Nonexistence of certain Fano subplanes of Figueroa planes

Julia M. Nowlin Brown

Abstract

Using Grundhöfer's construction of the Figueroa planes from Pappian planes which have an order three planar collineation $\widehat{\alpha}$, we show that any Figueroa plane (finite or infinite) of characteristic not two cannot have a Fano subplane which includes an $\widehat{\alpha}$ invariant triangle.

A class of non-Desarguesian, proper, finite projective planes of orders q^3 for prime powers $q \not\equiv 1 \pmod 3$ and $q > 2$ were defined by Figueroa [4] in 1982. This construction was generalized to all prime powers $q > 2$ by Hering and Schaeffer [6] later in the same year. We [1] gave a group-coset description of these finite Figueroa planes in 1983. The construction was extended to include infinite planes in 1984 by Dempwolff [3]. These constructions were all algebraic in the sense that they made essential use of collineation groups and coordinates. In 1986 Grundhöfer [5] gave a beautiful synthetic construction which included all these Figueroa planes.

The question of what projective subplanes a given projective plane possesses is always of interest and usually is not trivial for non-Desarguesian planes. The Fano plane is the smallest projective plane (having order two and exactly seven points and seven lines). Hanna Neumann has conjectured that all finite non-Desarguesian planes have a Fano subplane (see [7] for some of her early work on the subject). We show that many Figueroa planes do not possess a Fano subplane of a certain particular nice type.

We remind readers of Theo Grundhöfer's elegant synthetic definition of the Figueroa planes. Consider a Pappian plane which has an order 3 planar collinear $\widehat{\alpha}$. The point set (line set) of the Figueroa plane is the same as the point set (line set)

Bull. Belg. Math. Soc. 12 (2005), 675–684

of the Pappian plane, but incidence is changed. Letting $\mathrm{I}^{\mathcal{P}}$ and $\mathrm{I}^{\mathcal{F}}$ denote Pappian and Figueroa incidence, respectively, $\mathrm{I}^{\mathcal{F}}$ is defined in terms of $\mathrm{I}^{\mathcal{P}}$ as follows: if either $P^{<\widehat{\alpha}>}$ or $\ell^{<\widehat{\alpha}>}$ is not a proper triangle, then $P \ \mathrm{I}^{\mathcal{F}} \ \ell \Leftrightarrow P \ \mathrm{I}^{\mathcal{P}} \ \ell$; if both $P^{<\widehat{\alpha}>}$ and $\ell^{<\widehat{\alpha}>}$ are proper triangles, then $P \ \mathrm{I}^{\mathcal{F}} \ \ell \Leftrightarrow$ the "vertex" opposite ℓ in $\ell^{<\widehat{\alpha}>}$ $\mathrm{I}^{\mathcal{P}}$ the "side" opposite P in $P^{<\widehat{\alpha}>}$.

The map $\widehat{\alpha}$ which is a planar collineation of the Pappian plane remains a planar collineation of the Figueroa plane. Any collineation or polarity of the Pappian plane which commutes with $\widehat{\alpha}$ remains a collineation or polarity, respectively, of the Figueroa plane. Letting α denote the field automorphism associated with the planar automorphism $\widehat{\alpha}$, the Figueroa plane inherits a collineation group isomorphic to $\mathrm{PGL}(3, \mathrm{Fix}(\alpha))$. We describe the orbits of this group.

The set of points P for which $P^{\langle\widehat{\alpha}\rangle}$ is a single point is an orbit. We call points in this orbit *type* I. The set of points P for which $P^{\langle\widehat{\alpha}\rangle}$ is three collinear points is an orbit. We call points in this orbit *type* II. The set of points P for which $P^{\langle\widehat{\alpha}\rangle}$ is a proper triangle is an orbit. We call points in this orbit *type* III. Line orbits have dual descriptions and types of lines have dual definitions. Figueroa incidence induces on the orbits of type I points and lines the structure of a subplane isomorphic to $\mathrm{PG}(2, \mathrm{Fix}(\alpha))$.

