On partial ovoids of Hermitian surfaces

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Abstract

Lower bounds for the size of a complete partial ovoid in a non-degenerate Hermitian surface are obtained. For even characteristic, a sharp bound is obtained and all examples of this size are described. Next, a general construction method for locally hermitian partial ovoids is explained, which leads to interesting small examples. Finally, a conjecture is given for the size of the largest complete strictly partial ovoid. By using partial derivation, several examples of complete strictly partial ovoids of this size are provided.

1 Introduction

Let $\mathcal{H} = \mathcal{H}(3, q^2)$ be a Hermitian surface of the projective space $\mathsf{PG}(3, q^2)$, where q is any prime power. The lines lying on $\mathcal{H}$ are called its *generators*, and an *ovoid* of $\mathcal{H}$ is defined to be a point set in $\mathcal{H}$ having exactly one common point with every generator. Thus any ovoid must have $q^3 + 1$ points. Any non-tangent plane of $\mathcal{H}$ cuts out on $\mathcal{H}$ a Hermitian curve, which is an ovoid of $\mathcal{H}$ called the *classical* ovoid. Non-classical ovoids of $\mathcal{H}$ were first constructed by Payne and Thas [14], and are now known to exist in abundance.

A *partial ovoid* or *cap* of $\mathcal{H}$ is any point set in $\mathcal{H}$ which has at most one common point with every generator. A partial ovoid is called *complete* if it is not contained in a larger partial ovoid of $\mathcal{H}$. Since ovoids are known to exist, we will be most interested in *complete strictly partial ovoids*; that is, complete partial ovoids which are not ovoids. In particular, we will be interested in the spectrum of sizes for such

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objects. Examples of complete strictly partial ovoids will be given, and the results of various computer searches will be reported.

2 Lower Bounds

Let $\mathcal{O}$ be a complete partial ovoid of $\mathcal{H}$. Any generator of $\mathcal{H}$ which contains no point of $\mathcal{O}$ will be called a *free generator* of $\mathcal{H}$. If X is a point of $\mathcal{H}$ that does not belong to $\mathcal{O}$, then X must be collinear with at least one point of $\mathcal{O}$, for otherwise $\mathcal{O} \cup \{X\}$ would be a partial ovoid properly containing $\mathcal{O}$, which contradicts the completeness of $\mathcal{O}$. The number of points of $\mathcal{O}$ collinear with X will be called the *strength* of X. Note that $\mathcal{O}$ is an ovoid of $\mathcal{H}$ if and only if all points of $\mathcal{H} \setminus \mathcal{O}$ have strength $q+1$.

Theorem 2.1. *Let $\mathcal{O}$ be a complete strictly partial ovoid of the Hermitian surface $\mathcal{H} = \mathcal{H}(3, q^2)$ in $\Sigma = \mathsf{PG}(3, q^2)$. Then the number of points in $\mathcal{O}$ is at least $q^2 + 1$.*

Proof. Let g be a free generator of $\mathcal{H}$, the existence of which follows from the assumption that $\mathcal{O}$ is not an ovoid of $\mathcal{H}$. Let P be a point of g. Then there must be a generator through P, say g_P, whose intersection with $\mathcal{O}$ is not empty, since P has strength at least 1.

Now the set $\{g_P | P \in g\}$ consists of q^2+1 skew generators of $\mathcal{H}$ as the Hermitian surface does not contain any triangle. Since each of these generators meets $\mathcal{O}$, we see that $|\mathcal{O}| \geq q^2 + 1$. ∎

It should be noted that in [8] it is shown that the above lower bound holds in $\mathcal{H}(n, q^2)$ for all dimensions $n \geq 3$.

If q is even, then there are complete partial ovoids of size $q^2 + 1$. For instance, let $\mathcal{Q}$ be an elliptic quadric of a Baer subspace $\Sigma_0 = \mathsf{PG}(3, q)$ of Σ, and let $\mathcal{L}$ be the set of tangent lines to $\mathcal{Q}$ in Σ_0. Since q is even, $\mathcal{L}$ is a general linear complex of $\mathsf{PG}(3, q)$. As shown in [1], the lines of any general linear complex $\mathcal{L}$, when extended over $GF(q^2)$, cover the points of a Hermitian surface $\mathcal{H}$. Moreover, the generators of the resulting Hermitian surface are either extended lines of $\mathcal{L}$ or they are skew to the Baer subspace Σ_0. Therefore, no generator meets $\mathcal{Q}$ in more than one point, and $\mathcal{Q}$ is a partial ovoid of size $q^2 + 1$. To show that $\mathcal{Q}$ is complete, observe that every point of Σ_0 lies on $q + 1$ tangents to $\mathcal{Q}$ and every point of $\mathcal{H} \setminus \Sigma_0$ lies on a unique (extended) tangent line to $\mathcal{Q}$.

