Harmonicity and minimality of vector fields and distributions on locally conformal Kähler and hyperkähler manifolds

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Abstract

We show that on any locally conformal Kähler (l.c.K.) manifold (M, J, g) with parallel Lee form the unit anti-Lee vector field is harmonic and minimal and determines a harmonic map into the unit tangent bundle. Moreover, the canonical distribution locally generated by the Lee and anti-Lee vector fields is also harmonic and minimal when seen as a map from (M, g) with values in the Grassmannian $G_2^{or}(M)$ endowed with the Sasaki metric. As a particular case of l.c.K. manifolds, we look at locally conformal hyperkähler manifolds and show that, if the Lee form is parallel (that is always the case if the manifold is compact), the naturally associated three- and four-dimensional distributions are harmonic and minimal when regarded as maps with values in appropriate Grassmannians. As for l.c.K. manifolds without parallel Lee form, we consider the Tricerri metric on an Inoue surface and prove that the unit Lee and anti-Lee vector fields are harmonic and minimal and the canonical distribution is critical for the energy functional, but when seen as a map with values in $G_2^{or}(M)$ it is minimal, but not harmonic.

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1 Introduction

The theory of harmonic maps is by now well settled: existence and uniqueness theorems were proved, examples were produced in any dimension of the domain and codomain. Still, it is important to continue enlarging the class of explicit examples.

Oriented distributions on Riemannian manifolds, unit vector fields in particular, proved to be a very fruitful source of such examples in two directions. On the one hand, they can provide examples of critical points of natural generalizations of the usual energy and volume functionals. On the other hand, they can provide examples of harmonic maps and/or minimal immersions in appropriate oriented Grassmannians endowed with the Sasaki metric (in the tangent or unit tangent bundle when we reduce to vector fields). Based on the fundamental work done previously in [6], [8], [19], such approach was systematically developed by one of the present authors and by his collaborators (see, for example, [3], [7] and [10 – 12] and the references therein). We give the necessary definitions in §2.

One class of Riemannian manifolds naturally endowed with distinguished vector fields and distributions is the locally conformal Kähler (l.c.K.) class (see [4]). A leading example of such a manifold is the Hopf manifold. Definitions, examples and basic properties of l.c.K. structures are given in §3. The Hermitian-Weyl structure of a l.c.K. manifold canonically determines a one-form (called the Lee form) and two vector fields, the Lee vector field and its orthogonal by the complex structure, the anti-Lee vector field. Together they generate a distribution which in some cases is a foliation (for example when the Lee form is parallel or for the Tricerri metric on the Inoue surface). It is natural to ask for their Riemannian properties with respect to the harmonicity and minimality.

The aim of this paper is to use the l.c.K. manifolds, with and without parallel Lee form, to exhibit new examples of harmonic and minimal unit vector fields and distributions.

In §4, we discuss l.c.K. manifolds with parallel Lee form. The Lee vector field being parallel, it trivially has all desired properties. But the anti-Lee vector field is never parallel and we show that it is harmonic and minimal, but unstable for both associated functionals and determines a harmonic map from the manifold into its unit tangent bundle endowed with the Sasaki metric. When the Lee field of a compact l.c.K. manifold is regular, the manifold fibers (and the projection is a Riemannian submersion) in circles over an α-Sasakian manifold whose characteristic field is the projection of the anti-Lee field. Even if the characteristic field of a Sasakian manifold is known to be harmonic and minimal, one cannot derive directly the conclusion for the anti-Lee field because a theory of the behaviour of harmonic and minimality properties of vector fields and distributions in a Riemannian submersion is still lacking. Moreover, most of the known examples of l.c.K. manifolds with parallel Lee form are non-regular.

A particular case of l.c.K. manifolds with parallel Lee form is formed by the locally conformal hyperkähler (l.c.h.K.) manifolds. They bear three "nested" l.c.K. structures, thus giving rise to a three-dimensional and a four-dimensional distribution which can be shown to be harmonic and minimal as maps with values in the appropriate Grassmannians.

In the last section of this paper, we work on an Inoue surface endowed with the

Tricerri metric. This is an explicit example of a l.c.K. metric without parallel Lee form on a compact manifold. Still we can prove that the Lee and anti-Lee vector fields are harmonic and minimal and the canonical distribution is critical for the energy functional and, when seen as a map with values in $G_2^{or}(M)$, it is minimal, but not harmonic.

We stress that usually l.c.K. manifolds are regarded in the framework of conformal geometry and the properties of the complex conformal structure are studied by means of the Weyl connection. But here we are interested merely in the Riemannian geometry of a fixed metric, the one which is locally conformal with Kähler ones, hence we shall neglect all conformal setting.

