On the structure of parabolic subgroups

Boudjemaa Anchouche

Abstract

Let G be a compact connected semisimple Lie group, $G^{\mathbb{C}}$ its complexification and let P be a parabolic subgroup of G^C. Let $P = L.R_u(P)$ be the Levi decomposition of P, where L is the Levi component of P and $R_u(P)$ is the unipotent part of P. The group L acts by the adjoint representation on the successive quotients of the central series

$$\mathfrak{u}(\mathfrak{p}) = \mathfrak{u}^{(0)}(\mathfrak{p}) \supset \mathfrak{u}^{(1)}(\mathfrak{p}) \supset \cdots \supset \mathfrak{u}^{(i)}(\mathfrak{p}) \supset \cdots \supset \mathfrak{u}^{(r-1)}(\mathfrak{p}) \supset \mathfrak{u}^{(r)}(\mathfrak{p}) = 0,$$

where $\mathfrak{u}(\mathfrak{p})$ is the Lie algebra of $R_u(P)$. We determine for each $0 \le i \le r-1$ the irreducible components $V_i^{(n_1,\,\ldots,\,n_\nu)}$ of the adjoint action of L on $\mathfrak{u}^{(i)}(\mathfrak{p})/\mathfrak{u}^{(i+1)}(\mathfrak{p})$.

1 Introduction.

Let G be a compact connected semisimple Lie group, $G^{\mathbb{C}}$ its complexification and let P be a parabolic subgroup of G^C. Let $P = L.R_u(P)$ be the Levi decomposition of P, where L is the Levi component of P and $R_u(P)$ is the unipotent part of P. The group L acts by the adjoint representation on the successive quotients of the central series

$$\mathfrak{u}(\mathfrak{p}) = \mathfrak{u}^{(0)}(\mathfrak{p}) \supset \mathfrak{u}^{(1)}(\mathfrak{p}) \supset \cdots \supset \mathfrak{u}^{(i)}(\mathfrak{p}) \supset \cdots \supset \mathfrak{u}^{(r-1)}(\mathfrak{p}) \supset \mathfrak{u}^{(r)}(\mathfrak{p}) = 0.$$

where $\mathfrak{u}(\mathfrak{p})$ is the Lie algebra of $R_u(P)$, $\mathfrak{u}^{(0)} = \mathfrak{u}$, $\mathfrak{u}^{(j+1)}(\mathfrak{p}) = [\mathfrak{u}(\mathfrak{p}), \mathfrak{u}^{(j)}(\mathfrak{p})]$, for $j = 0, 1, \ldots$, and $[.,.]$ is the Lie algebra bracket . In this paper we determine for each $0 \le i \le r-1$ the irreducible components $V_i^{(n_1,\,\ldots,\,n_\nu)}$ of the adjoint action of L on

Received by the editors October 2003.
Communicated by M. Van den Bergh.
1991 *Mathematics Subject Classification :* Primary 22E46 ; Secondary 14M15.
Key words and phrases : parabolic subgroups, central series, irreducible representations.

the successive quotients $\mathfrak{u}^{(i)}(\mathfrak{p})/\mathfrak{u}^{(i+1)}(\mathfrak{p})$ (Theorem 1). The case $i=0$ was settled in [1] and was a key ingredient in the proof of the Harder Narasimhan reduction for principal bundles over Kahler manifolds. We were informed that the main result in this note in the case of fields of characteristic not necessarily zero was first proved by A. Borel and J. Tits (unpublished) and by Azad-Barry-Seitz [2]. The aim of this paper is to give a short and different proof of the result of Borel-Tits and Azad-Barry-Seitz cited above.

2 The irreducible components of the action of L on $\mathfrak{u}^{(i)}(\mathfrak{p})/\mathfrak{u}^{(i+1)}(\mathfrak{p})$.