We will proceed to show that there are no other examples of size $q^2 + 1$, which will imply an increase of the lower bound if q is odd.

Lemma 2.2. *Let $\mathcal{O}$ be a complete partial ovoid of $\mathcal{H}$. Then $|\mathcal{O}| = q^2+1$ if and only if all points of $\mathcal{H} \setminus \mathcal{O}$ have strength 1 or $q+1$, where both strengths occur.*

Proof. Let $\mathcal{O}$ be a complete partial ovoid of $\mathcal{H}$ of size $q^2 + 1$. As $\mathcal{O}$ is not an ovoid, there exists a free generator g of $\mathcal{H}$, and every point of g is collinear with at least one point of $\mathcal{O}$. Since g has $q^2 + 1$ points and $|\mathcal{O}| = q^2 + 1$, it follows that every point of g must be collinear with exactly one point of $\mathcal{O}$, so all points of g have strength 1. In particular, there exist points of strength 1.

Let P, Q be two points of $\mathcal{O}$ and denote the polarity associated with $\mathcal{H}$ by τ. Then every point of $P^\tau \cap Q^\tau$ has strength at least 2. Consider a point X of $\mathcal{H} \setminus \mathcal{O}$ with strength at least 2. If there exists a free generator g on X, then the completeness of

$\mathcal{O}$ implies that every point of g must be collinear with at least one point of $\mathcal{O}$, and consequently $|\mathcal{O}| \geq q^2 + 2$. Since $|\mathcal{O}| = q^2 + 1$ by assumption, there cannot exist a free generator containing X and X must have strength $q + 1$. In particular, there also exist points of strength $q + 1$.

Conversely, let $\mathcal{O}$ be a complete strictly partial ovoid of $\mathcal{H}$ with the property that all points of $\mathcal{H} \setminus \mathcal{O}$ have strength 1 or $q + 1$, and assume that both strengths occur. Consider a point $X \notin \mathcal{O}$ with strength 1 and let g be a free generator on X. The $q^2 + 1$ points on g cannot have strength $q + 1$, so they must all have strength 1. On the other hand, every point of $\mathcal{O}$ must be collinear with exactly one point of g, since $\mathcal{H}$ is a generalized quadrangle. Hence $\mathcal{O}$ has size $q^2 + 1$. ■

Theorem 2.3. *Let $\mathcal{O}$ be a complete partial ovoid of $\mathcal{H}$. Then $|\mathcal{O}| = q^2 + 1$ if and only if q is even and $\mathcal{O}$ is an ovoid of some $W_3(q)$, which is the intersection of $\mathcal{H}$ with a Baer subspace $\mathsf{PG}(3, q)$ of $\mathsf{PG}(3, q^2)$ as in the model for $\mathcal{H}$ described above.*