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2 Harmonic and minimal sections of tensor bundles

Let (N, g) be a Riemannian manifold and denote by ∇ its Levi Civita connection. The tangent bundle TN is naturally endowed with the Sasaki metric g^S which is also naturally induced on the hypersurface T_1N of unit vectors. Now, any section $\xi \in \Gamma(Q)$ (where Q stands here for TN or T_1N) may be understood as a map $\sigma : (N, g) \to (Q, g^S)$ between Riemannian manifolds. As such, one may ask about some of its specific properties: harmonicity, shape and volume of its image as an immersed submanifold, etc. We define the operators φ_ξ, $L_\xi \in \mathrm{End}(TN)$ (see [3], [6], [8] for details) by

$$\begin{aligned}\varphi_\xi &:= -\nabla\xi,\\ L_\xi &:= \mathrm{Id} + \varphi_\xi^t \circ \varphi_\xi\end{aligned}$$

and may compute, for compact N, the energy and volume of ξ:

$$\begin{aligned}E(\xi) &= \frac{1}{2}\int_N \mathrm{Tr} L_\xi \, \mu_g,\\ \mathrm{Vol}(\xi) &= \int_N \sqrt{\det L_\xi}\, \mu_g,\end{aligned}$$

where μ_g is the volume form of (N, g). The critical point conditions for the two functionals were found in [19] and [8]. Defining $K_\xi = -\sqrt{\det L_\xi} L_\xi^{-1} \circ \varphi_\xi^t$, these conditions read respectively:

$$\mathrm{Tr}(Z \mapsto (\nabla_Z \varphi_\xi^t)) \text{ vanishes on } \xi^\perp, \tag{2.1}$$

$$\mathrm{Tr}(Z \mapsto (\nabla_Z K_\xi)) \text{ vanishes on } \xi^\perp. \tag{2.2}$$

A unit vector field ξ is then called:

- a *harmonic vector field* if (2.1) is satisfied. If moreover, $\mathrm{Tr}(Z \mapsto R_{\xi\varphi_\xi Z}X) = 0$ for all X, then ξ is a *harmonic map* from (N, g) into (T_1N, g^S). Here and in the sequel we use the conventions $R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, $R(X, Y, Z, W) = g(R_{XY}Z, W)$.
- a *minimal vector field* if it satisfies (2.2) (this is equivalent to the image of ξ being a minimal submanifold in (T_1N, g^S)).

A stronger condition can be imposed, namely $g((\nabla_X\varphi_\xi)Y, Z) = 0$ for any $X, Y, Z \perp \xi$. In this case we say that ξ is *strongly normal*. The motivation of considering this condition and naming it like this can be found in [10]. It has been proved that *a strongly normal unit geodesic vector field is harmonic and minimal* (see [10], [11]).

These notions can be generalized for sections of any tensor bundle $\pi : Q \to N$ over N. One may endow Q with a generalization g^S of the Sasaki metric (in the sense that for $Q = TN$, g^S is exactly the Sasaki metric). It can be defined using Dombrowski's connection map K of the Levi Civita connection of g as (see [7], for example):

$$g^S(\xi_1, \xi_2) = g(\pi_*\xi_1, \pi_*\xi_2) + g(K\xi_1, K\xi_2).$$

We now give the necessary definitions and formulas for these Riemannian properties of σ according to [7].

A p-dimensional oriented distribution σ on N can be viewed as a map $\sigma : N \to G^{or}_p(N)$. If $\{E_1, \dots, E_n\}$ is a positive orthonormal local frame such that σ is locally generated by $\{E_1, \dots, E_p\}$, then σ can be identified with the p-vector $E_1 \wedge \cdots \wedge E_p$ and, as such, it can be considered as a section of the tensor bundle $\Lambda^p(N)$. Define

$$\chi_\sigma = \sum_{i,j=1}^{n} g(R_{E_iE_j}\sigma, \nabla_{E_j}\sigma)E_i,$$
$$\eta_\sigma = \sum_{i=1}^{n}(\nabla^2\sigma)(E_i, E_i).$$

Moreover, let $S^0_{\sigma(x)}$ be the subspace generated in $\Lambda^p(T_xN)$ by $\sigma(x)$ and let $S^1_{\sigma(x)}$, $S^2_{\sigma(x)}$ be the subspaces S generated respectively by the multivectors

$$\sigma^a_j(x) = E_1 \wedge \cdots \wedge E_{a-1} \wedge E_{p+j} \wedge E_{a+1} \wedge \cdots \wedge E_p,$$
$$\sigma^{ab}_{ij}(x) = E_1 \wedge \cdots \wedge E_{a-1} \wedge E_{p+i} \wedge E_{a+1} \wedge \cdots \wedge E_{b-1} \wedge E_{p+j} \wedge E_{b+1} \wedge \cdots \wedge E_p,$$

where $a, b = 1, \dots p$ and $i, j = 1, \dots n - p$. We then have:

