alpha_m$ be all the points in the image of $f : \mathbb{S}^2(K) \to \mathbb{S}^1(K)$ and consider the map $\rho \circ f : \mathbb{S}^2(K) \to U(K)$, where $\rho : \mathbb{S}^1(K) \to U(K)$ is the isomorphism described above.

The polynomial map $F = (\rho \circ f - \rho(\alpha_1)) \cdots (\rho \circ f - \rho(\alpha_m))$ obviously vanishes on $\mathbb{S}^2(K)$ and so F is a multiple of the polynomial $X_0^2 + X_1^2 + X_2^2 - 1$. Thus for some j, the polynomial $\rho \circ f - \rho(\alpha_j)$ is a multiple of $X_0^2 + X_1^2 + X_2^2 - 1$ because the principal ideal $(X_0^2 + X_1^2 + X_2^2 - 1)$ is prime. Therefore $\rho(f(x_0, x_1, x_2)) = \rho(\alpha_j))$ for all $(x_0, x_1, x_2) \in \mathbb{S}^2(K)$ and so $f(x_0, x_1, x_2) = \alpha_j$ for all $(x_0, x_1, x_2) \in \mathbb{S}^2(K)$. ∎

Next, we show

Lemma 1.6. *If K is an algebraically closed field of characteristic different from two then any polynomial map $f : \mathbb{S}^2(K) \to \mathbb{S}^1(K)$ is constant.*

Proof. First we show that any two orthogonal vectors $u, v \in \mathbb{S}^2(K)$ (for the form $x_0^2 + x_1^2 + x_2^2$) differ only by multiplication with a power of I.

In fact, consider the polynomial self-map g of $\mathbb{S}^1(K)$ given by $g(x, y) = f(xu+yv)$ for any $(x, y) \in \mathbb{S}^1$. By Theorem 1.4 there exist $\alpha \in U(K)$ and an integer k such that $g(x, y) = \alpha(x, y)^k$. Therefore, $f(u) = \alpha$, $f(v) = \alpha I^k$ and so $f(v) = f(u)I^k$.

Now, given $x \in \mathbb{S}^2$, by Witt Theorem there exists $y \in \mathbb{S}^2$ orthogonal to the vectors x and $e_0 = (1, 0, 0)$. By means of the above, there are integers k and l such that $f(y) = f(e_0)I^k$ and $f(y) = f(x)I^l$. Therefore, $f(x) = f(e_0)I^{k-l}$ and the image of f is finite. Thus, by Lemma 1.5 the result follows. ∎

Whence, we are in a position to show our next main result.

Theorem 1.7. *Let K be a field, $f : \mathbb{S}^n(K) \to \mathbb{S}^1(K)$ a polynomial map and $n \geq 2$. Then:*

(1) *if K is of characteristic two, $f = (f_0, 1 + f_0)$;*

(2) *if K is finite, any set map $\mathbb{S}^n(K) \to \mathbb{S}^1(K)$ is a polynomial one;*

(3) *if K is an infinite field of characteristic different from two, f is constant.*

Proof. The cases (1) and (2) are straightforward.

(3) In the light of Proposition 1.3 we can assume that K is algebraically closed. Let $(x_0, \dots, x_n) \in \mathbb{S}^n(K)$ with $x_i^2 + x_j^2 \neq 1$ for some couple of indexes i, j such that

$0 \leq i < j \leq n$. Choose one of the two square roots of $1 - x_i^2 - x_j^2$ and define $\iota : \mathbb{S}^2(K) \to \mathbb{S}^n(K)$ by

$$\iota(y_0, y_1, y_2) = y_0 e_i + y_1 e_j + \frac{y_2}{\sqrt{1 - x_i^2 - x_j^2}} \sum_{\substack{k=0 \\ k \neq i,j}}^{n} x_k e_k,$$

where $e_0, \dots, e_{n+1}$ denote vectors of the standard basis in K^{n+1}.