Proof. By Lemma 2.2, $\mathcal{O}$ has size $q^2 + 1$ if and only if every point of $\mathcal{H}$ off $\mathcal{O}$ has strength 1 or $q + 1$. The generalized quadrangle $\mathcal{H}$ is isomorphic to the dual of the generalized quadrangle $Q^-(5, q)$, and $\mathcal{O}$ corresponds to a maximal partial spread $\mathcal{S}$ of $Q^-(5, q)$ of size $q^2 + 1$. Denote by $\tilde{\mathcal{S}}$ the set of all points of $Q^-(5, q)$ on the lines of $\mathcal{S}$. Then every line of $Q^-(5, q)$ has either 1 or $q + 1$ points in common with $\tilde{\mathcal{S}}$, again by Lemma 2.2. Define a substructure $\mathcal{T} := (\mathcal{P}, \mathcal{B}, \mathrm{I})$ of the generalized quadrangle $Q^-(5, q)$ as follows. The point set $\mathcal{P}$ coincides with $\tilde{\mathcal{S}}$, the set $\mathcal{B}$ of lines of $\mathcal{T}$ consists of all lines of $Q^-(5, q)$ having $q + 1$ points in common with $\tilde{\mathcal{S}}$, and incidence I is the incidence of $Q^-(5, q)$. Since every line of $Q^-(5, q)$ either contains exactly one point of $\tilde{\mathcal{S}}$ or is a line of $\mathcal{B}$, it follows that $\mathcal{T} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ satisfies the conditions of [10, Theorem 2.3.1]. Since the lines of $\mathcal{S}$ are pairwise disjoint, one concludes from [10, Theorem 2.3.1] that $\mathcal{T}$ is a subquadrangle of $Q^-(5, q)$. Moreover, the lines of $\mathcal{S}$ partition the point set of this subquadrangle and hence $\mathcal{S}$ is a spread of $\mathcal{T}$, so that $|\mathcal{P}| = (q + 1)(q^2 + 1)$. Consequently, $\mathcal{T}$ has order (q, q) and it is a generalized quadrangle $Q(4, q)$ having a spread $\mathcal{S}$. By [11], a spread of $Q(4, q)$ exists if and only if q is even. Since $Q(4, q)$ corresponds to a subquadrangle $W_3(q)$ which is the intersection of $\mathcal{H}$ with a Baer subspace $\mathsf{PG}(3, q)$, the theorem follows. ■

Corollary 2.4. *If q is odd, then a complete partial ovoid of $\mathcal{H}$ has at least $q^2 + 2$ points.*

In fact, one can show that a complete partial ovoid of $\mathcal{H}$ cannot have exactly $q^2 + 2$ points. This was first pointed out to the authors by J. A. Thas [13], and can easily be shown as follows.

Theorem 2.5. *There are no complete partial ovoids of $\mathcal{H}$ with size $q^2 + 2$. In particular, if q is odd, then every complete partial ovoid has size at least $q^2 + 3$.*

Proof. Suppose $\mathcal{O}$ is a complete partial ovoid in $\mathcal{H}$ of size $q^2 + 2$. Then every free generator has q^2 points of strength 1 and one point of strength 2. Moreover, each point of $\mathcal{H} \setminus \mathcal{O}$ with strength 2 lies on exactly $q - 1$ free generators. Since there are $(q + 1)(q^3 - q^2 - 1)$ free generators, we see that there are precisely $(q + 1)(q^3 - q^2 - 1)/(q - 1)$ points of strength 2. But this is not an integer unless $q = 2$ or $q = 3$, and computer searches show no such examples exist for these values of q. ■

We now improve the lower bound for complete partial ovoids under some very special circumstances, when q is odd.

Theorem 2.6. *Let q be an odd prime power, and let $\mathcal{H}$ be a Hermitian surface in $\Sigma = \mathsf{PG}(3, q^2)$ obtained by extending over $GF(q^2)$ the lines of a general linear complex $\mathcal{L}$ in a Baer subspace $\Sigma_0 = \mathsf{PG}(3, q)$. Let $\mathcal{O}$ be a complete partial ovoid of $\mathcal{H}$, and let $\mathcal{O}' = \mathcal{O} \cap \Sigma_0$. Then $|\mathcal{O}'| \leq q^2 - q + 1$. Moreover, if $|\mathcal{O}'| = q^2 - q + 1$, then*

$$|\mathcal{O}| \geq \frac{3}{2}q^2 - \frac{1}{2}q + 1.$$

Proof. Let $W(3, q)$ denote the symplectic geometry (polar space) consisting of the points of Σ_0 and the lines in $\mathcal{L}$. Since the (extended) generators of $W(3, q)$ are also generators of $\mathcal{H}$, no two points of $\mathcal{O}'$ determine a generator of $W(3, q)$ and $\mathcal{O}'$ is a partial ovoid of $W(3, q)$. As q is odd, $|\mathcal{O}'| \leq q^2 - q + 1$ by a result of Tallini [11].

Now suppose that $|\mathcal{O}'| = q^2 - q + 1$. The number of free generators of $W = W(3, q)$ is $q(q+1)$. These generators may or may not intersect in Σ_0, but they are certainly mutually skew in $\Sigma \setminus \Sigma_0$. Thus the number of points of $\mathcal{H} \setminus \Sigma_0$ lying on these free (extended) generators of W is $q(q+1)(q^2 - q) = q^2(q^2 - 1)$. We define these points to be "free points".