Proposition 2.1. [7, Prop. 3.2]

i) The map $\sigma : (N, g) \to (G^{or}_p(N), g^S)$ is a harmonic map if and only if $\chi_\sigma = 0$ and $\eta_{\sigma(x)}$ belongs to the subspace $S^0_{\sigma(x)} \oplus S^2_{\sigma(x)}$ for all $x \in N$. σ is a harmonic distribution if and only if $\eta_{\sigma(x)}$ belongs to the subspace $S^0_{\sigma(x)} \oplus S^2_{\sigma(x)}$ for all $x \in N$.

ii) The immersion $\sigma : (N, g) \to (G^{or}_p(N), g^S)$ is minimal if and only if

$$\sum_{i=1}^{n}\{\nabla_{E_i}\nabla_{PE_i}\sigma - \nabla_{P\nabla_{E_i}E_i}\sigma\}$$

belongs to the subspace $S^0_{\sigma(x)} \oplus S^2_{\sigma(x)}$ for all $x \in N$ and for $P := L^{-1}_{\sigma^*g^S} = \sqrt{\det A}\, A^{-1}$, where $(\sigma^*g^S)(X, Y) = g(AX, Y) = g(X, Y) + g(\nabla_X\sigma, \nabla_Y\sigma)$.

3 Locally conformal Kähler manifolds

Let (M, J, g) be a connected Hermitian manifold of complex dimension $n \geq 2$. We denote by Ω its fundamental two-form given by $\Omega(X, Y) = g(X, JY)$.

(M, J, g) is called *locally conformal Kähler*, *l.c.K.* for short, if for each point x of M there exists an open neighbourhood U of x and a positive function f_U on U so that the local metric $g_U = e^{-f_U} g_{|U}$ is Kähler. We refer to [4] for a general overview. Equivalently, (M, J, g) is l.c.K. if and only if there exists a *closed* one-form ω such that

$$d\Omega = \omega \wedge \Omega. \tag{3.1}$$

Of course, locally, $\omega_{|U} = df_U$.

The one-form ω is called *the Lee form* and its metrically equivalent vector field $B = \omega^\sharp$ is called *the Lee vector field*. We shall also consider the *anti-Lee vector field* JB. Using them, one can give a third equivalent definition in terms of the Levi Civita connection ∇ of the metric g. Namely, (M, J, g) is l.c.K. if and only if the following equation is satisfied for any $X, Y \in \mathcal{X}(M)$:

$$(\nabla_X J)Y = \frac{1}{2}\{\omega(JY)X - \omega(Y)JX + g(X, Y)JB - \Omega(X, Y)B\}. \tag{3.2}$$

Note that the above equation shows that l.c.K. manifolds belong to the class W_4 of the celebrated Gray-Hervella classification [13].

A strictly smaller class of l.c.K. manifolds is the one formed by those with parallel (with respect to the Levi Civita connection) Lee form, also called *Vaisman manifolds* because I. Vaisman was the first to study them systematically under the name of generalized Hopf manifolds [18]. On such a manifold, the length of the Lee vector field is constant and we shall always assume it is nonzero. Hence, in what follows, we shall normalize and consider that on a Vaisman manifold $\|B\| = \|JB\| = 1$. The next proposition gathers the essential facts we shall need.

Proposition 3.1. *Let (M, J, g) be a Vaisman manifold. Then the Lee and anti-Lee vector fields commute ($[B, JB] = 0$), are Killing ($\mathcal{L}_B g = \mathcal{L}_{JB} g = 0$) and holomorphic ($\mathcal{L}_B J = \mathcal{L}_{JB} J = 0$). Consequently, the distribution generated by B and JB is a holomorphic Riemannian foliation.*

We shall denote by $\mathcal{F}$ the foliation generated by B and JB. We also note that the leaves of the foliation generated by the nullity of the Lee form carry an induced α-Sasakian structure (see [2] as concerns metric contact manifolds) with JB as characteristic vector field.

Examples of (compact), non-Kähler, l.c.K. manifolds are now abundant. Let $\lambda_i \in \mathbb{C}$, $i = 1, \dots n$, $1 <| \lambda_1^{-1} |\leq \cdots \leq| \lambda_n^{-1} |$ and let $\Lambda = (\lambda_1, \dots, \lambda_n)$. Then all the Hopf manifolds $(\mathbb{C}^n \setminus 0)/\Gamma_\Lambda$, with Γ_Λ generated by $z_i \mapsto \lambda_i^{-1} z_i$, are known to admit Vaisman metrics (see [14] for the general case and [5] for the surface case). Note that these manifolds are diffeomorphic to $S^1 \times S^{2n-1}$ and hence cannot be Kähler. In the simplest case, when $\lambda_i = 1/2$, one recovers the standard Hopf manifold with l.c.K. metric (read on $\mathbb{C}^n$) $g_0 = (\sum |z_i|^2)^{-1} \sum dz_i \otimes d\bar{z}_i$ and Lee form $\omega_0 = -(\sum |z_i|^2)^{-1} \sum(\bar{z}_i dz_i + z_i d\bar{z}_i)$; here the Lee field is the one tangent to the S^1 factor.