Let G be a compact connected semisimple Lie group, and let $G^{\mathbb{C}}$ be its complexification. Fix a Borel subgroup B of $G^{\mathbb{C}}$ and let $\Pi=\{\alpha_1,\alpha_2,\dots\alpha_k\}$ be the corresponding set of simple roots of $(\mathfrak{g}^{\mathbb{C}},\mathfrak{h})$ where $\mathfrak{g}^{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{g}$ of G and $\mathfrak{h}=\mathfrak{t}^{\mathbb{C}}$ is the complexification of a maximal torus $\mathfrak{t}$ of $\mathfrak{g}$. Denote by $\Phi(\mathfrak{g}^{\mathbb{C}},\mathfrak{h})$ (resp. $\Phi^+(\mathfrak{g}^{\mathbb{C}},\mathfrak{h})$) the set of roots (resp. positive roots) defined by Π. Let $\Pi_1=\{\alpha_{i_1},\dots,\alpha_{i_l}\}\subset\Pi$ be a subset of simple roots and let P be the parabolic subgroup of $G^{\mathbb{C}}$ defined by Π_1 i.e., the Lie algebra $\mathfrak{p}$ of P is given by $\mathfrak{p}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Gamma}\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha=\left\{X\in\mathfrak{g}^{\mathbb{C}}\mid [H,\ X]=\alpha\,(H)\,X,\ \text{for all } H\in\mathfrak{h}\right\}$, $\Gamma=\Phi^+[I]\cup\{\alpha\in\Phi(\mathfrak{g}^{\mathbb{C}},\mathfrak{t}^{\mathbb{C}})\,/\,\alpha=\sum_{\beta\in\Pi_1}n^\alpha_\beta\beta,\ n^\alpha_\beta\ \text{ are integers of the same sign}\}$, and $\Phi^+[I]$ is the set of positive roots supported outside of Π_1.
Let $I=\{1,2,\dots,k\}\backslash\{i_1,i_2,\dots,i_l\}$. For any root $\alpha=\sum_{i=1}^k n_i\alpha_i$, define $n_I(\alpha)=\sum_{i\in I}n_i$. An equivalent description of the Lie algebra $\mathfrak{p}$ of P is

$$\mathfrak{p}=\mathfrak{h}\oplus\bigoplus_{n_I(\alpha)\geq 0}\mathfrak{g}_\alpha.$$

The Lie algebra $\mathfrak{l}$ of the Levi component L of P is given by $\mathfrak{l}=\mathfrak{h}\oplus\bigoplus_{n_I(\alpha)=0}\mathfrak{g}_\alpha$ and the nilpotent radical $\mathfrak{u}(\mathfrak{p})$ of $\mathfrak{p}$ is given by $\mathfrak{u}(\mathfrak{p})=\bigoplus_{n_I(\alpha)>0}\mathfrak{g}_\alpha$ hence the roots of $\mathfrak{u}(\mathfrak{p})$ are the positive roots whose support lies outside Π_1 i.e., the roots γ of $\mathfrak{u}(\mathfrak{p})$ are the positive roots which satisfy $n_I(\gamma)>0$. Recall that $\Phi(\mathfrak{g}^{\mathbb{C}},\mathfrak{h})\subset\sqrt{-1}\mathfrak{t}^*$. Let $\xi_i\in\mathfrak{h}$ be defined by $\alpha_j(\xi_i)=\sqrt{-1}\delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker symbol, and let $\xi=\sum_{j=1}^l\xi_{i_j}$. By construction, the eigenvalues of $ad\,\xi$ lie in $\sqrt{-1}\,\mathbb{Z}$, more precisely $ad\,(\xi)$ has eigenvalue $\sqrt{-1}n_I(\alpha)$ on $\mathfrak{g}_\alpha$, in particular ξ centralizes $\mathfrak{l}$. The element ξ is called the canonical element of the parabolic subgroup P of $G^{\mathbb{C}}$. Consider the flag manifold $M=G^{\mathbb{C}}/P$ where $G^{\mathbb{C}}$ and P are as above. The height of $G^{\mathbb{C}}/P$ is defined to be the smallest positive integer r such that $\mathfrak{u}^{(r)}(\mathfrak{p})=0$ i.e., if $\mathfrak{u}(\mathfrak{p})$ is r-step nilpotent then we say that M is of *height* r and denote this by $ht(M)=r$.
The group L operates by the adjoint representation on the nilpotent radical $\mathfrak{u}(\mathfrak{p})$ of the Lie algebra $\mathfrak{p}$, hence on the successive quotients of the central series