Let P be a point of $\mathcal{O} \setminus \mathcal{O}'$. Then, by the construction of $\mathcal{H}$, we know that P lies on a unique free generator of W, say ℓ, and thus P is a free point. The $q^2 - q$ points of $\ell \setminus \Sigma_0$, one of which is P, are free points that are now blocked by the addition of P. The remaining q generators of $\mathcal{H}$ through P are all skew to Σ_0 (see [1]), and we let m be any one of them. There are exactly $q^2 + 1$ W-generators that meet m and they form a regular symplectic spread $\mathcal{S}$ of Σ_0.

Since $|\mathcal{O}'| = q^2 - q + 1$, exactly $q^2 - q + 1$ lines of $\mathcal{S}$ meet $\mathcal{O}'$ (in one point each), and hence q lines of $\mathcal{S}$ are skew to $\mathcal{O}'$. Therefore there are q free generators of W meeting m, one of which is ℓ. Thus we get another $q - 1$ points on m that are now blocked by the addition of P to $\mathcal{O}'$. Allowing m to vary over the q generators through P skew to Σ_0, we get a total of $q(q-1) = q^2 - q$ free points not on ℓ that are now blocked by P. So, adjoining P to $\mathcal{O}'$ blocks a total of $q^2 - q + q^2 - q = 2(q^2 - q)$ previously free points.

Adding another point P' will block another $2(q^2 - q)$ free points, not necessarily disjoint from the above set of $2(q^2 - q)$ free points. Now we must eventually block all the free points since $\mathcal{O}$ is complete. Therefore we must adjoin to $\mathcal{O}'$ at least $\frac{q^2(q^2-1)}{2(q^2-q)} = \frac{1}{2}q(q + 1)$ points. That is,

$$|\mathcal{O} \setminus \mathcal{O}'| \geq \frac{1}{2}q(q + 1)$$

and hence

$$|\mathcal{O}| \geq q^2 - q + 1 + \frac{1}{2}q(q + 1) = \frac{3}{2}q^2 - \frac{1}{2}q + 1,$$

proving the result. ∎

It should be noted that if $\mathcal{O}'$ is "large", say of size greater than $\frac{1}{2}q^2$, then the above lower bound for $\mathcal{O}$ still holds. In fact, the same argument gives an even stronger bound. However, if $\mathcal{O}'$ is "small", this counting argument breaks down and the lower bound gets weaker. Eventually, it degenerates into the general lower bound given in Theorem 2.1.

Perhaps more importantly, for odd q we have not been able to construct complete partial ovoids of any size close to the bound given in Theorem 2.6. For instance, for $q = 5$, the smallest complete partial ovoid we have been able to construct has size 61. It appears that much work remains in improving the general lower bound when q is odd.

3 Complete Caps from Maximal Partial Spreads of $\mathsf{PG}(3, q)$

In this section we present a construction method for maximal partial spreads of $Q^-(5, q)$, starting from a maximal partial spread of $\mathsf{PG}(3, q)$. As the generalized quadrangles $\mathcal{H} = \mathcal{H}(3, q^2)$ and $Q^-(5, q)$ are dual to each other, this is equivalent to constructing complete caps of $\mathcal{H}$. All partial spreads of $Q^-(5, q)$ we obtain will be *locally hermitian* at some line L, which means that they are the union of (3–dimensional) reguli pairwise meeting in L. The method is based on a known construction for locally hermitian spreads of $Q^-(5, q)$, see [12], and was suggested to the authors by J. A. Thas.

Consider the elliptic quadric $Q^-(5, q)$, with associated polarity $\perp$. Let L be a line of $Q^-(5, q)$, and thus $L^\perp$ is a 3-dimensional projective space $\mathsf{PG}(3, q)$ intersecting $Q^-(5, q)$ exactly in L. Consider a maximal partial spread S in $L^\perp$, such that $L \in S$. For every line $M \in S \setminus \{L\}$, $M^\perp$ is a 3-dimensional projective space containing L and meeting $Q^-(5, q)$ in a hyperbolic quadric $Q^+(3, q)$. If $\mathcal{R}_M$ denotes the regulus through L of this hyperbolic quadric, we let

$$\mathcal{S}_S := \bigcup_{M \in S \setminus \{L\}} \mathcal{R}_M .$$

Theorem 3.1. *With the above notation, $\mathcal{S}_S$ is a maximal partial spread of $Q^-(5, q)$.*

Proof. Let K and M be any two distinct lines of $S \setminus \{L\}$. Then we have that $K^\perp \cap M^\perp = \langle K, M\rangle^\perp = (L^\perp)^\perp = L$. As the reguli $\mathcal{R}_K$ and $\mathcal{R}_M$ share the line L, it follows that no two lines of $\mathcal{S}_S$ intersect. Hence $\mathcal{S}_S$ is a partial spread of $Q^-(5, q)$.

