Spectral asymptotics for the Laplacian under an eigenvalue dependent boundary condition

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Abstract

This paper deals with a spectral problem for the Laplacian stemming from a parabolic problem in a bounded domain under a dynamical boundary condition. As a distinctive feature the eigenvalue parameter appears here also in the boundary condition:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \partial_\nu u = \lambda \sigma u & \text{on } \partial\Omega. \end{cases}$$

By variational techniques the resulting eigenvalue sequence can be compared with the spectra under Dirichlet or Neumann boundary conditions and with the spectrum of the Steklov problem in order to get upper bounds for the spectral growth. For continuous positive σ, the growth order is determined and upper and lower bounds for the leading asymptotic coefficient are obtained. Moreover, the exact asymptotic behavior of the eigenvalue sequence is determined in the one–dimensional case.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$, and let $\nu : \partial\Omega \to \mathbb{R}^n$ denote its outer normal unit vector field and ∂_ν the outer normal derivative. For

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the Laplacian Δ in Ω we consider the initial boundary value problem

$$\begin{cases} \partial_t v = \Delta v & \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t v + \partial_\nu v = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ v(\cdot, 0) = v_0 \in L^2(\Omega), \end{cases} \tag{1}$$

where σ is a bounded nonnegative function defined on $\partial\Omega$ and where a *dynamical* condition on the time lateral boundary relating the outer normal derivative to the time derivative is imposed. The classical separation ansatz $v(x,t) = u(x)\delta(t)$ in the differential equation and in the boundary condition leads to the elliptic eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \partial_\nu u = \lambda \sigma u & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where, as a distinctive feature, the eigenvalue parameter λ appears also in the boundary condition. It is evident that this complicates the application of many functional analytic techniques used under Neumann, Dirichlet or linear eigenvalue independent boundary conditions. The aim of this paper is to investigate the effect of the above boundary condition on the eigenvalues λ of (2), especially on the asymptotic behavior. In the one–dimensional case the latter resembles the one in the classical cases, while in higher dimensions the growth exponent decreases with respect to the classical cases. But, in any case, roughly speaking, a spectral shift to the left in comparison with the Neumann condition can be observed. Nevertheless, the control of the spectral deviation in detail seems to be quite complicated, see also [11].

As for the parabolic problem (1), using the present results, its solution can be obtained as an eigenfunction expansion

$$v(\cdot, t) = \sum_{k\in\mathbb{N}} (v_0, \varphi_k)_\sigma \varphi_k e^{-\lambda_k t}$$

using the Fourier coefficients in $L^2(\Omega) \times L^2_\sigma(\partial\Omega)$ bearing the weight σ on the boundary. As for more general parabolic problems including dynamical boundary conditions and applications of these, we refer to [4], [5], [6], [7] and the references therein.

Let us recall a classical result concerning the asymptotic behavior of eigenvalues of the Laplace operator under homogeneous Dirichlet boundary conditions. Let $(\omega_k)_{k\in\mathbb{N}}$ be the nondecreasing sequence of the eigenvalues of $-\Delta$ in $H^1_0(\Omega)$. Then $(\omega_k)_{k\in\mathbb{N}}$ behaves like

$$\omega_k = C(\Omega)\, k^{2/n} + o(k^{2/n}), \qquad C(\Omega) = \frac{4\pi^2}{\left(v_n \operatorname{mes}_n(\Omega)\right)^{2/n}}, \tag{3}$$

where $C(\Omega)$ is the Weyl constant of Ω and where v_n is the volume of the unit ball of $\mathbb{R}^n$. This behavior is due to the existence of a Hilbert basis of $L^2(\Omega)$ consisting in eigenfunctions of the Dirichlet problem and to the extremal variational property of the eigenvalues $(\omega_k)_{k\in\mathbb{N}}$. One might expect an analogue asymptotic behavior to hold for the eigenvalues $\left(\lambda_k(\sigma)\right)_{k\in\mathbb{N}}$ of (2) with a constant independent of the function σ. But for positive σ this is only true in dimension 1.