In general, collinearity of three points (concurrency of three lines) may be different with respect to $\mathrm{I}^{\mathcal{F}}$ and to $\mathrm{I}^{\mathcal{P}}$. But for $\{P, P^{\widehat{\alpha}}, P^{\widehat{\alpha}^2}\}$ (for $\{\ell, \ell^{\widehat{\alpha}}, \ell^{\widehat{\alpha}^2}\}$) it is the same. So the above definition of types is unambiguous. In what follows Pappian collinearity, Pappian concurrency, Pappian incidence, etc. will be with respect to $\mathrm{I}^{\mathcal{P}}$. "Unmodified" collinearity, concurrency, incidence, etc. will be with respect to $\mathrm{I}^{\mathcal{F}}$. Note that a point (line) of type III can never be incident with a line (point) of type I.

If the Pappian plane has an $\widehat{\alpha}$ invariant subplane on which the restriction of $\widehat{\alpha}$ is an order three planar collineation, the Figueroa plane constructed using the subplane and the restriction of $\widehat{\alpha}$ to the subplane is a Figueroa subplane of the (first) big Figueroa plane. From this it follows that the Figueroa plane of order q^3 is a subplane of the Figueroa plane of order q^{3r} for any prime power q and any $r \equiv 1$ or $2 \pmod 3$.

We define the *characteristic* of a Figueroa plane to be the the *characteristic* of the field which coordinatises the Pappian plane from which the Figueroa plane is constructed.

We use Grundhöfer's construction together with homogeneous coordinates to show the following

THEOREM: Every (finite or infinite) Figueroa plane, of characteristic not two, cannot have a Fano subplane which includes an $\hat{\alpha}$-invariant proper triangle.

Remark: The only other existence or nonexistence result, known to the author, about Fano subplanes of a Figueroa plane of characteristic not two derive from the Fano subplane of the Figueroa plane of order 27 which was discovered by Cherowitzo

[2]. (This construction was not self dual so Cherowitzo discovered two embedded Fano subplanes up to equivalence under collineations.) Thus there exist Fano subplanes in any "super" plane of the Figueroa plane of order 27. This includes Figueroa planes of order 3^{3r} for any $r \equiv 1$ or 2 (mod 3).

Proof:

A Fano plane is the incidence geometry $PG(2,2)$. It has exactly seven points and seven lines. We denote the point set by $\{V_1, V_2, V_3, M_1, M_2, M_3, C\}$ and the line set by $\{s_1, s_2, s_3, m_1, m_2, m_3, c\}$ where s_2 I V_1, V_3, M_2; s_1 I V_3, V_2, M_1; m_2 I V_2, M_2, C; c I M_2, M_1, M_3; m_1 I M_1, C, V_1; m_3 I C, M_3, V_3; s_3 I M_3, V_1, V_2.

We will call the points V_1, V_2, V_3 *vertices* and suggest that the reader imagine a picture of the Fano plane with these three points as the vertices of the "outer triangle". The lines s_1, s_2, s_3 then become the sides of the "outer triangle" and we call these lines *sides*. The nonvertex points M_1, M_2, M_3 on the sides s_1, s_2, s_3, respectively, will be called *median points* . The nonside lines m_1, m_2, m_3 through the vertices V_1, V_2, V_3, respectively, will be called *median lines* . Median point–median line flags M_i-m_i will be called *median flags*. The "interior" point C will be called the *centre point*. The "circular" line c will be called the *circle line*.

Let α be an order three automorphism of the field coordinatising the Pappian plane from which our Figueroa plane is constructed. We choose as our planar automorphism the map $\widehat{\alpha}$ which acts on points as $\langle(x, y, z)\rangle \mapsto \langle(z^\alpha, x^\alpha, y^\alpha)\rangle$ and on lines as $\left\langle\begin{pmatrix} d \\ e \\ f \end{pmatrix}\right\rangle \mapsto \left\langle\begin{pmatrix} f^\alpha \\ d^\alpha \\ e^\alpha \end{pmatrix}\right\rangle$. A point $P = \langle(x, y, z)\rangle$ is of type III $\Leftrightarrow$ $\det\begin{bmatrix} x & y & z \\ z^\alpha & x^\alpha & y^\alpha \\ y^{\alpha^2} & z^{\alpha^2} & x^{\alpha^2} \end{bmatrix} \neq 0$. A dual (and transpose) statement holds for lines. The point action of the PGL group acting on this representation of the Figueroa plane becomes the group of all invertible matrices $\begin{bmatrix} a & b & c \\ c^\alpha & a^\alpha & b^\alpha \\ b^{\alpha^2} & c^{\alpha^2} & a^{\alpha^2} \end{bmatrix}$ (acting by right matrix multiplication on the point coordinates) modulo the invertible $\mathrm{Fix}(\langle\alpha\rangle)$-multiples of the identity matrix. The group induced by all matrices of this form with $b = c = 0$ will be used very often and will be denoted G. The map $\pi : \langle(x, y, z)\rangle \leftrightarrow \left\langle\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\rangle$ is a polarity of this Figueroa plane. Duality under π will be called π-*duality*.