In order to show that $\mathcal{S}_S$ is maximal, suppose by way of contradiction that there exists a line A of $Q^-(5, q)$ which is skew to all lines of $\mathcal{S}_S$. For every line $M \in S \setminus \{L\}$, the lines of $\mathcal{R}_M$ cover all points of $M^\perp \cap Q^-(5, q)$. Hence A is also skew to $M^\perp$, and consequently $A^\perp$ is disjoint from M. But then $A^\perp \cap L^\perp$ is a line of $L^\perp$, skew to all lines of S, contradicting the maximality of S. ∎

By this construction, one can associate a maximal partial spread $\mathcal{S}_S$ of $Q^-(5, q)$, and equivalently a complete cap $\mathcal{O}_S$ of $\mathcal{H}$, with every maximal partial spread S of $\mathsf{PG}(3, q)$. The resulting complete cap has size $|\mathcal{O}_S| = |\mathcal{S}_S| = (|S| - 1)q + 1$. By relying on the known results concerning small maximal partial spreads of $\mathsf{PG}(3, q)$, we obtain examples of “small” complete caps on $\mathcal{H}$, as follows.

In [6], maximal partial spreads of sizes $13, 14, 15, \ldots, 22$ in $\mathsf{PG}(3, 5)$ are constructed by computer. Using the smallest one, our method above produces a maximal partial spread of size 61 on $Q^-(5, 5)$. This in turn yields a complete cap of size 61 on $\mathcal{H}(3, 25)$, the smallest one we have constructed so far. Similarly, in [5], maximal partial spreads of sizes $23, 24, 25$ are constructed by computer in $\mathsf{PG}(3, 7)$. The smallest one produces a maximal partial spread of size 155 on $Q^-(5,7)$ and hence a complete cap of the same size on $\mathcal{H}(3,49)$. Again, this is the smallest complete cap we have been able to construct on this Hermitian surface. In general, maximal partial spreads of size n in $\mathsf{PG}(3,q)$ for odd $q \geq 7$ have been constructed for all $\frac{q^2+1}{2}+6 \leq n \leq q^2-q+2$ (see [7]). In fact, for certain values of q this result can be slightly improved (again, see [7]). For each of these maximal partial spreads there is a corresponding complete cap on $\mathcal{H}$ of size $(n-1)q+1$, see also [2].

The general construction producing the smallest maximal partial spreads in $\mathsf{PG}(3,q)$ known to the authors is the one presented in [3]. In that paper (see the comments at the end of Section 3 in [3]) a certain randomized selection process guarantees the existence of a maximal partial spread in $\mathsf{PG}(3,q)$, for odd q, of size $(m+1)q+1$, where m is the smallest integer greater than or equal to $2\log_2(q)$. Thus, using our method above, one obtains complete caps of size $(m+1)q^2+1$ on $\mathcal{H}(3,q^2)$ for any odd q, where $m = \lceil 2\log_2(q) \rceil$. Of course, this is still much larger than the general lower bound given in Theorem 2.5 for odd q.

4 Construction of Large Complete Caps

Randomized and biased computer searching in [4] found complete partial ovoids of $\mathcal{H}$ for $q=5$ with sizes between 78 and 119, inclusive, as well as sizes $121 = q^3-q+1$ and $126 = q^3+1$. For $q=7$ the computer searches in [4] found complete partial ovoids of various sizes between 195 and $337 = q^3-q+1$, as well as $344 = q^3+1$. For $q=3$ our random searching found complete partial ovoids of sizes $28 = q^3+1$ and $25 = q^3-q+1$ through 16, inclusive. There seems to be strong evidence to conjecture that there are no complete partial ovoids of size between q^3-q+1 and q^3+1. In fact, the authors have recently been told that this has been proven in [9]. In this section we provide several (related) construction methods for complete strictly partial ovoids, valid for any prime power q. In particular, we produce several examples of size q^3-q+1, which is now known to be the largest possible size for a complete strictly partial ovoid of $\mathcal{H}(3,q^2)$.