The present paper is organized as follows: Section 2 is devoted to the one–dimensional case for which the coefficient of the leading asymptotic term is determined. Moreover, it turns out that in the lower asymptotic terms, the parameters σ

interfere in such a way that the spectral shift due to the dynamical boundary condition is well displayed. In Section 3 we discuss the variational setting of Problem (2) for arbitrary dimensions. It turns out that the associated Green operator is compact and self–adjoint in $L^2(\Omega) \times L^2_\sigma(\partial\Omega)$ and, thereby, the eigenvalues of Problem (2) form a countably infinite set in $\mathbb{R}_+$ without finite accumulation point. In Section 4 we combine these results with extremal principles in order to deduce

$$\limsup_{k\to\infty} \frac{\lambda_k(\sigma)}{k^{2/n}} \leqslant C(\Omega),$$

while for continuous positive functions σ, a comparison with the eigenvalue sequence of the Steklov problem yields

$$\frac{1}{2^{1/(n-1)}} \frac{C_{\mathrm{Stek}}(\Omega)}{\max\limits_{\partial\Omega} \sigma} \leqslant \liminf_{k\to\infty} \frac{\lambda_k}{k^{1/(n-1)}} \leqslant \limsup_{k\to\infty} \frac{\lambda_k}{k^{1/(n-1)}} \leqslant \frac{C_{\mathrm{Stek}}(\Omega)}{\min\limits_{\partial\Omega} \sigma}$$

for $n \geqslant 3$ and

$$\frac{1}{2} \frac{C(\Omega)\, C_{\mathrm{Stek}}(\Omega)}{C(\Omega) \max\limits_{\partial\Omega} \sigma + C_{\mathrm{Stek}}(\Omega)} \leqslant \liminf_{k\to\infty} \frac{\lambda_k}{k} \leqslant \limsup_{k\to\infty} \frac{\lambda_k}{k} \leqslant \min\left\{ C(\Omega), \frac{C_{\mathrm{Stek}}(\Omega)}{\min\limits_{\partial\Omega} \sigma} \right\}$$

for $n = 2$. In the final Section 5 the leading term in the asymptotic for Problem (2) on the unit disk for constant $\sigma > 0$ is estimated and shown to depend on σ.

It should be noted that a domain decomposition in connection with the Courant-Fischer min–max principle does not yield the asymptotic behavior of the eigenvalue sequence as it does under Dirichlet and Neumann boundary conditions. This is essentially due to the boundary integral in the Rayleigh quotient of Problem 2 (see Section 4). Thus, the Rayleigh quotient is different for different dynamical boundary conditions, while in the Dirichlet and Neumann cases it is the same functional and does not depend on the boundary. Moreover, a key step of the domain decomposition technique is based on an explicit calculation of the eigenvalues in a cube of $\mathbb{R}^n$. But such an explicit calculation is neither available nor applicable in the dynamical case. For the same reason, the heat kernel method, see e.g. [17], for the Dirichlet case, based on the asymptotic expansion

$$\sum_{k=0}^{\infty} e^{-t\,\omega_k} = \frac{1}{t^{n/2}} \Big((4\pi)^{-n/2} \operatorname{mes}_n(\Omega) + a_1 t^{1/2} + a_2 t + a_3 t^{3/2} + \cdots + a_n t^{n/2} \Big)$$

as $t \to 0+$ and an explicit formula for the eigenvalues $(\omega_k)_{k\in\mathbb{N}}$, seems not to apply to the dynamical case, though the maximum principle for the corresponding parabolic problem holds [5]. Moreover, it does not seem that the theory of S–hermitian eigenvalue problems developed by Schäfke and Schneider [16] applies to our context, though it applies to a large class of systems of ordinary differential equations under eigenvalue dependent boundary conditions, see also [4].

2 The one–dimensional case

For $\Omega = (0,1)$ and nonnegative parameters σ_0, σ_1, let us consider the following problem [4], [11]:

$$\begin{cases} u'' = -\lambda u & \text{in } (0,1) \\ u'(0) = -\lambda\sigma_0 u(0) \\ u'(1) = \lambda\sigma_1 u(1) \end{cases} \tag{4}$$

We readily deduce that $\lambda \geqslant 0$, so that any solution of the differential equation of (4) is of the form

$$u(x) = a\cos\Big(x\sqrt{\lambda}\Big) + b\sin\Big(x\sqrt{\lambda}\Big).$$

Taking into account the boundary conditions, we see that the problem (4) has a non trivial solution if and only if

$$\tan\sqrt{\lambda} = \frac{(\sigma_0+\sigma_1)\sqrt{\lambda}}{\sigma_0\sigma_1\lambda - 1}. \tag{5}$$

The increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ formed by the roots of this characteristic equation tends to infinity and behaves asymptotically like

$$\lambda_k = \pi^2 k^2 + o(k^2). \tag{6}$$

Therefore, the eigenvalues of Problem (4) display the same asymptotic behavior as in the case of the Dirichlet or Neumann boundary conditions. The formula (6) can be improved since the following asymptotic expansion holds ($\sigma_0\sigma_1 > 0$):

$$\lambda_k = \pi^2 k^2 - 2\pi k + \pi^2 + \frac{2(\sigma_0+\sigma_1)}{\sigma_0\sigma_1} + o(1).$$

We omit the details and refer to [11].

In the case $\sigma_0 = \sigma_1 =: \sigma$, let us write $\lambda_k = \lambda_k(\sigma)$ as a function of the parameter σ. By a careful analysis of the characteristic equation (5), we obtain the following result about the variation of the $\lambda_k(\sigma)$ with respect to σ:

$$0 \leqslant \sigma < \tilde{\sigma} \implies \forall k \in \mathbb{N}^* : \lambda_k(\tilde{\sigma}) < \lambda_k(\sigma).$$

We omit the details and refer to [11]. In a certain sense, the boundary condition in (4) interpolates between the Dirichlet boundary condition and the Neumann one. For $\sigma = 0$, Problem (4) is the Neumann problem while by letting σ tend to infinity, the limit problem in the class of bounded functions becomes $u'' = -\lambda u$ in $(0,1)$ under Dirichlet boundary condition.

Closing this section, we note that the same results hold mutatis mutandis on connected networks built up by a finite number of one–dimensional $\mathcal{C}^2$–parametrized edges $\{e_1,\dots,e_N\}$ in $\mathbb{R}^m$ of arc lengths $\{l_1,\dots,l_N\}$ respectively, for the precise definitions see [2],[3],[4]. In Chapter 20 of [4] it has been shown for the general Sturm–Liouville problems on such a network that the eigenvalues grow quadratically and that the leading asymptotic coefficient is completely determined by the Weyl constants on the singles edges. Especially, for the Laplacian under dynamical

Kirchhoff conditions at all nodes and continuity conditions at the ramification nodes the eigenvalue problem in question reads

$$\begin{cases} u \in \mathcal{C}^2(G) \\ \partial_j^2 u_j = -\alpha u_j & \text{on each edge } \ e_j,\ 1 \le j \le N, \\ \sum_{j=1}^{N} d_{ij}\partial_j u_j(v_i) = \alpha\sigma_i u(v_i) & \text{at all vertices } v_i \in V(G). \end{cases} \tag{7}$$

Then using the results in [2],[3], we obtain, see [4],

$$\lim_{k\to\infty} \frac{\alpha_k}{k^2} = \pi^2 \left(\sum_{j=1}^{N} l_j \right)^{-2},$$

that reduces to the formula

$$\lim_{k\to\infty} \frac{\alpha_k}{k^2} = \frac{\pi^2}{N^2}$$

in the case of N edges all of arc length 1. Thus, for Problem (7), we conclude that the eigenvalues grow quadratically as on a single interval and that the dynamical coefficients σ_i in the Kirchhoff conditions do not affect the leading eigenvalue asymptotic of (7).

3 The spectral problem for the Laplacian

For higher dimensions the Problem (2)

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \partial_\nu u = \lambda\sigma u & \text{on } \partial\Omega, \end{cases}$$

cannot be treated as easily with the aid of fundamental solutions as in the one–dimensional case. As under classical boundary conditions, the variational formulation of the problem will help to reveal the true nature of the problem. For a test function $v \in H^1(\Omega)$, Green's formula in $H^1(\Omega)$ yields

$$\int_\Omega (\nabla u, \overline{\nabla v})_{\mathbb{C}^n}\, dx = \lambda \left(\int_\Omega u\overline{v}\, dx + \int_{\partial\Omega} \sigma\gamma_0 u \overline{\gamma_0 v}\, ds \right), \tag{8}$$