Assume there is a Fano subplane of the type described.

Because points of type III form an orbit and $\langle(1, 0, 0)\rangle$ is of type III, we may assume that the vertices of the $\widehat{\alpha}$-invariant triangle in the Fano subplane have coordinates $\langle(1, 0, 0)\rangle, \langle(0, 1, 0)\rangle, \langle(0, 0, 1)\rangle$. Because of well known symmetries of the Fano plane, we may assume that the vertices V_1, V_2, V_3 have these coordinates, respectively. Note that the triangle with vertices V_1, V_2, V_3 and sides $s_1 = V_2V_3, s_2 = V_3V_1, s_3 = V_1V_2$ is invariant under $\langle\pi\rangle$ and that $\pi : V_i \leftrightarrow s_i$.

Non-vertex points on side $s_3 = V_1V_2 = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ are $\langle (a, 1, 0) \rangle$ with $N(a) = aa^{\alpha}a^{\alpha^2} = -1$ (each of these points is of type II) and $\langle (x^{\alpha^2+\alpha}, 1, -x^{\alpha}) \rangle$ with $N(x) \neq 0, -1$ (each of these points is of type III and has "opposite side" $\left\langle \begin{pmatrix} x \\ 1 \\ 0 \end{pmatrix} \right\rangle$). (The function N is the usual (multiplicative) relative norm function.)

Non-side lines through point $V_3 = s_1s_2$ are $\left\langle \begin{pmatrix} d \\ 1 \\ 0 \end{pmatrix} \right\rangle$ with $N(d) = -1$ (each of these lines is of type II) and $\left\langle \begin{pmatrix} y^{\alpha^2+\alpha} \\ 1 \\ -y^{\alpha} \end{pmatrix} \right\rangle$ with $N(y) \neq 0, -1$ (each of these lines is of type III and has "opposite vertex" $< (y, 1, 0) >$).

Using Grundhöfer's definition of incidence to determine all possible M_3-m_3 median flags, we learn the following. There are no M_3-m_3 median flags of type II-II (because $ad = -1$ is impossible for $N(a) = N(d) = -1$). There are possible M_3-m_3 median flags of type II-III: $\langle (a, 1, 0) \rangle$ - $\left\langle \begin{pmatrix} a^{\alpha^2+\alpha} \\ 1 \\ a^{\alpha} \end{pmatrix} \right\rangle$ where $N(a) = -1$. There are possible M_3-m_3 median flags of type III-II: $\langle (d^{\alpha^2+\alpha}, 1, d^{\alpha}) \rangle$ - $\left\langle \begin{pmatrix} d \\ 1 \\ 0 \end{pmatrix} \right\rangle$ where $N(d) = -1$. There are possible M_3-m_3 median flags of type III-III: $\langle (x^{\alpha+1}, 1, -x) \rangle$ - $\left\langle \begin{pmatrix} 1 \\ x^{\alpha+1} \\ x^{\alpha} \end{pmatrix} \right\rangle$ where $N(x) \neq 0 \pm 1$.

So the types of M_3-m_3 median flags are II-III, III-II, and III-III.

By the action of $\langle \widehat{\alpha} \rangle$, the same statement holds for M_1-m_1 and M_2-m_2 median flags. The form of these M_1-m_1 and M_2-m_2 median flags for various types may be found by applying $\langle \widehat{\alpha} \rangle$.

We shall determine all Fano subplanes for each possible case of median flag types.

LEMMA 1: A Fano subplane of the type described in the theorem cannot have two or more median flags of type II-III. A Fano subplane of the type described in the theorem cannot have two or more two median flags of type III-II.

Proof of Lemma: By π-duality it is sufficient to prove the first claim. Suppose the first claim is false. By the action of $\langle \widehat{\alpha} \rangle$ we may assume that median flags M_1-m_1 and M_3-m_3 are of type II-III. Also, using the action of the group G, we may assume that the M_3-m_3 median flag is $\langle(-1,1,0)\rangle$ - $\left\langle\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\rangle$ and that the M_1-m_1 median flag is $\langle(0,b,1)\rangle$ - $\left\langle\begin{pmatrix}b^{\alpha+1}\\-1\\b\end{pmatrix}\right\rangle$ with $N(b)=-1$. These flags are both of type II-III.