Construction I

Let σ be a non-tangent plane of the Hermitian surface $\mathcal{H}$, and let $\mathcal{U}$ be the Hermitian curve cut out on $\mathcal{H}$ by σ. Consider two chords of $\mathcal{U}$ through a given point $P \in \mathcal{U}$, say ℓ and m. Then $\mathcal{S} := \mathcal{U} \setminus (\ell \cup m)$ is a partial ovoid of $\mathcal{H}$ of size q^3-2q. We will complete $\mathcal{S}$ to a complete strictly partial ovoid $\mathcal{O}$ of $\mathcal{U}$ of size q^3-q+1.

If a point X of $\mathcal{H} \setminus \mathcal{S}$ is not collinear with any point of $\mathcal{S}$, then either X is a point of $\ell \cup m$, or $X^\tau \cap \sigma \in \{\ell, m\}$, where τ denotes the polarity associated with $\mathcal{H}$. This means that the points of $\mathcal{H} \setminus \mathcal{S}$ which are not collinear with any point of $\mathcal{S}$ are precisely the points of $(\ell \cup m \cup \ell^\tau \cup m^\tau) \cap \mathcal{H}$. Since ℓ and m meet in P, the chords ℓ^τ and m^τ of $\mathcal{H}$ lie in the plane P^τ, and $\ell^\tau \cap m^\tau$ is some point that does not belong to $\mathcal{H}$. Let $g_1, g_2, \ldots, g_{q+1}$ denote the $q+1$ generators of $\mathcal{H}$ containing P, and define $X_1 := g_1 \cap \ell^\tau$ and $Y_i := g_i \cap m^\tau$ for $i = 2, 3, \ldots, q+1$. Then the points $X_1, Y_2, Y_3, \ldots, Y_{q+1}$ are pairwise noncollinear in $\mathcal{H}$, and they are contained in $(\ell^\tau \cup m^\tau) \cap \mathcal{H}$. As a consequence, they are not collinear with any point of the partial ovoid $\mathcal{S}$. So $\mathcal{O} := \mathcal{S} \cup \{X_1, Y_2, Y_3, \ldots, Y_{q+1}\}$ is a partial ovoid of $\mathcal{H}$ of size q^3-q+1.

In order to show that $\mathcal{O}$ is complete, recall that $(\ell \cup m \cup \ell^\tau \cup m^\tau) \cap \mathcal{H}$ consists of all the points of $\mathcal{H} \setminus \mathcal{S}$ which are not collinear with any point of $\mathcal{S}$. Every point of $\ell \cap \mathcal{H}$ is collinear with the point X_1 of $\mathcal{O}$, while every point of $m \cap \mathcal{H}$ is collinear with the q points $Y_2, Y_3, \ldots, Y_{q+1}$ of $\mathcal{O}$. Finally, every point of $(\ell^\tau \cup m^\tau) \cap \mathcal{H}$ either lies on g_1 and hence is collinear with $X_1 \in \mathcal{O}$, or lies on the generator g_i for some $i \in \{2, 3, \ldots, q+1\}$ and hence is collinear with the point $Y_i \in \mathcal{O}$. Thus we see that all points of $\mathcal{H} \setminus \mathcal{O}$ are collinear with at least one point of $\mathcal{O}$, and the partial ovoid $\mathcal{O}$ is complete.

This construction can be modified to obtain many examples of complete strictly partial ovoids of $\mathcal{H}$, all with the same size. If we define $X_i := g_i \cap \ell^\tau$ and $Y_i := g_i \cap m^\tau$ for all $i \in \{1, 2, \ldots, q+1\}$, then it is obvious that for any two nonempty subsets $I \subseteq \{1, 2, \ldots, q+1\}$ and $J := \{1, 2, \ldots, q+1\} \setminus I$, the set $\mathcal{O}' := \mathcal{S} \cup \{X_i \mid i \in I\} \cup \{Y_j \mid j \in J\}$ also is a complete strictly partial ovoid of $\mathcal{H}$ of size $q^3 - q + 1$.