More generally, the total space of a flat principal circle bundle over a compact Sasakian manifold carries a Vaisman metric whose Lee form is identified with the connection form of the bundle.

The full list of compact complex surfaces which admit l.c.K. metrics with parallel Lee form was given in [1]. It includes the proper elliptic surfaces, the primary and secondary Kodaira surfaces and the elliptic Hopf surfaces.

Belgun also proved that the Inoue surfaces cannot admit Vaisman metrics. However, it was shown by Tricerri in [17] that some Inoue surfaces admit l.c.K. metrics with non-parallel Lee form. We briefly recall this construction. Let $H = \{w = (w_1, w_2) \in \mathbb{C} \mid w_2 > 0\}$ and let $A = (a_{ij}) \in \mathrm{SL}(3, \mathbb{Z})$ having one real eigenvalue $\alpha > 1$ with eigenvector (a_1, a_2, a_3), and a non-real complex eigenvalue β, with eigenvector (b_1, b_2, b_3). The group Γ_A generated by the transformations

$$(w, z) \mapsto (\alpha w, \beta z),$$
$$(w, z) \mapsto (w + a_j, z + b_j)$$

acts on $H \times \mathbb{C}$ and the quotient is a compact complex surface, the Inoue surface S_A. The metric $g = w_2^{-2} dw \otimes d\bar{w} + w_2 dz \otimes d\bar{z}$ on $H \times \mathbb{C}$ is globally conformal Kähler with Lee form $\omega = d \log w_2$. Being compatible with the action of Γ_A, it induces a l.c.K. metric on S_A.

A l.c.K. manifold is naturally endowed with two distinguished vector fields, B and JB, which also generate a two-dimensional distribution. It is thus natural to look for their properties of minimality and harmonicity. Note that if the Lee form is parallel, the properties of B are trivial, so in that case we restrict to looking only at JB.

A particularly significant class of l.c.K. manifolds appears in the context of quaternion Hermitian geometry. Namely, a hyperhermitian manifold $(M^{4n}, g, J_1, J_2, J_3)$ is called *locally conformal hyperkähler*, *l.c.h.K.* for short, if for each point x of M there exists an open neighbourhood U of x and a positive function f_U on U so that the local metric $g_U = e^{-f_U} g_{|U}$ is hyperkähler (see [15] for the fundamental properties, formulas and examples). The Lee form locally defined by $\omega_{|U} = df_U$ here satisfies the equation

$$d\Theta = \omega \wedge \Theta, \tag{3.3}$$

where $\Theta = \sum_{i=1}^{3} \Omega_i \wedge \Omega_i$ and Ω_i is the fundamental 2-form of the Hermitian structure (M, g, J_a), $a = 1, 2, 3$. It can be shown that each of the Hermitian structures (g, J_a) is l.c.K. Moreover, if M is compact, in the conformal class of a l.c.h.K. metric there is always a metric whose associated Lee form is parallel (see [16]) and hence, when working on compact l.c.h.K. manifolds, we shall assume that the Lee form is parallel (and normalized such that the Lee field has length 1).

For a l.c.h.K. manifold with parallel Lee form, (3.2) holds for each J_a. Moreover,

$$[B, J_a B] = 0, \quad [J_a B, J_b B] = J_c B,$$

where (a, b, c) is any cyclic permutation of $(1, 2, 3)$. Hence, in addition to the three two-dimensional foliations $\mathcal{F}_a$, associated to each single Vaisman structure (g, J_a), we have a three-dimensional foliation $\mathcal{D}$ locally generated by $J_a B$ and a four-dimensional foliation $\bar{\mathcal{D}}$ locally generated by B and $J_a B$, $a = 1, 2, 3$. We shall study the harmonicity and minimality properties of the corresponding distributions at the end of the next section.

4 Harmonicity and minimality on Vaisman manifolds

Let (M, J, g) be a connected Vaisman manifold of real dimension $2n$. Recall that B is a parallel unit vector field.

4.1 The anti-Lee vector field

We shall study the Riemannian properties of the anti-Lee vector field JB. For simplicity, denote $\varphi := \varphi_{JB} = -\nabla(JB)$. Then (3.2) together with $(\nabla J)B = \nabla(JB)$ imply

$$\varphi X = \frac{1}{2}\{JX - \omega(JX)B - \omega(X)JB\}. \tag{4.1}$$

Note that $\varphi B = \varphi(JB) = 0$.