where γ_0 denotes the trace operator $H^1(\Omega) \to H^{1/2}(\partial\Omega)$. This identity defines a weak solution u belonging to $H^1(\Omega)$. A complex number λ is said to be an *eigenvalue* of the Problem (2) in $H^1(\Omega)$ if there exists $0 \neq u \in H^1(\Omega)$ such that (8) holds for all $v \in H^1(\Omega)$. Of course, such a function u is said to be an *eigenfunction* of (2) with respect to λ. The set of all eigenvalues of (2) in this sense will be denoted by Λ. Note that (8) is the appropriate weak formulation of the equation

$$P \begin{pmatrix} u \\ \gamma_0 u \end{pmatrix} = \lambda \begin{pmatrix} u \\ \gamma_0 u \end{pmatrix} \quad \text{with} \quad P = \begin{pmatrix} -\Delta & 0 \\ 0 & \sigma^{-1}\partial_\nu \end{pmatrix},$$

where the operator P is not densely defined in $L^2(\Omega) \times L^2_\sigma(\partial\Omega)$, and, therefore, requires the weak approach presented below. We shall show that Λ is a countable set that can be arranged in an increasing sequence which tends to infinity. Under higher regularity assumptions on Ω, we then shall be able to treat the boundary value problem (2) in the strong sense, showing that the eigenfunctions of (2) belong to the space $C^2(\overline{\Omega})$.

On the one side, the variational formulation (8) takes well into account the presence of the eigenvalues in the boundary condition of (2) by a simple boundary potential integral that will be added in the denominator of the corresponding Rayleigh quotient. On the other, this boundary integral requires that the variation takes place in the following product space H. Let E_σ be the Hilbert space $L^2(\Omega) \times L^2_\sigma(\partial\Omega)$ endowed with the scalar product

$$(F,G)_0 = \Big((f,\alpha),(g,\beta)\Big)_0 = \int_\Omega f\overline{g}\,dx + \int_{\partial\Omega} \sigma\alpha\overline{\beta}\,ds$$

with $F = (f,\alpha), G = (g,\beta) \in L^2(\Omega) \times L^2_\sigma(\partial\Omega)$. Moreover, we put $\|F\|_0 = \sqrt{(F,F)_0}$. Let H denote the following subspace of E_σ

$$H = \Big\{U = (u,\gamma_0 u) \mid u \in H^1(\Omega)\Big\},$$

which is a Hilbert space with the scalar product

$$(U,V)_1 = (u,v)_{H^1(\Omega)} + \int_{\partial\Omega} \sigma\gamma_0 u\overline{\gamma_0 v}\,ds.$$

For $U \in H$, we put $\|U\|_1 = \sqrt{(U,U)_1}$. As a first result concerning the eigenvalue distribution we show the following.

Theorem 3.1. *The set of eigenvalues Λ of Problem* (2) *forms a countably infinite set $\{\lambda_k \mid k \in \mathbb{N}\} \subset \mathbb{R}_+$ without finite accumulation point, thus its elements can be arranged in an increasing sequence such that*

$$0 = \lambda_0 < \lambda_1 \leqslant \ldots \leqslant \lambda_k \leqslant \lambda_{k+1} \leqslant \ldots$$

and

$$\lim_{k\to\infty} \lambda_k = \infty.$$

Proof. We can easily see that $0 \in \Lambda$. In fact 0 is a simple eigenvalue, since an eigenfunction u belonging to it is constant owing to $u = v$ in (8). Let $F = (f,\alpha) \in E_\sigma$. The weak formulation of the shifted problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \sigma^{-1}\partial_\nu u + u = \alpha & \text{on } \partial\Omega \end{cases}$$

is to find $u \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$,

$$\int_\Omega (\nabla u, \overline{\nabla v})_{\mathbb{C}^n}\,dx + \int_\Omega u\overline{v}\,dx + \int_{\partial\Omega} \sigma\gamma_0 u\overline{\gamma_0 v}\,ds = \int_\Omega f\overline{v}\,dx + \int_{\partial\Omega} \sigma\alpha\overline{\gamma_0 v}\,ds.$$