Using now Lemma 1.6, we know that $f\iota$ is constant. In particular

$$f(e_i) = f(\iota(1,0,0)) = f(\iota(0,1,0)) = f(e_j) =$$

$$f(\iota(x_i, x_j, \sqrt{1 - x_i^2 - x_j^2})) = f(x_0, \dots, x_n).$$

Therefore, by taking as $(x_0, \dots, x_n)$ the points $e_0, \dots, e_n$ respectively, we deduce that $f(e_0) = f(e_1) = \dots = f(e_n) = f(x_0, \dots, x_n)$ for any $(x_0, \dots, x_n) \in \mathbb{S}^n(K)$ with the possible exception of those $(x_0, \dots, x_n)$ such that $x_0^2 = \dots = x_n^2$ and $2x_0^2 = 1$. Therefore f is constant with a possible exception of a finite number of points in $\mathbb{S}^n(K)$. But since K is infinite, being an algebraically closed field, we derive that f is constant. ∎

Let now S_1, S_2, S_3 be any sets and $f : S_1 \times S_2 \to S_3$ a map. It is obvious that f is constant provided the maps $f(-, s_2) : S_1 \to S_3$ and $f(s_1, -) : S_2 \to S_3$ are constant for any $s_1 \in S_1$ and $s_2 \in S_2$. Write $\mathbb{T}^n(K)$ for the n-th cartesian power of $\mathbb{S}^1(K)$ for any $n \geq 1$ called the *n-torus* over K. Then, the results above lead to

Corollary 1.8. *Let K be an infinite field and $f = (f_1, \dots, f_n) : \mathbb{S}^{m_1}(K) \times \dots \times \mathbb{S}^{m_k}(K) \to \mathbb{T}^n(K)$ a polynomial map with $m_1, \dots, m_k \geq 1$. Then:*

(1) *$f_j = (f'_j, f'_j + 1)$, where f'_j is a polynomial function on the sphere $\mathbb{S}^{m_j}(K)$ for $j = 1, \dots, n$ provided K is of characteristic two;*

(2)

$$f_j(x_1, \dots, x_k) = \alpha_j x_1^{l_{1j}} \cdots x_k^{l_{kj}}$$

for any point $(x_1, \dots, x_k) \in \mathbb{S}^{m_1}(K) \times \dots \times \mathbb{S}^{m_k}(K)$ and $j = 1, \dots, n$ with some $\alpha_1, \dots, \alpha_n \in \mathbb{S}^1(K)$ and integers l_{uv} such that $l_{uv} = 0$ provided $m_u > 1$ for $u = 1, \dots, k$, $v = 1, \dots, n$ and K is of characteristic different from two. We put $x^0 = 1$ whenever $x \in \mathbb{S}^d(K)$ with $d \geq 1$ and identify points of $\mathbb{S}^1(K)$ with elements of the group $U(\widetilde{K})$.

For the field $\mathbb{R}$ of reals by [7] (see also [4, Chapter 13]) any polynomial map $\mathbb{S}^n(\mathbb{R}) \to \mathbb{S}^{n-1}(\mathbb{R})$ of spheres is constant provided n is a power of 2. It follows that all polynomial maps $\mathbb{S}^m(\mathbb{R}) \to \mathbb{S}^n(\mathbb{R})$ are constant if $m \geq 2n$. On the other hand, if K is a field with $i \in K$ then any polynomial $F \in K[X_0, \dots, X_m]$ determines the non-constant polynomial map $(f, if, 1, 0, \dots, 0) : \mathbb{S}^m(K) \to \mathbb{S}^n(K)$ provided $n \geq 2$, where $f(x) = F(x)$ for any $x \in \mathbb{S}^m(K)$. In particular, for the field of complex numbers $\mathbb{C}$ and any positive integers m, n with $n \geq 2$ there are non-constant polynomial maps $\mathbb{S}^m(\mathbb{C}) \to \mathbb{S}^n(\mathbb{C})$.

2 Maps of tori over reals

Throughout of this section the n-sphere $\mathbb{S}^n(\mathbb{R})$ (resp. the n-torus $\mathbb{T}^n(\mathbb{R})$) over the reals $\mathbb{R}$ will be denoted simply by $\mathbb{S}^n$ (resp. $\mathbb{T}^n$) for $n \geq 0$. Observe that any polynomial $F(X) \in \mathbb{C}[X]$ yields two polynomials $F_0(X_0, X_1), F_1(X_0, X_1) \in \mathbb{R}[X_0, X_1]$, where $F(X_0 + iX_1) = F_0(X_0, X_1) + iF_1(X_0, X_1)$. If the associated polynomial self-map $\mathbb{R}^2 \to \mathbb{R}^2$ restricts to such a map $f : \mathbb{S}^1 \to \mathbb{S}^1$ then by Theorem 1.4 there are $\alpha \in \mathbb{S}^1$ and a non-negative integer k with $f(z) = \alpha z^k$ for any $z \in \mathbb{S}^1$ and consequently $F(X) = \alpha X^k$. Whence, given a polynomial $F(X_1, \dots, X_n) \in \mathbb{C}[X_1, \dots, X_n]$ restricting to a map $\mathbb{T}^n \to \mathbb{S}^1$ we get $F(X_1, \dots, X_n) = \alpha X_1^{k_1} \cdots X_n^{k_n}$ for some $\alpha \in \mathbb{S}^1$ and non-negative integers $k_1, \dots, k_n$.