By Grundhöfer's construction (because the points M_1 and M_3 are of type II), the line M_1M_3 is $\left\langle\begin{pmatrix}1\\1\\-b\end{pmatrix}\right\rangle$ and the point $\langle(b,0,1)\rangle$ is of type II and is on M_1M_3 and on s_2. Therefore the M_2-m_2 flag must be $\langle(b,0,1)\rangle$ - $\left\langle\begin{pmatrix}-1\\b^{\alpha^2+1}\\b\end{pmatrix}\right\rangle$ which is of type II-III.

We now introduce the notations $T(b)=b+b^{\alpha}+b^{\alpha^2}$ and $B(b)=b^{\alpha+1}+b^{\alpha^2+\alpha}+b^{1+\alpha^2}$. (The function T is the usual (additive) relative trace function. The function B has been called *bitrace* for degree three extensions by Sherk [8].)

The median lines m_1,m_2,m_3, all of which are of type III, cannot be concurrent in a point of type III because the opposite vertex points are, respectively, $\langle(0,b,-1)\rangle,\langle(1,0,b^{\alpha^2+\alpha})\rangle,\langle(1,1,0)\rangle$ and these points cannot be Pappian collinear because $2\neq 0$. If the median lines are concurrent in a point of type II then $m_1m_3=\langle(1-b,b(1+b^{\alpha}),b^{\alpha+1}+1)\rangle$ must be a point of type II. This is equivalent to $T(b)=1$. We must also have $m_1m_3I^{\mathcal{F}}m_2 \Leftrightarrow m_1m_3I^{\mathcal{P}}m_2$ which is equivalent to $B(b)=-1$. Thus b is a root of the polynomial $X^3-T(b)X^2+B(b)X-N(b)=X^3-X^2-X+1=(X-1)^2(X+1)$. So $b=\pm1$. Both of these values contradict $B(b)=-1$ for $2\neq 0$.

This proves Lemma 1.

LEMMA 2 : A Fano plane of the type described in the Theorem cannot have exactly one median flag of type II-III and exactly one median flag of type III-II.

Proof of Lemma 2: Suppose this Lemma is false.

By π-duality and the actions of the groups $\langle\hat{\alpha}\rangle$ and G we may assume that the median flags are M_3-$m_3=\langle(-1,1,0\rangle$ -$\left\langle\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\rangle$ of type II-III, M_1-$m_1=\langle(b^{\alpha+1},-1,b)\rangle$-$\left\langle\begin{pmatrix}0\\b\\1\end{pmatrix}\right\rangle$ with $N(b)=-1$ of type III-II and M_2-$m_2=\langle(1,-z,z^{\alpha+1})\rangle$ - $\left\langle\begin{pmatrix}z^{\alpha+1}\\z^{\alpha}\\1\end{pmatrix}\right\rangle$ with $N(z)\neq 0,\pm1$ of type III-III.

Note that M_2 has opposite side $\left\langle\begin{pmatrix}1\\0\\z^{\alpha^2}\end{pmatrix}\right\rangle$, M_1 has opposite side $\left\langle\begin{pmatrix}0\\b\\-1\end{pmatrix}\right\rangle$ and m_2 has opposite vertex $\langle(z^{\alpha^2},0,-1)\rangle$.

If $B(b)+T(b)\neq -1$ then the line M_1M_3 is $\left\langle\begin{pmatrix}b^{\alpha^2+1}+b+1\\b^{\alpha^2+1}+b+1\\-(b^{1+\alpha}+b^{\alpha+\alpha^2}+b^{\alpha}+b^{\alpha^2})\end{pmatrix}\right\rangle$ which is of type III and has opposite vertex $\langle(b^{1+\alpha}+b^{\alpha}+1,b^{\alpha+\alpha^2}+b^{\alpha^2}+1,b^{\alpha^2+1}+b-1)\rangle$. Because M_2 has opposite side $\left\langle\begin{pmatrix}1\\0\\z^{\alpha^2}\end{pmatrix}\right\rangle$, for M_2 to be on this potential circle line, it is necessary that $z^{\alpha^2}=-\frac{b^{\alpha+1}+b^{\alpha}+1}{b(b^{\alpha+\alpha^2}+b^{\alpha^2}+1)}$. Thus $1\neq N(z)=N(-1)N(b^{-1})=1$. Contradiction.