$$\mathfrak{u}(\mathfrak{p})\,=\,\mathfrak{u}^{(0)}(\mathfrak{p})\,\supset\,\mathfrak{u}^{(1)}(\mathfrak{p})\,\supset\,\cdots\,\supset\,\mathfrak{u}^{(i)}(\mathfrak{p})\,\supset\,\cdots\,\supset\,\mathfrak{u}^{(r-1)}(\mathfrak{p})\supset\mathfrak{u}^{(r)}(\mathfrak{p})\,=\,0\,.$$

Lemma 1. *For all nonnegative integers i such that $i+1\leq ht(M)$ we have*

$$\mathfrak{u}^{(i)}(\mathfrak{p})/\mathfrak{u}^{(i+1)}(\mathfrak{p})=\bigoplus_{\gamma\in\Gamma_i}\mathfrak{g}_\gamma$$

where $\Gamma_i=\{\gamma\in\Phi^+(\Pi)\mid n_I(\gamma)=i+1\}$, and $\mathfrak{g}_\gamma$ is the one dimensional vector space defined as above.

Proof. From the identity $\mathfrak{u}^{(i)}(\mathfrak{p}) = \sum_{j\geq i+1}\mathfrak{g}^j$ where $\mathfrak{g}^j = \{X \in \mathfrak{g}^{\mathbb{C}} \mid ad(\xi)X = \sqrt{-1}jX\}$ - it is not difficult to see that $\mathfrak{u}^{(i)}(\mathfrak{p}) \subset \sum_{j\geq i+1}\mathfrak{g}^j$, (for the converse see [3], Theorem 4.4) - we deduce that $\mathfrak{u}^{(i)}(\mathfrak{p})/\mathfrak{u}^{(i+1)}(\mathfrak{p}) = \mathfrak{g}^{i+1}$. Therefore if $X_\gamma \in \mathfrak{u}^{(i)}(\mathfrak{p})/\mathfrak{u}^{(i+1)}(\mathfrak{p})$, then $ad(\xi)X_\gamma = \sqrt{-1}(i+1)X_\gamma$ and from the definition of $X_\gamma \in \mathfrak{g}_\gamma$, we deduce that $n_I(\gamma) = i+1$. ■

Let $\rho_i : L \longrightarrow GL(\mathfrak{u}^{(i)}(\mathfrak{p})/\mathfrak{u}^{(i+1)}(\mathfrak{p}))$ be the induced representation. Write $I = \{j_1, ..., j_\nu\}$ where $I = \{1, 2, \ldots, k\} \setminus \{i_1, i_2, \ldots, i_l\}$ as above, with $j_1 < ... < j_\nu$, and for nonnegative integers $n_1, ..., n_\nu$ with $n_1 + ... + n_\nu = i+1$ consider the subspace $V_i^{(n_1,\, ...,\, n_\nu)} = \bigoplus_{\gamma\in\Gamma_i^{(n_1,\, ...,\, n_\nu)}} \mathfrak{g}_\gamma$ where $\Gamma_i^{(n_1,\, ...,\, n_\nu)} = \{\gamma \in \Phi^+(\Pi) \mid \gamma \equiv \sum_{k=1}^{\nu} n_k\alpha_{j_k}$ $[\Pi_1]$, $n_I(\gamma) = i+1\}$. If $\Gamma_i^{(n_1,\, ...,\, n_\nu)} = \emptyset$ then we put $V_i^{(n_1,\, ...,\, n_\nu)} = 0$. Therefore

$$\mathfrak{u}^{(i)}(\mathfrak{p})/\mathfrak{u}^{(i+1)}(\mathfrak{p}) = \bigoplus_{\substack{(n_1,\, ...,\, n_\nu)\in\mathbb{Z}^\nu_{\geq 0}\\ n_1+...+n_\nu=i+1}} V_i^{(n_1,\, ...,\, n_\nu)}$$

Let $(.\ ,\ .)$ be a scalar product on $\mathfrak{h}^*$ invariant under the action of the Weyl group $W(\Pi)$ of G generated by the reflections s_α where $\alpha \in \Pi$ and $||\ .\ ||$ be the corresponding norm. Denote by $W(\Pi_1)$ the Weyl group generated by the simple reflections s_α where $\alpha \in \Pi_1$.