We aim to state a more general result. Then, write $D = \{z \in \mathbb{C}; |z| \leq 1\}$ for the closed disc in the complex plane $\mathbb{C}$.

Proposition 2.1. *The following conditions are equivalent for a continuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$:*

(1) *f is the restriction of a meromorphic map $\tilde{f} : U \to \mathbb{C}$ on some neighbourhood U of $\mathbb{S}^1$;*

(2) *there exist $\alpha, z_1, \dots, z_n \in \mathbb{C}$ such that $|\alpha| = 1$, $|z_k| < 1$ for $k = 1, \dots, n$ and f is the restriction of the rational map $\tilde{f}$ given by*

$$\tilde{f}(z) = \alpha \prod_{k=1}^{n} \left(\frac{z - z_k}{\bar{z}_k z - 1} \right)^{\varepsilon_k}$$

with $\varepsilon_k = \pm 1$ and $k = 1, \dots, n$;

(3) *f is the restriction of a rational map $\tilde{f}$;*

(4) *there exists an integer n_0 such that either $a_n = 0$ for all $n > n_0$ or $a_n = 0$ for all $n < n_0$, the radius of convergence of $\sum_{n<0} a_n z^{-n}$ in the first case and the radius of convergence of $\sum_{n>0} a_n z^n$ in the second case is greater than one, where*

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} dt$$

are the Fourier coefficient of f for all integers n.

Proof. The implications (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1) are clear and so why we only check (1) $\Longrightarrow$ (2).

Let $z_1, \dots, z_n$ be the zeroes or poles of $\tilde{f}$ on D, where we count each zero or pole as many times as indicated by its order. We consider then the rational map

$$g(z) = \prod_{k=1}^{n} \left(\frac{z - z_k}{\bar{z}_k z - 1} \right)^{\varepsilon_k},$$

where $\varepsilon_k = 1$ if z_k is a zero and $\varepsilon_k = -1$ if z_k is a pole.

It is clear that g restricts to a map $\mathbb{S}^1 \to \mathbb{S}^1$ and it has exactly the same zeros and poles on D as the meromorphic map $\tilde{f}$. By the lifting homotopy property there exists a holomorphic map θ defined on a neighbourhood of D such that $\frac{\tilde{f}}{g} = e^{\theta}$.

But $|z| = 1$ implies $|\frac{\tilde{f}(z)}{g(z)}| = 1$. Whence, $|e^{(\theta(z))}| = 1$ provided $|z| = 1$. If e^{θ} would be non-constant then by the maximum-modulus principle:

(a) $|z| < 1$ implies $|e^{\theta(z)}| < 1$;

(b) because $e^{\theta(z)} \neq 0$ for any $z \in D$ so there is $z_0 \in \mathbb{S}^1$ with $|e^{\theta(z_0)}| = 1$ as the minimum of the map $|e^\theta|$.

This contradiction shows that e^θ must be constant and of course $|e^\theta| = 1$. Hence

$$\tilde{f}(z) = \alpha \prod_{k=1}^{n} \Big(\frac{z - z_k}{\bar{z}_k z - 1}\Big)^{\varepsilon_k},$$

for any $z \in \mathbb{C}$ with $\alpha \in \mathbb{S}^1$. ∎

In particular, if a holomorphic map $\tilde{f} : \mathbb{C} \to \mathbb{C}$ restricts to a map $f : \mathbb{S}^1 \to \mathbb{S}^1$ then all its possible zeros on the disc D coincide with 0 and so by (2) in Proposition 2.1 we get $\alpha \in \mathbb{S}^1$ and a non-negative integer k with $\tilde{f}(z) = \alpha z^k$ for any $z \in \mathbb{S}^1$ and consequently $f(z) = \alpha z^k$ for any $z \in \mathbb{C}$.