Construction II

Consider three chords ℓ, m and n in σ, such that ℓ and m are as in the first example, $l \cap n$ is a point Q of $\mathcal{H}$, and $m \cap n$ is a point R not on $\mathcal{H}$. Further, let $\mathcal{O}$ be the complete partial ovoid in the first construction above, and define $\mathcal{T} := \mathcal{O} \setminus n$. Then $\mathcal{T}$ has size $q^3 - 2q + 1$. Suppose that the generators g_i and the points X_i and Y_i, $i = 1, 2, \ldots, q+1$, are defined as above, and let the generators of $\mathcal{H}$ through Q be denoted by $h_1, h_2, \ldots, h_{q+1}$. Here we assume that the numbering has been chosen such that X_i is a point of h_i for all $i \in \{1, 2, \ldots, q+1\}$. Finally, let Z_i be the point $n^\tau \cap h_i$ for $i = 1, 2, \ldots, q+1$. Since the common point R of m and n is not a point of $\mathcal{H}$, m^τ and n^τ are chords of the hermitian curve $\mathcal{H} \cap R^\tau$, which implies that the points $Y_1, Y_2, \ldots, Y_{q+1}, Z_1, Z_2, \ldots, Z_{q+1}$ are pairwise noncollinear. It thus follows by similar arguments to those given above that the set $\mathcal{O}'' := \mathcal{T} \cup \{Z_2, Z_3, \ldots, Z_{q+1}\}$ is a complete strictly partial ovoid of $\mathcal{H}$ of size $q^3 - q + 1$. Note that this procedure can be applied repeatedly by considering other chords of $\mathcal{H}$ in σ which meet m in R and meet ℓ in some point of $\mathcal{H}$ other than Q.

Construction III

Let $\ell_1, \ell_2, \ldots, \ell_{q+1}$ be $q+1$ distinct chords of $\mathcal{H}$ in σ through a common point P of $\mathcal{H}$. Then $\ell_1^\tau, \ell_2^\tau, \ldots, \ell_{q+1}^\tau$ are chords of $\mathcal{H}$ in P^τ containing a common point $S = \sigma^\tau$, which is not a point of $\mathcal{H}$. For any nonempty proper subset $I \subset \{1, 2, \ldots, q+1\}$, it is possible to find $q+1$ points $X_1, X_2, \ldots, X_{q+1}$ of $\mathcal{H}$ in the plane P^τ which are pairwise noncollinear, and such that for every $i \in I$, ℓ_i^τ contains at least one point X_j. By defining $\mathcal{O} := (\mathcal{U} \setminus \{l_i \mid i \in I\}) \cup \{X_1, X_2, \ldots, X_{q+1}\}$, one obtains a complete partial ovoid of $\mathcal{H}$ of size $q^3 - kq + 1$, where $|I| = k + 1$.

Construction IV

Consider three chords ℓ, m and n of $\mathcal{H}$ in σ such that $P := \ell \cap m$, $Q := \ell \cap n$, and $R := m \cap n$ are distinct points of $\mathcal{H}$. In P^τ we pick one point X of $\ell^\tau \cap \mathcal{H}$ and q points of $m^\tau \cap \mathcal{H}$ which are not collinear with X. In Q^τ we pick the unique point Z of $n^\tau \cap \mathcal{H}$ which is not collinear with any of the chosen points of $m^\tau \cap \mathcal{H}$, thus obtaining $q+2$ points in total. If one removes the $3q$ points of $(\ell \cup m \cup n) \cap \mathcal{H}$ from

$\mathcal{U}$ and adds these $q+2$ points, one obtains a complete strictly partial ovoid of $\mathcal{H}$ of size q^3-2q+3. Note that X and Z are not collinear in $\mathcal{H}$ for they are both collinear with the point Y, which is the intersection of ℓ^τ and the generator PX of $\mathcal{H}$, and $\mathcal{H}$ does not contain triangles.

Remark 4.1. *The general method underlying all of the above constructions may be thought of as "partial derivation" of the classical ovoid of $\mathcal{H}$, in the sense that chords of $\mathcal{H}$ are replaced by parts of their images under the polarity of $\mathcal{H}$. This replacement procedure can be applied consecutively many times, thus yielding a wealth of complete (strictly) partial ovoids of $\mathcal{H}$. Moreover, partial derivation can be combined with the usual derivation (see [14]) of ovoids of $\mathcal{H}$, provided that one derives with respect to chords which have not been affected by the partial derivation. Finally, partial derivation can also be applied to nonclassical ovoids of $\mathcal{H}$ which contain a suitable configuration of chords.*

Extension of the method to $T_3(\mathcal{O})$

The construction of complete caps of $\mathcal{H}(3,q^2)$ by partial derivation can be formulated for maximal partial spreads of the generalized quadrangle $T_3(\mathcal{O})$ as well, where $\mathcal{O}$ is an arbitrary ovoid of $\mathsf{PG}(3,q)$. This generalized quadrangle can be described as follows, see [10].