If $B(b)+T(b)\neq 1$ then the point m_1m_3 is $\langle(b^{\alpha+\alpha^2}+b^{\alpha^2+1}+b+b^{\alpha^2},b^{\alpha+\alpha^2}-b^{\alpha^2}-1,b^{\alpha^2+1}+b+1)\rangle$ which is of type III and has opposite side $\left\langle\begin{pmatrix}b^{\alpha^2+1}-b-1\\-b^{\alpha^2+1}+b+1\\b^{\alpha^2+1}+b-1\end{pmatrix}\right\rangle$. For m_2 to be through this potential centre point, it is necesary that $z^{\alpha^2}=\frac{b^{\alpha^2+1}+b-1}{b(b^{\alpha+\alpha^2}+b^{\alpha^2}-1)}$. Thus $-1\neq N(z)=N(b^{-1})=-1$. Contradiction.

This proves Lemma 2.

LEMMA 3: There cannot be a Fano subplane of the type described in the Theorem with exactly two median flags of type III-III.

Proof of the Lemma 3: Suppose this Lemma is false.

By π-duality and the actions of the groups $\langle\widehat{\alpha}\rangle$ and G, we may assume that the median flags are

M_1-$m_1=\langle(-y,y^{\alpha+1},1)\rangle-\left\langle\begin{pmatrix}y^{\alpha}\\1\\y^{\alpha+1}\end{pmatrix}\right\rangle$ of type III-III, M_2-$m_2=\langle(1,0,-1)\rangle-\left\langle\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\rangle$ of type II-III and M_3-$m_3=\langle(x^{\alpha+1},1,-x)\rangle-\left\langle\begin{pmatrix}1\\x^{\alpha+1}\\x^{\alpha}\end{pmatrix}\right\rangle$ of type III-III with $N(y),N(x)\neq 0,\pm1$.

First suppose the centre point is of type III, i.e. the median lines m_1,m_2,m_3 are concurrent in a point of III . This means that the vertices opposite these median lines, respectively, $\langle(0,-1,y^{\alpha^2})\rangle,\langle(1,0,1)\rangle,\langle(1,-x^{\alpha^2},0)\rangle$ are Pappian collinear in a type III line. This is equivalent to $xy=-1$ and $B(x)\neq -1$. If the circle line is of type II, then Pappian collinearity of M_1, M_2, M_3 implies (using $y = -x^{-1}$) $0 = x^\alpha + x$ which is impossible for $x \neq 0$ and $2 \neq 0$. If the circle line is of type III then the point of Pappian intersection of the sides opposite M_1 and M_3 is $\langle(-1, x^{\alpha^2}, 1,)\rangle$. So the line M_1M_3 is $\left\langle\begin{pmatrix} 1-x \\ 1+x^{\alpha+1} \\ 1+x^\alpha \end{pmatrix}\right\rangle$. But then $M_2 I^{\mathcal{F}} M_1M_3 \Leftrightarrow M_2 I^{\mathcal{P}} M_1M_3 \Leftrightarrow 0 = x^\alpha + x$ which again is a contradiction.

Now suppose the centre point is of type II, i.e. the median lines are concurrent in a point of type II. Then m_3m_2 is not of type III which implies that the line which is Pappian incident with the points opposite m_3 and m_2 is not of type III which is equivalent to $\left\langle\begin{pmatrix} x^{\alpha^2} \\ 1 \\ -x^{\alpha^2} \end{pmatrix}\right\rangle$ is not of type III which is equivalent to $B(x) = -1$. Similarly m_1m_2 is not of type III implies $T(y) = -N(y)$. Also concurrency of m_1, m_2, m_3 implies Pappian concurrency of m_1, m_2, m_3 implies $x^{\alpha+1}y^{\alpha+1} - x^\alpha y^\alpha + y^{\alpha+1} - x^{\alpha+1}y^\alpha - x^\alpha + 1 = 0$ (we will call this the "centre II" equation).