More generally, observe first that

$$\mathbb{S}^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1};\ \sum_{k=0}^{n} |z_k|^2 = 1\}$$

for any $n \geq 1$ and

$$\mathbb{S}^n = \{(\lambda, z_1, \dots, z_{\frac{n}{2}}) \in \mathbb{R} \times\ \mathbb{C}^{\frac{n}{2}};\ \lambda^2 + \sum_{k=1}^{\frac{n}{2}} |z_k|^2 = 1\}$$

for n even. If now a holomorphic map $\tilde{f} : (\tilde{f}_1, \dots, \tilde{f}_n) : \mathbb{C} \to \mathbb{C}^n$ restricts to a map $f : \mathbb{S}^1 \to \mathbb{S}^{2n-1}$ then by the maximum-modulus principle we get that $|\tilde{f}_j(z)| \leq 1$ for any $z \in D$ and $j = 1, \dots, n$. Then as above we derive that there are $\alpha_j \in \mathbb{C}$ and non-negative integers k_j such that $f_j(z) = \alpha_j z^{k_j}$ for any $z \in \mathbb{C}$ and $j = 1, \dots, n$.

Corollary 2.2. *If a map $f : \mathbb{T}^m \to \mathbb{S}^{2n-1}$ with $m, n \geq 1$ is the restriction of a holomorphic map $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_n) : \mathbb{C}^m \to \mathbb{C}^n$ then*

$$\tilde{f}(z_1, \dots, z_m) = (\alpha_1 z_1^{k_{11}} \cdots z_m^{k_{m1}}, \dots, \alpha_n z_1^{k_{1n}} \cdots z_m^{k_{mn}})$$

for any $(z_1, \dots, z_m) \in \mathbb{C}^m$ with some $(\alpha_1, \dots, \alpha_n) \in \mathbb{S}^{2n-1}$ and non-negative integers k_{st} for $s = 1, \dots, m$ and $t = 1, \dots, n$.

Now, let $f : \mathbb{T}^m \to \mathbb{S}^{2n_1-1} \times \cdots \times \mathbb{S}^{2n_k-1}$ with $m, n_1, \dots, n_k \geq 1$ be the restriction of a holomorphic map $\tilde{f} : \mathbb{C}^m \to \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k}$. Then it might be easily derived from the result above. In particular, any complex polynomial $\mathbb{T}^m \to \mathbb{S}^{2n_1-1} \times \cdots \times \mathbb{S}^{2n_k-1}$ is fully described. Obviously it follows that any such a map is null-homotopic through a complex polynomial homotopy provided $n_1, \dots, n_k > 1$ (see [1, 3, 6] for the real case).

Furthermore, the following result holds.

Corollary 2.3. (1) *Let $f' : \mathbb{C}^m \to \mathbb{C}$ be a holomorphic map with $m > 1$. If f' restricts to a map $\mathbb{S}^{2m-1} \to \mathbb{S}^1$ then f' is constant.*

(2) *Let $f'' : \mathbb{R} \times \mathbb{C}^m \to \mathbb{C}$ be a map such that $f''(\lambda, -) : \mathbb{C}^m \to \mathbb{C}$ is holomorphic for any $\lambda \in \mathbb{R}$ and $f''(-, z_1, \dots, z_m) : \mathbb{R} \to \mathbb{C}$ is a polynomial map for any $(z_1, \dots, z_m) \in \mathbb{C}^m$ with $m \geq 1$. If f'' restricts to a map $\mathbb{S}^{2m} \to \mathbb{S}^1$ then f'' is constant.*

Proof. (1) For a fixed point $x \in \mathbb{S}^{2m-1}$ consider the map

$$f'_x : \mathbb{C} \longrightarrow \mathbb{C}$$

given by $f'_x(z) = f'(zx)$ for any $z \in \mathbb{C}$. Then f'_x restricts to a map $\tilde{f}'_x : \mathbb{S}^1 \to \mathbb{S}^1$ with Brouwer degree zero. By Corollary 2.2, the map $\tilde{f}'_x$ is constant on the whole $\mathbb{C}$ and so $f'(zx) = f'(x) = f'(0)$ for any $z \in \mathbb{C}$. Now, for $y \in \mathbb{C}^m$ with $y \neq 0$ we get $f'(y) = f'(||y|| \frac{y}{||y||}) = f'(\frac{y}{||y||}) = f'(0)$ and consequently f' is constant.