For each $w \in W(\Pi_1)$, there exists elements $\alpha_1, ..., \alpha_k \in \Pi_1$ such that $w = s_{\alpha_1}.s_{\alpha_2}....s_{\alpha_k}$. We define the length of w by $l(w) = \min\{k \mid w = s_{\alpha_1}.s_{\alpha_2}....s_{\alpha_k}\}$. If $w \in W(\Pi_1)$ is written as $w = s_{\alpha_1}.s_{\alpha_2}....s_{\alpha_k}$ with $k = l(w)$ then we say that $s_{\alpha_1}.s_{\alpha_2}....s_{\alpha_k}$ is a reduced expression for w. It can be proved that $l(w)$ is the number of roots $\alpha \in \Pi_1$ such that $w\alpha < 0$.
The following theorem is the main result of this paper.

Theorem 1. *Let $V_i^{(n_1,\, ...,\, n_\nu)}$ be as above and suppose that $V_i^{(n_1,\, ...,\, n_\nu)} \neq 0$. Then $V_i^{(n_1,\, ...,\, n_\nu)}$ is an irreducible L-submodule of the L-module $\mathfrak{u}^{(i)}(\mathfrak{p})/\mathfrak{u}^{(i+1)}(\mathfrak{p})$.*

Proof. Let $V_{i,\,\mathfrak{b}(\mathfrak{l})}^{(n_1,\, ...,\, n_\nu)} = \{v \in V_i^{(n_1,\, ...,\, n_\nu)} \mid ad(X)v = 0$ for all $X \in \mathfrak{b}(\mathfrak{l})\}$, where $\mathfrak{b}(\mathfrak{l}) = \bigoplus_{\alpha\in\Phi^+(\Pi_1)}\mathfrak{g}_\alpha$. To prove that $V_i^{(n_1,\, ...,\, n_\nu)}$ is an irreducible L-submodule it is enough to prove that $\dim_{\mathbb{C}} V_{i,\,\mathfrak{b}(\mathfrak{l})}^{(n_1,\, ...,\, n_\nu)} = 1$. Obviously $V_{i,\,\mathfrak{b}(\mathfrak{l})}^{(n_1,\, ...,\, n_\nu)} \neq 0$. Suppose that $\dim_{\mathbb{C}} V_{i,\,\mathfrak{b}(\mathfrak{l})}^{(n_1,\, ...,\, n_\nu)} \geq 2$ and let γ_1 and γ_2 be two elements of $\Gamma_i^{(n_1,\, ...,\, n_\nu)}$ such that $\mathfrak{g}_{\gamma_1} \otimes \mathfrak{g}_{\gamma_2} \subseteq V_{i,\,\mathfrak{b}(\mathfrak{l})}^{(n_1,\, ...,\, n_\nu)}$ and $\gamma_1 \neq \gamma_2$. Then $v_1 \in \mathfrak{g}_{\gamma_1}$ and $v_2 \in \mathfrak{g}_{\gamma_2}$ where $\gamma_1 \neq \gamma_2$ and $\gamma_i \equiv \sum_{k=1}^{\nu} n_k\alpha_{j_k}$ $[\Pi_1]$, $i = 1, 2$. If $\gamma_1$and γ_2 have the same length then with a modification of ([[4], Proposition 11, page 151]) or by [[2], Lemma 1, 553], we deduce that there exists an element $w \in W(\Pi_1)$ such that $\gamma_2 = w(\gamma_1)$. From the definition of $\gamma_1$and γ_2 we have $(\gamma_1, \alpha) \geqslant 0$ and $(\gamma_2, \alpha) \geqslant 0$ for all $\alpha \in \Pi_1$. Let $\Phi^w(\Pi_1) = \{\alpha \in \Phi^+(\Pi_1) \mid w(\alpha) \in -\Phi^+(\Pi_1)\}$. If $\Phi^w(\Pi_1) = \varnothing$ then $w = id$ and $\gamma_1 = \gamma_2$. So suppose that $\Phi^w(\Pi_1) \neq \varnothing$ and let α be an element of $\Phi^w(\Pi_1)$, then