Let $\mathsf{PG}(3,q)$ be embedded as a hyperplane H in $\mathsf{PG}(4,q)$ and consider an ovoid $\mathcal{O}$ of $\mathsf{PG}(3,q)$. Then points of $T_3(\mathcal{O})$ are of three types:

(i) the symbol (∞);

(ii) the 3-dimensional subspaces of $\mathsf{PG}(4,q)$ which meet H in the tangent plane of $\mathcal{O}$ at some point;

(iii) the points of $\mathsf{PG}(4,q)\setminus H$.

Lines of $T_3(\mathcal{O})$ are of two types:

(a) the points of $\mathcal{O}$;

(b) the lines, not contained in H, through a point of $\mathcal{O}$.

A point of type (iii) is only incident with lines of type (b); the incidence is inherited from $\mathsf{PG}(4,q)$. A point of type (ii) is incident with all lines of type (b) that are contained in it and with the unique line of type (a) corresponding to the point of tangency. The point (∞) is incident with no lines of type (b) and with all lines of type (a).

The generalized quadrangle $T_3(\mathcal{O})$, with $\mathcal{O}$ an arbitrary ovoid of $\mathsf{PG}(3,q)$, is known to have spreads which are constructed in the following way, see [10]. Let x be a point of $\mathcal{O}$, and let $\pi\subseteq H$ be a plane not containing x. Consider a 3-dimensional subspace $\delta\subseteq\mathsf{PG}(4,q)$ such that $\delta\cap H=\pi$. Define $L=\pi\cap\pi_x$, where π_x is the tangent plane to $\mathcal{O}$ at x, and consider a spread S of δ containing L. For every point $x_i\in\mathcal{O}\setminus\{x\}$, $i=1,2,\ldots,q^2$, let y_i be the point $xx_i\cap\pi$ and denote the line of S incident with y_i by L_i. If the lines of the plane $\langle x,x_i,L_i\rangle$, different from xx_i, that are incident with x_i are labelled M_{ij}, $j = 1, 2, \ldots, q$, then $\mathcal{S} := \{x\} \cup \{M_{ij} \mid i = 1, 2, \ldots, q^2; j = 1, 2, \ldots, q\}$ is a spread of $T_3(\mathcal{O})$. Similarly, if S is a maximal partial spread of $\mathsf{PG}(3, q)$, by this construction one obtains a maximal partial spread of $T_3(\mathcal{O})$, as was described in [2]. If $\mathcal{O} \cong Q^-(3, q)$, then this example corresponds to the construction of Section 3.

In order to extend the notion of partial derivation to the setting of $T_3(\mathcal{O})$, we need to introduce some more notation. Let the lines of the plane $\langle x, x_i, L_i \rangle$, different from xx_i, that are incident with the point x be denoted by $\overline{M}_{ij}$, $j = 1, 2, \ldots, q$. Assume furthermore that the labelling has been chosen such that for a fixed $j \in \{1, 2, \ldots, q\}$, the 3-spaces $\langle \pi_x, \overline{M}_{ij} \rangle$ coincide for all $i \in \{1, 2, \ldots, q^2\}$. Now we pick two distinct points x_k and x_l of $\mathcal{O} \setminus \{x\}$, and define $\mathcal{S}_{kl} := \mathcal{S} \setminus \Big(\{M_{kj}, M_{lj} \mid j = 1, 2, \ldots, q\} \cup \{x\} \Big)$. In order to extend $\mathcal{S}_{kl}$ to a maximal partial spread of $T_3(\mathcal{O})$, consider a nontrivial subset $I \subset \{1, 2, \ldots, q\}$. Then the set of lines $\mathcal{S}' := \mathcal{S}_{kl} \cup \{\overline{M}_{kj} \mid j \in I\} \cup \{\overline{M}_{lj} \mid j \in \{1, 2, \ldots, q\} \setminus I\} \cup \{x_k\}$ is a maximal partial spread of $T_3(\mathcal{O})$. If $\mathcal{O} \cong Q^-(3, q)$ and S is the regular spread of δ, then this maximal partial spread is the dual of the complete cap of $\mathcal{H}(3, q^2)$ described in Construction I.

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