If the circle line is of type III (as well as the centre point is of type II) then the point opposite M_1M_3 is $\langle(1, -x^{\alpha^2}, (xy)^{\alpha^2})\rangle$ and the line M_1M_3 is $\left\langle\begin{pmatrix} 1+x^{\alpha+1}y^\alpha \\ x^{\alpha+1} - xy \\ x^{\alpha+1}y^{\alpha+1} + x^\alpha \end{pmatrix}\right\rangle$. The other median point M_2 is on this line $\Leftrightarrow x^{\alpha+1}y^{\alpha+1} - x^{\alpha+1}y^\alpha + x^\alpha - 1 = 0$.

Adding this equation to the centre II equation and solving for y yields $y = \frac{x^\alpha(2x+1)}{2x^{\alpha+1}+1}$. (Note that $B(x) = -1$ implies $2x^{1+\alpha} + 1 \neq 0$.) Substituting this in the last equation gives $\frac{(x^\alpha-1)}{(2x^{\alpha+1}+1)^{\alpha+1}}\left(6N(x)x^\alpha + N(x) + 2x^{\alpha^2+\alpha} + 2x^{1+\alpha} + 1\right) = 0$. Thus $0 = (6N(x)x + N(x) + 2x^{\alpha+1} + 2x^{\alpha^2+1} + 1) - (6N(x)x + N(x) + 2x^{\alpha+1} + 2x^{\alpha^2+1} + 1)^\alpha = 2(x - x^\alpha)(3N(x) + x^{\alpha^2})$. From this it follows that $x^\alpha = x$ and thus $y^\alpha = y$.

Now $-1 = B(x) = 3x^2$ (so $3 \neq 0$) and $3y = T(y) = -N(y) = -y^3$ so $y^2 = -3$ and $x^2 = -1/3$. Thus $y = \frac{2x^2+x}{2x^2+1} = \frac{-2/3+x}{-2/3+1} = 3x-2$. Thus $0 = -3-y^2 = 12(x-1/3)$. Thus $x = 1/3$. This contradicts $x^2 = -1/3$ for $2 \neq 0$.

If the circle line is of type II (as well as the centre point is of type II) then the line M_2M_3 is $\left\langle\begin{pmatrix} 1 \\ x - x^{\alpha+1} \\ 1 \end{pmatrix}\right\rangle$ and this line is of type II $\Leftrightarrow$ (using $B(x) = -1$) $0 = (N(x)+1)(Nx) + T(x) - 1)$ so $T(x) = 1 - N(x)$. Analogous considerations of the line M_1M_2 (using $T(y) = -N(y)$) shows $B(y) = N(y) - 1$. Finally the fact that M_1 is on M_2M_3 yields $x^{\alpha+1}y^{\alpha+1} - xy^{\alpha+1} = 1 - y$. Thus $x^{\alpha+1} - x = \frac{y^{\alpha^2} - y^{\alpha^2+1}}{N(y)}$. So $N(x) - 2 = B(x) - T(x) = \frac{T(y)-B(y)}{N(y)} = \frac{-N(y)-(N(y)-1)}{N(y)} = -2 + \frac{1}{N(y)}$ so $N(y) = \frac{1}{N(x)}$.

Define a polynomial r by $r(Z) = Z^3 - T(x)Z^2 + B(x)Z - N(x) = -N(x)Z^3\Big(Z^{-3}-$

$T(y)Z^{-2}+B(y)Z^{-1}-N(y)\Big)$. We have shown that the set of roots of r is $\{x, x^{\alpha}, x^{\alpha^2}\}$ $= \{y^{-1}, y^{-\alpha}, y^{-\alpha^2}\}$. Thus $y = x^{-\alpha^i}$ for some i. Substituting this in the equation $x^{\alpha+1}y^{\alpha+1}-xy^{\alpha+1} = 1-y$ we get $x^{\alpha+1}-x = x^{\alpha^{i+1}+\alpha^i}-x^{\alpha^{i+1}}$. For $i = 0$ or 2 this (using $\alpha^3 = 1$) leads directly to $x^{\alpha} = x$. For $i = 1$ this becomes $x(x^{\alpha} - 1) = x^{\alpha^2}(x^{\alpha} - 1)$ which (using $x \neq 1$) again leads to $x^{\alpha} = x$. This gives $3x^2 = B(x) = -1$ so $3 \neq 0$, $x^2 = -1/3$ and $y = x^{-\alpha^i} = x^{-1}$. Using these in the centre II equation gives $0 = 1 - 1 - 3 - x - x + 1 = -2x - 2$ so $x = -1$ which contradicts $N(x) \neq -1$.