$$0 \geqslant (\gamma_2, w(\alpha)) = (\gamma_1, \alpha) \geqslant 0,$$

hence $(\gamma_1, \alpha) = 0$ for all α in $\Phi^w(\Pi_1)$, therefore $\sigma_\alpha(\gamma_1) = \gamma_1$ for all α in $\Phi^w(\Pi_1)$. If $w = \sigma_{\alpha_{i_1}}\sigma_{\alpha_{i_2}}...\sigma_{\alpha_{i_j}}$ is a reduced expression of w then α_{i_k} is a root of $\Phi^w(\Pi_1)$, for, if $\alpha_{i_k} \notin \Phi^w(\Pi_1)$ then it is not difficult to see that this leads to a contradiction to the minimality of the length of w. Hence $\gamma_2 = w(\gamma_1) = w_1(\gamma_1)$ where $w_1 = \sigma_{\alpha_{i_1}}\sigma_{\alpha_{i_2}}...\sigma_{\alpha_{i_{k-1}}}$.

By induction we deduce that $\gamma_1 = \gamma_2$ and therefore $\dim_{\mathbb{C}} V_{i,\, \mathfrak{b}(\mathfrak{l})}^{(n_1,\, \ldots,\, n_\nu)} = 1$.
Suppose now that $\gamma_1$and γ_2 are of different lengths. Let w_1 and w_2 be two elements of $W(\Pi_1)$ such that $w_1(\gamma_1)$ and $w_2(\gamma_2)$ are of minimal height in their $W(\Pi_1)-$orbit. So $(w_1(\gamma_1),\, \alpha) \leq 0$ and $(w_2(\gamma_2),\, \alpha) \leq 0$ for all $\alpha \in \Pi_1$. Therefore $(w_1(\gamma_1),\, w_2(\gamma_2)) > 0$, so $\gamma_1 - \, w_1^{-1} w_2(\gamma_2) = \beta \in \Phi(\Pi_1)$. If β is negative then $\gamma_1 + (-\beta) = w_1^{-1} w_2(\gamma_2)$ is a root, a fact which is in contradiction with the definition of γ_1. Without loss of generality we can assume that $w_1 = w_2$. If β is positive then $\gamma_1 = w_1^{-1} w_2(\gamma_2) + \beta = \gamma_2 + \beta$ where $\beta = \sum\limits_{\alpha \in \Pi_1} n_\alpha^\beta \alpha$ with $n_\alpha^\beta \geqslant 0$. If at least one of the n_α^β is strictly positive then we get a contradiction to the definition of γ_2. Therefore $\gamma_1 = \gamma_2$. Hence the Theorem. ■

References

[1] Anchouche, B.; Azad, H.; Biswas, I.: "Harder-Narasimhan Reduction For Principal Bundles Over a Compact Kähler Manifold". Math Ann. Volume 323, Issue 4 (2002) pp 693-712.

[2] Azad, H.; Barry, M.; Seitz, G.: On the structure of parabolic subgroups. Comm. Algebra 18 (1990), no. 2, 551–562

[3] Burstall, Francis E.; Rawnsley, John H: Twistor theory for Riemannian symmetric spaces. Lecture Notes in Mathematics, 1424. Springer-Verlag, Berlin, 1990.

[4] Bourbaki, N: Elements de mathematiques, Groupes et Algebres de Lie, Chapitres 4, 5 et 6. Masson (1981).

Department of Mathematics
Sultan Qaboos University
PO Box 36
Al-Khod PC123
Sultanate of Oman
email: anchouch@squ.edu.om