Repeated use of (3.2) and (4.1) gives the formula for the covariant derivative of φ:

$$(\nabla_X\varphi)Y = \frac{1}{4}\{\omega(JY)X - \omega(X)\omega(JY)B + [g(X,Y) - \omega(X)\omega(Y)]JB\}. \tag{4.2}$$

Consequently, we obtain

$$g((\nabla_X\varphi)Y, Z) = 0 \quad \text{for any } X, Y, Z \perp JB,$$

proving that JB *is a strongly normal (since Killing) and geodesic vector field.* This, moreover, implies that JB *is a harmonic and minimal vector field* [10], [11].

We now show that JB, viewed as a map from M to T_1M, is a harmonic map. To this end, we have to show (in the notations of §2) that $\sum g(R_{A_{JB}E_iJB}JB, E_i) = 0$ for any local orthonormal frame $\{E_i\}$. But since

$$g(R_{A_{JB}E_iJB}JB, E_i) = R(\varphi E_i, JB, JB, E_i) = g(R_{JB\varphi E_i}JB, E_i),$$

it is enough to show that $\sum g(R_{JBE_i}JB, \varphi E_i) = 0$. Since JB is a Killing vector field, we get $R_{JBX}Y = -\nabla^2_{XY}JB = (\nabla_X\varphi)Y$ and using (4.1) and (4.2), we obtain

$$\begin{aligned} R_{JBE_i}JB &= \frac{1}{4}\{-E_i + \omega(E_i)B + g(E_i, JB)JB\}, \\ \varphi E_i &= \frac{1}{2}\{JE_i - \omega(JE_i)B - \omega(E_i)JB\}. \end{aligned}$$

So, the desired result follows at once. Summing up, we have proved:

Proposition 4.1. *On a Vaisman manifold (M, g), the anti-Lee vector field is a harmonic and minimal vector field. Moreover, it is a harmonic map from (M, g) into the unit tangent bundle (T_1M, g^S).*

Remark 4.1. For a compact M, we may also determine the volume and energy of JB. As φ is skew-symmetric, we have $L = I - \varphi^t\varphi = I - \varphi^2$. But, since $\varphi B = \varphi JB = 0$, (4.1) gives $\varphi^2 X = \frac{1}{2}\varphi(JX)$ and hence we have

$$\varphi^2 X = \frac{1}{4}\{-X + \omega(X)B - \omega(JX)JB\}. \tag{4.3}$$

Then one easily computes:

$$
\begin{aligned}
L &= \frac{5}{4}\,\mathrm{Id} - \frac{1}{4}\omega \otimes B + \frac{1}{4}\omega \circ J \otimes JB, \\
L^{-1} &= \frac{4}{5}\,\mathrm{Id} + \frac{1}{5}\omega \otimes B - \frac{1}{5}\omega \circ J \otimes JB.
\end{aligned} \tag{4.4}
$$

Note that in an adapted local frame of the form

$$\{E_1, \dots, E_{2n-2}, E_{2n-1} = B, E_{2n} = JB\}, \tag{4.5}$$

the matrix of L_{JB} is $\mathrm{diag}(\frac{5}{4}, \dots, \frac{5}{4}, 1, 1)$. Hence, we have

$$
\begin{aligned}
E(JB) &= \frac{1}{2}\int_M \mathrm{Tr} L_{JB}\, \mu_g = \frac{5n-1}{4}\,\mathrm{Vol}(M), \\
\mathrm{Vol}(JB) &= \int_M \sqrt{\det L_{JB}}\, \mu_g = \left(\frac{5}{4}\right)^{n-1} \mathrm{Vol}(M).
\end{aligned}
$$

Moreover, the Hessian forms for the energy and volume were computed in [19] and [9], respectively. (See also [12].) We have:

$$
\begin{aligned}
(\mathrm{Hess}\, E)_\xi(X) &= \int_M (\|\nabla X\|^2 - \|X\|^2 \|\varphi_\xi\|^2)\mu_g, \\
(\mathrm{Hess}\, \mathrm{Vol})_\xi(X) &= \int_M \Big[\|X\|^2 \alpha_\xi(\xi) + (\det L_\xi)^{-\frac{1}{2}}((\mathrm{Tr}(K_\xi \circ \nabla X))^2 - \mathrm{Tr}(K_\xi \circ \nabla X)^2) \\
&\qquad + \mathrm{Tr}(L_\xi^{-1} \circ (\nabla X)^t \circ \varphi_\xi \circ K_\xi \circ \nabla X) \\
&\qquad\qquad + (\det L_\xi)^{\frac{1}{2}}\, \mathrm{Tr}(L_\xi^{-1} \circ (\nabla X)^t \circ \nabla X)\Big]\, \mu_g,
\end{aligned} \tag{4.6}
$$

where we have put $\alpha_\xi(X) = \mathrm{Tr}(Z \mapsto (\nabla_Z K_\xi)X)$ and $X \perp \xi$. In general, a unit harmonic (respectively minimal) ξ is called *stable* if $(\mathrm{Hess}\, E)_\xi(X) \geq 0$ (respectively $(\mathrm{Hess}\, \mathrm{Vol})_\xi(X) \geq 0$) for any $X \perp \xi$. In our case, with $\xi = JB$, it is easily seen that for $X = B$ one obtains $(\mathrm{Hess}\, E)_{JB}(B) < 0$ and $(\mathrm{Hess}\, \mathrm{Vol})_{JB}(B) < 0$, hence JB is not stable neither as a harmonic map nor as a minimal submanifold.