This proves Lemma 3.

LEMMA 4: There cannot be a Fano subplane of the type described in the Theorem with three median flags of type III-III.

Proof of the Lemma: Suppose the Lemma is false.

We may assume that the median flags are M_1-$m_1 = \langle(-y, y^{\alpha+1}, 1)\rangle - \left\langle\begin{pmatrix} y^{\alpha} \\ 1 \\ y^{\alpha+1}\end{pmatrix}\right\rangle$, M_2-$m_2 = \langle(1, -z, z^{\alpha+1})\rangle - \left\langle\begin{pmatrix} z^{\alpha+1} \\ z^{\alpha} \\ 1\end{pmatrix}\right\rangle$, and M_3-$m_3 = \langle(x^{\alpha+1}, 1, -x)\rangle - \left\langle\begin{pmatrix} 1 \\ x^{\alpha+1} \\ x^{\alpha}\end{pmatrix}\right\rangle$ all of type III-III with $N(y), N(x), N(z) \neq 0, \pm 1$.

The sides opposite M_1, M_2, M_3 are $\left\langle\begin{pmatrix} 0 \\ y^{\alpha^2} \\ 1\end{pmatrix}\right\rangle$, $\left\langle\begin{pmatrix} 1 \\ 0 \\ z^{\alpha^2}\end{pmatrix}\right\rangle$, $\left\langle\begin{pmatrix} x^{\alpha^2} \\ 1 \\ 0\end{pmatrix}\right\rangle$ respectively. The vertices opposite m_1, m_2, m_3 are $\langle(0, -1, y^{\alpha^2})\rangle$, $\langle(z^{\alpha^2}, 0, -1)\rangle$, $\langle(-1, x^{\alpha^2}, 0)\rangle$ respectively.

If the circle line is of type III then $xyz = -1$. If the centre point is of type III then $xyz = 1$. These cannot both occur in the same example because $2 \neq 0$.

Consider the case where the circle line is of type III (so that $z = -(xy)^{-1}$) and the centre point is of type II. The points m_1m_3 and m_2m_1 are $\langle(x^{\alpha}(xy^{\alpha+1} - 1), y^{\alpha}(x^{\alpha} - y), 1-x^{\alpha+1}y^{\alpha})\rangle$ and $\langle(1-y^{\alpha+1}z^{\alpha}, y^{\alpha}(yz^{\alpha+1}-1), z^{\alpha}(y^{\alpha}-z))\rangle$; the conditions that these points are of type II are $0 = N(xy)+N(y)+1-T(xy^{1+\alpha})$ and $0 = N(yz)+N(z)+1-T(yz^{1+\alpha}) =$ (using $z = -x^{-1}y^{-1}$) $N(z)\Big(N(y)+1-N(xy)+T(x^{\alpha^2}y^{\alpha^2+1})\Big)$ which are equivalent to $0 = N(xy)+N(y)+1-T(xy^{1+\alpha})$ and $0 = N(y)+1-N(xy)+T(xy^{1+\alpha})$, respectively. Adding these last two equations gives $0 = 2(N(y)+1)$ which contradicts $2 \neq 0$ and $N(y) \neq -1$.

By π-duality, the only remaining case is where the circle line is of type II and the centre point is of type II. Applying the determinant condition for the Pappian collinearity of M_1, M_2, M_3 and the Pappian concurrency of the lines m_1, m_2, m_3 gives the equations:

$c = 0$ where $c = xyz(xyz)^{\alpha} + 1 - xyz + xyy^{\alpha} + yzz^{\alpha} + zxx^{\alpha}$

$d = 0$ where $d = 1 + (xyz)^\alpha + xyz(xyz)^\alpha - x(xy)^\alpha - y(yz)^\alpha - z(zx)^\alpha$

Note that, as above, the conditions that m_3m_1, m_1m_2, m_2m_3 are points of type II are equivalent to $T(xy^{1+\alpha}) = N(xy) + N(y) + 1,\ T(yz^{1+\alpha}) = N(yz) + N(z) + 1,\ T(zx^{1+\alpha}) = N(zx) + N(x) + 1$.