(2) For a fixed point $(\lambda, x) \in \mathbb{S}^{2m}$ consider the map

$$f''_{(\lambda,x)} : \mathbb{C} \longrightarrow \mathbb{C}$$

given by $f''_{(\lambda,x)}(z) = f''(\lambda, zx)$ for any $z \in \mathbb{C}$. Then $f''_{(\lambda,x)}$ restricts to a map $\mathbb{S}^1 \to \mathbb{S}^1$ with Brouwer degree zero. By means of Corollary 2.2, we conclude as above that $f''(\lambda, zx) = f''(\lambda, x) = f''(\lambda, 0)$ for any $z \in \mathbb{C}$. Now, for any $y \in \mathbb{C}^m$ with $y \neq 0$ and $|\lambda| < 1$ we get $f''(\lambda, y) = f''(\lambda, (\frac{||y||}{\sqrt{1-\lambda^2}})(\frac{\sqrt{1-\lambda^2}}{||y||} y)) = f''(\lambda, 0)$ because $(\lambda, \frac{\sqrt{1-\lambda^2}}{||y||} y) \in \mathbb{S}^{2m}$. Therefore there are $a_0, \ldots, a_k \in \mathbb{C}$ with $a_k \neq 0$ and such that

$$f''(\lambda, y) = f''(\lambda, 0) = a_k \lambda^k + \cdots + a_1 \lambda + a_0$$

for any $y \in \mathbb{C}^m$ provided $|\lambda| < 1$. Because $f''(\lambda, y) \in \mathbb{S}^{2m}$ and $f''(\mathbb{S}^{2m}) \subseteq \mathbb{S}^1$ so $(a_k \lambda^k + \cdots + a_1 \lambda + a_0)(\bar{a_k} \lambda^k + \cdots + \bar{a_1} \lambda + \bar{a_0}) = 1$ for any $\lambda < 1$, where $-$ denotes the conjugation in $\mathbb{C}$. Whence $k = 0$ and the polynomial map $f''(-, y)$ is constant for any $y \in \mathbb{C}^m$. Finally we derive that f'' is a constant map. ■

Because any complex polynomial map $\mathbb{R}^l \to \mathbb{S}^1$ is constant for $l \geq 0$ so the following complex version of Corollary 1.8 arises.

Proposition 2.4. *Let $\tilde{f} : \mathbb{R}^l \times \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_k} \to \mathbb{C}^n$ be a map with $m_1, \ldots, m_k \geq 1$, $l \geq 0$ and such that:*

(1) *the map $\tilde{f}(\lambda, -, \cdots, -) : \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_k} \to \mathbb{C}^n$ is holomorphic for any $\lambda \in \mathbb{R}^l$;*

(2) *the map $\tilde{f}(-, z_1, \ldots, z_m) : \mathbb{R}^l \to \mathbb{C}^n$ is polynomial for any $(z_1, \ldots, z_m) \in \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_k}$.*

If $\tilde{f}$ restricts to a map $f : \mathbb{S}^{2m_1} \times \cdots \times \mathbb{S}^{2m_l} \times \mathbb{S}^{2m_{l+1}-1} \times \cdots \times \mathbb{S}^{2m_k-1} \to \mathbb{T}^n$ for $l < k$ and to a map $f : \mathbb{S}^{2m_1} \times \cdots \times \mathbb{S}^{2m_l} \times \mathbb{R}^{l-k} \to \mathbb{T}^n$ for $l \geq k$ then

$$f(\lambda, z_1, \ldots, z_k) = (\alpha_1 z_1^{j_{11}} \cdots z_k^{j_{k1}}, \ldots, \alpha_n z_1^{j_{1n}} \cdots z_k^{j_{kn}})$$

for any $(z_1, \ldots, z_k) \in \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_k}$ with some $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ and non-negative integers j_{st} such that $j_{st} = 0$ provided $m_s > 1$ for $s = 1, \ldots, m$ and $t = 1, \ldots, n$. We put $z^0 = 1$ whenever $z \in \mathbb{C}^d$ with $d \geq 1$.

Of course, in virtue of the same methods, similar results could be shown replacing holomorphic maps by antiholomorphic ones.

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