4.2 Harmonicity and minimality of the distribution associated to the foliation $\mathcal{F}$

We shall denote by σ the bivector $B \wedge JB$.

We first compute $\eta := \eta_\sigma = \sum \nabla^2_{E_i E_i}\sigma$ for any local orthonormal frame $\{E_i\}$. We successively have:

$$
\begin{aligned}
\nabla_X \sigma &= -B \wedge \varphi X = \frac{1}{2} B \wedge \{-JX + \omega(X) JB\}, \\
\nabla_Y \nabla_X \sigma &= \frac{1}{2} B \wedge \{-\nabla_Y(JX) - \frac{1}{2}\omega(X) JY + [Y(\omega(X) + \frac{1}{2}\omega(X)\omega(Y)] JB\}, \\
\nabla_{\nabla_X Y} \sigma &= \frac{1}{2} B \wedge \{-J\nabla_X Y + \omega(\nabla_X Y) JB\}.
\end{aligned} \tag{4.7}
$$

We use these formulas and a local frame of the form (4.5) to compute $\sum \nabla^2_{E_iE_i}\sigma = \sum(\nabla_{E_i}\nabla_{E_i}\sigma - \nabla_{\nabla_{E_i}E_i}\sigma)$. We obtain

$$\sum \nabla_{E_i}\nabla_{E_i}\sigma = \frac{1}{2}B \wedge \sum\{E_i(\omega(E_i))JB - \nabla_{E_i}(JE_i)\},$$
$$\sum \nabla_{\nabla_{E_i}E_i}\sigma = \frac{1}{2}B \wedge \sum\{-J\nabla_{E_i}E_i + \omega(\nabla_{E_i}E_i)JB\}.$$

Hence $\eta = -\frac{1}{2}B \wedge \sum(\nabla_{E_i}J)E_i$ which, using (3.2), gives

$$\eta = -\frac{n-1}{2}\sigma.$$

Next we show that $\chi := g(X, \chi_\sigma) = \sum g(R_{XE_i}\sigma, \nabla_{E_i}\sigma) = 0$ for all X.

As $R_{XE_i} = \nabla^2_{XE_i} - \nabla^2_{E_iX}$, we may use (4.7) combined with (3.2) to derive

$$R_{XE_i}\sigma = \frac{1}{4}B \wedge \{\omega(JX)E_i - \omega(JE_i)X\}.$$

Now, recall that $g(X_1 \wedge X_2, X_3 \wedge X_4) = \det(g(X_i, X_j), i = 1, 2, j = 3, 4$. Then, by a straightforward computation, it follows that $\chi = 0$.

Finally, compute $\rho := \sum\{\nabla_{E_i}\nabla_{PE_i}\sigma - \nabla_{P\nabla_{E_i}E_i}\sigma\}$, with P as in Proposition 2.1. A straightforward computation shows that A is given by putting $L := A$ in (4.4).

It will again be convenient to consider the local orthonormal basis of the form (4.5). Note that under this assumption, we have

$$\omega(\nabla_{E_i}E_i) = 0 \quad \text{as } \omega(E_i) = g(B, E_i) = \text{const.},$$
$$\omega(J\nabla_{E_i}E_i) = -g(E_i, \varphi E_i) = 0 \quad \text{as } \varphi \text{ is skew-symmetric.}$$

With this, we have for the second term:

$$A^{-1}\nabla_{E_i}E_i = \frac{4}{5}\nabla_{E_i}E_i,$$
$$\nabla_{A^{-1}\nabla_{E_i}E_i}(JB) = -\frac{2}{5}J\nabla_{E_i}E_i.$$

So, we obtain

$$\sum \nabla_{A^{-1}\nabla_{E_i}E_i}\sigma = \sum B \wedge \nabla_{A^{-1}\nabla_{E_i}E_i}(JB) = -\frac{2}{5}B \wedge \sum J\nabla_{E_i}E_i. \tag{4.8}$$

As for the first term, a similar computation, in which we take into account the formulas $\nabla_{A^{-1}E_i}(JB) = -\varphi(A^{-1}E_i)$ and $\varphi B = \varphi JB = 0$, yields

$$\nabla_{A^{-1}E_i}(JB) = -\frac{2}{5}\{JE_i - \omega(JE_i)B - \omega(E_i)JB\},$$
$$\nabla_{E_i}\nabla_{A^{-1}E_i}(JB) = -\frac{2}{5}\{\nabla_{E_i}(JE_i) + \omega(E_i)\varphi E_i\}.$$