Also note that the conditions that $M_3M_1,\ M_1M_2\ , M_2M_3$ are lines of type II are equivalent to $T(x^{1+\alpha}y^\alpha) = -N(xy) + N(x) - 1,\ T(y^{1+\alpha}z^\alpha) = -N(yz) + N(y) - 1,\ T(z^{1+\alpha}x^\alpha) = -N(zx) + N(z) - 1$.

Thus

$T(c) = B(xyz) + 3 - T(xyz) + \Big(N(xy) + N(yz) + N(zx)\Big) + \Big(N(x) + N(y) + N(z)\Big) + 3,$

$T(d) = B(xyz) + 3 + T(xyz) + \Big(N(xy) + N(yz) + N(zx)\Big) - \Big(N(x) + N(y) + N(z)\Big) + 3,$

$T((xyz)^{\alpha^2}c + c^{\alpha^2}) = 6N(xyz) + 2(N(xy) + N(yz) + N(zx)) + 2(N(x) + N(y) + N(z)) + 6.$ and

$T((xyz)^{\alpha^2}d - d^\alpha) = 6N(xyz) - 2(N(xy) + N(yz) + N(zx)) + 2(N(x) + N(y) + N(z)) - 6.$

We can deduce four useful facts:

$0 = T(c) + T(d)$ implies $B(xyz) = -(N(xy) + N(yz) + N(zx)) - 6$

$0 = T(c) - T(d)$ implies $T(xyz) = N(x) + N(y) + N(z)$

$0 = T((xyz)^{\alpha^2}d - d^\alpha - (xyz)^{\alpha^2}c - c^{\alpha^2})$ implies $N(xy) + N(yz) + N(zx) = -3$

$0 = T((xyz)^{\alpha^2}d - d^\alpha + (xyz)^{\alpha^2}c + c^{\alpha^2})$ implies $N(x) + N(y) + N(z) = -3N(xyz)$.

Thus $B(xyz) = -(-3) - 6 = -3 = N(x)N(y) + N(y)N(z) + N(z)N(x)$.

Thus $\{xyz, (xyz)^\alpha, (xyz)^{\alpha^2}\}$ is the set of roots of the polynoimial p where $p(W) = W^3 - T(xyz)W^2 + B(xyz)W - N(xyz) = W^3 - \Big(N(x) + N(y) + N(z)\Big)W^2 + \Big(N(x)N(y) + N(y)N(z) + N(z)N(x)\Big)W - N(x)N(y)N(z) = \Big(W - N(x)\Big)\Big(W - N(y)\Big)\Big(W - N(z)\Big)$ whose set of roots is $\{N(x), N(y), N(z)\}$. Thus xyz is fixed by α. So p has only the root xyz (but with multiplicity 3) and thus $xyz = N(x) = N(y) = N(z)$.

Now $N(x) = N(y)$ implies $x = \phi y^{\alpha^2}$ for some ϕ with $N(\phi) = 1$. Then $N(y) = xyz = \phi y^{\alpha^2}yz$ implies $z = \phi^{-1}y^\alpha$. Substituting these expressions for x, y, z in c and $T(xy^{1+\alpha})$ gives $N(y)^2 + N(y) + 1 = T(xy^{1+\alpha}) = N(y)T(\phi) = c - N(y)^2 + N(y) - 1 = -N(y)^2 + N(y) - 1$. So $2(N(y)^2 + 1) = 0$. Because $2 \neq 0$, $N(y)^2 = -1$.

Then $N(y)T(\phi) = T(xy^{1+\alpha}) = N(y)^2 + N(y) + 1 = N(y)$ so $T(\phi) = 1$. Also $N(y)T(\phi^{1+\alpha}) = T(x^{1+\alpha}y^\alpha)) = -N(x)^2 + N(x) - 1 = N(y)$ so $T(\phi^{1+\alpha}) = B(\phi) = 1$.

So ϕ is a root of $q(W) = W^3 - W^2 + W - 1 = (W-1)(W^2+1)$. If $\phi = 1$, then $3 = T(\phi) = 1$ which is impossible for $2 \neq 0$. If $\phi^2 = -1$ then ($\phi^\alpha = \phi$ and) $-3 = B(\phi) = 1$ which also is impossible for $2 \neq 0$.

This proves Lemma 4.

The author would like to thank Gary Ebert, Chris Fisher, Norm Johnson, Tim Penttila, and Alex Rosa for help with some of the references.

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Department of Mathematics and Statistics
York University
4700 Keele Street
North York, Ontario
Canada M3J 1P3
juliamnb@yorku.ca