With (3.2) and recalling the type of basis we are using, we find

$$\sum(\nabla_{E_i}J)E_i = (n-1)JB.$$

Hence, we obtain

$$\sum \nabla_{E_i}\nabla_{A^{-1}E_i}\sigma = -\frac{2(n-1)}{5}\sigma - \frac{2}{5}B \wedge \sum J\nabla_{E_i}E_i.$$

Together with (4.8), this yields

$$\rho = -\frac{2(n-1)}{5}\sigma.$$

Note that neither η nor ρ are zero because $n \geq 2$. According to the Proposition 2.1 we thus proved:

Proposition 4.2. *The map $\sigma : (M, g) \to (G_2^{or}(M), g^S)$ is harmonic and its image is a minimal submanifold.*

4.3 Harmonicity and minimality of the distributions $\mathcal{D}$ and $\bar{\mathcal{D}}$ on a l.c.h.K. manifold with parallel Lee form

Let now $(M^{4n}, g, J_1, J_2, J_3)$ be a l.c.h.K. manifold with parallel Lee form and denote with σ and $\bar{\sigma}$ the multivectors corresponding to the distributions $\mathcal{D}$, $\bar{\mathcal{D}}$. Hence, $\sigma = J_1B \wedge J_2B \wedge J_3B$ and $\bar{\sigma} = J_1B \wedge J_2B \wedge J_3B \wedge B$.

Performing, essentially, the same kind of computations as in the previous subsection, and taking into account Proposition 2.1, we may prove:

Proposition 4.3. *On a locally conformal hyperkähler manifold (M, g, J_1, J_2, J_3) with parallel Lee form, the distributions $\mathcal{D}$ and $\bar{\mathcal{D}}$ locally generated respectively by $\{J_aB\}$, $\{J_aB, B\}$, $a = 1, 2, 3$, determine harmonic maps and minimal immersions of (M, g) into $(G_3^{or}(M), g^S)$ and $(G_4^{or}(M), g^S)$, respectively.*

5 The Inoue surface

Let S_A be the Inoue surface endowed with the metric described in §3. Unless on a Vaisman manifold, were B is parallel and thus of no interest for our problem, here it has interesting properties. On the other hand, it turns out that also the anti-Lee vector field has good properties. Namely we prove:

Proposition 5.1. *On an Inoue surface S_A endowed with the Tricerri metric, the following properties hold:*

i) the Lee and anti-Lee vector fields are harmonic and minimal;

ii) the distribution locally generated by the Lee and anti-Lee vector fields is harmonic and determines a minimal immersion of (S_A, g) into $(G_2^{or}(S_A), g^S)$.

The proof is computational. We sketch it for i), the proof of ii) is similar to the one in §4.2. It is convenient to work locally, in the orthonormal frame (see also [4]) as follows:

$$E_1 = w_2\frac{\partial}{\partial w_1}, \quad E_2 = w_2\frac{\partial}{\partial w_2} = B, \quad E_3 = \frac{1}{\sqrt{w_2}}\frac{\partial}{\partial z_1}, \quad E_4 = \frac{1}{\sqrt{w_2}}\frac{\partial}{\partial z_2},$$

with dual frame

$$\theta^1 = \frac{dw_1}{w_2}, \quad \theta^2 = \frac{dw_2}{w_2}, \quad \theta^3 = \sqrt{w_2}dz_1, \quad \theta^4 = \sqrt{w_2}dz_2,$$

and use the Cartan structure equations[1]. The list of connection forms is: $\theta^1_2 = -\theta^2_1 = \theta^1$, $\theta^2_3 = -\theta^3_2 = \frac{1}{2}\theta^3$, $\theta^2_4 = -\theta^4_2 = \frac{1}{2}\theta^4$, the other ones being zero. Consequently we have: $\nabla_{E_1}B = E_1$, $\nabla_{E_2}B = 0$, $\nabla_{E_3}B = \frac{1}{2}E_3$, $\nabla_{E_4}B = \frac{1}{2}E_4$.

As $B = E_2$, we now set $\varphi_2 X = -\nabla_X B$. With the above formulas, one now checks that $\sum g((\nabla_{E_i}\varphi_2)E_i, Z) = 0$ for any $Z = E_j$ with $j = 1, 3, 4$, and hence B is a harmonic vector field.

Note that $\sum R(B, \varphi_2 E_i)E_i \neq 0$ and hence, *B is not a harmonic map from S_A to $T_1 S_A$*.

In order to show that B is minimal, we need to prove that $\sum(\nabla_{E_i}K_2)E_i$ is a multiple of B, with $K_2 = -\sqrt{\det L_2} \circ L_2^{-1} \circ \varphi_2^t$ and $L_2 = \mathrm{Id} + \varphi_2^t \circ \varphi_2$.

The matrix of φ_2 in the specified basis is $\mathrm{diag}(1, 0, -\frac{1}{2}, -\frac{1}{2})$ and hence $\varphi_2 = \varphi_2^t$. Then it is immediate that the matrix of L_2 is $\mathrm{diag}(2, 1, \frac{5}{4}, \frac{5}{4})$ and $\det L_2 = \frac{50}{16}$. Further, the matrix of L_2^{-1} is $\mathrm{diag}(\frac{1}{2}, 1, \frac{4}{5}, \frac{4}{5})$. All in all we find $K_2 X = -\frac{5\sqrt{2}}{8}\theta^1(X)E_1 + \frac{1}{\sqrt{2}}\theta^3(X)E_3 + \frac{1}{\sqrt{2}}\theta^4(X)E_4$. This gives $\sum \nabla_{E_i}(K_2 E_i) = -\frac{9\sqrt{2}}{8}E_2$. As for each i, $\nabla_{E_i}E_i$ is a multiple of E_2 and $K_2 E_2 = 0$, we finally find $\sum(\nabla_{E_i}K_2)E_i = -\frac{9\sqrt{2}}{8}E_2$, as desired.

As S_A is compact, from the previous computations we also obtain (as in Remark 4.1):

$$E(B) = \frac{1}{2}\int_{S_A} \mathrm{Tr} L_2 \, \mu_g = \frac{11}{4}\,\mathrm{Vol}(S_A), \quad \mathrm{Vol}(B) = \int_{S_A} \sqrt{\det L_2}\, \mu_g = \frac{5\sqrt{2}}{4}\,\mathrm{Vol}(S_A).$$

Finally, we discuss the stability for the energy and for the volume of B. We take $X = JB = E_1$ in the first formula of (4.6). As $\nabla_{E_i}E_1 = \theta^1(E_i)E_2$, we obtain $\|\nabla E_1\| = 1$. We thus have $\|\varphi_2\|^2 = \frac{3}{2}$. Hence, by (4.6) we get $(\mathrm{Hess}(E)_B)(JB) = -\frac{1}{2}\,\mathrm{Vol}(S_A) < 0$ and thus B is not stable for the energy functional.

As for the volume functional (the second formula of (4.6)), we first observe that the image of the endomorphism ∇E_i is in the span of E_2 for $i = 1, 3, 4$. As $K_2 E_2 = 0$, the second and third terms in the integrand are zero. For the first term, we have $\alpha_B(B) = \mathrm{Tr}(Z \mapsto (\nabla_Z K_2)B) = \sum g(-K_2 \nabla_{E_i}E_2, E_i) = -\frac{9\sqrt{2}}{8}$. On the other hand, letting $X = JB$, for the last term of the integrand we obtain $\frac{5\sqrt{2}}{8}$ and so, finally we get $(\mathrm{Hess}(\mathrm{Vol})_B)(JB) = -\frac{\sqrt{2}}{2}\,\mathrm{Vol}(S_A) < 0$. Hence B is not stable for the volume functional.

As regards $JB = E_1$, we set $\varphi_1 = -\nabla E_1$ and let L_1, K_1 be the associated operators. We find as above $\sum(\nabla_{E_i}\varphi_1)E_i = E_1$, and hence JB is a harmonic vector field. As K_1 acts as follows: $K_1 E_2 = -\frac{1}{\sqrt{2}}E_1, K_1 E_i = 0$ for $i = 1, 3, 4$, we obtain that $\sum(\nabla_{E_i}K_1)E_i = 0$, proving that JB is a minimal vector field.

Also, $E(JB) = \frac{5}{2}\,\mathrm{Vol}(S_A)$ and $\mathrm{Vol}(JB) = \sqrt{2}\,\mathrm{Vol}(S_A)$.

As for the stability of JB, $\|\varphi_1\| = 1$. Taking $X = E_3$, we find $\|\nabla E_3\|^2 = \frac{1}{4}$, hence $(\mathrm{Hess}(E)_{JB})(E_3) = -\frac{3}{4}\,\mathrm{Vol}(S_A) < 0$ and thus JB is not stable for the energy

[1] For the structure equations, we use the convention $d\theta^i = \theta^i_k \wedge \theta^k$, $d\theta^i_j = -\theta^k_j \wedge \theta^i_k + R^i_j$, with connection forms given by $\nabla_X E_j = -\theta^k_j(X)E_k$ and where $R^i_j = \sum_{k<l} R_{ijkl}\theta^k \wedge \theta^l$, all indices running from 1 to 4.

functional. The stability problem for the volume functional is more difficult and up to now we did not obtain a result.

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