In other words, we have to find $U \in H$ such that

$$(U,V)_1 = (F,V)_0 \quad \text{for all } V \in H, \tag{9}$$

where $F = (f, \alpha) \in E_\sigma$. It is obvious that the linear form $V \mapsto (F, V)_0$ is continuous on H. Therefore, an application of Riesz's representation Theorem guarantees the existence of a unique $U \in H$ such that (9) holds. Introduce the operator T_1 from E_σ to H defined by $T_1 F = U$. Since $(T_1 F, V)_1 = (F, V)_0$ holds for all $V \in H$, we infer that $\|T_1 F\|_1 \leqslant \|F\|_0$, so that T_1 is a continuous operator. Let T_2 be the operator on H defined by $T_2 = T_1 \circ i$, where $i : H \hookrightarrow E_\sigma$ is the canonical injection between the indicated spaces. Owing to the compactness of the three injections $H^1(\Omega) \hookrightarrow L^2(\Omega)$, $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ and i [1], T_2 is a compact operator, and the non-zero elements of its spectrum are its eigenvalues. By

$$\forall U, V \in H : \quad (T_2 U, V)_1 = (U, V)_0 = \overline{(V, U)_0} = \overline{(T_2 V, U)_1} = (U, T_2 V)_1,$$

we conclude that T_2 is a symmetric operator. Therefore, each μ_k is a real number. Moreover, the spectrum of T_2 is countable with 0 as only possible accumulation point, and its elements can be arranged in a decreasing sequence $(\mu_k)_{k \in \mathbb{N}}$ in such a way that $\lim_{k \to \infty} \mu_k = 0$, where each μ_k is counted according to its multiplicity. Now, let $U \neq 0$ be an eigenfunction associated to the eigenvalue μ. Then

$$(T_2 U, U)_1 = (U, U)_0 = \mu (U, U)_1$$

and

$$0 < \mu = \frac{\|U\|_0^2}{\|U\|_1^2} \leqslant 1.$$

Furthermore, $\mu = 1$ if and only if $U = (u, \gamma_0 u)$, where u is a constant function. Obviously

$$\mu \in \{\mu_k \mid k \in \mathbb{N}\} \iff \frac{1}{\mu} - 1 \in \Lambda.$$

Accordingly, the set Λ can be arranged as an increasing sequence $(\lambda_k)_{k \in \mathbb{N}}$ with

$$\forall k \in \mathbb{N} : \quad \lambda_k = \frac{1}{\mu_k} - 1.$$

This achieves the proof. ∎

Remark 3.1 The spectral theorem applied to the compact operator T_2 in the preceding proof yields some more results. Firstly, every eigenvalue μ of T_2 is of finite multiplicity $\dim \ker(T_2 - \mu I) < \infty$. Note that $\ker T_2 = \{0\}$. Secondly, there exists a Hilbert basis of H consisting in eigenfunctions of T_2. Let $(\psi_k)_{k \in \mathbb{N}}$ be such a basis and set $(\varphi_k, \gamma_0 \varphi_k) = \frac{1}{\sqrt{\mu_k}} \psi_k$ for all $k \in \mathbb{N}$. Then φ_k is an eigenfunction of (2) associated to the eigenvalue $\lambda_k = \frac{1}{\mu_k} - 1 \in \Lambda$, and the Fourier expansion

$$f = \sum_{k=0}^{\infty} (f, \varphi_k)_\sigma \varphi_k$$

holds for each $f \in H^1(\Omega)$ with respect to the scalar product in $H^1(\Omega)$ defined by

$$(f, g)_\sigma = \int_\Omega f(x) \overline{g(x)} \, dx + \int_{\partial\Omega} \sigma(s) \gamma_0 f(s) \overline{\gamma_0 g(s)} \, ds. \qquad \diamondsuit$$

Some regularity considerations allow us to write the relation (8) as a boundary value problem. Let $k \in \mathbb{N}$. Let φ_k be an eigenfunction of (2) with respect to the eigenvalue λ_k. According to the above results, the following holds

$$\int_\Omega (\nabla\varphi_k, \overline{\nabla v})_{\mathbb{C}^n}\, dx = \lambda_k \left(\int_\Omega \varphi_k \overline{v}\, dx + \int_{\partial\Omega} \sigma\gamma_0\varphi_k \overline{\gamma_0 v}\, ds \right) \tag{10}$$

for all $v \in H^1(\Omega)$, which implies

$$\forall v \in H^1_0(\Omega): \quad \int_\Omega (\nabla\varphi_k, \overline{\nabla v})_{\mathbb{C}^n}\, dx = \int_\Omega \lambda_k\varphi_k\overline{v}\, dx.$$

Evidently, φ_k is a weak solution of the following partial differential equation

$$-\Delta\varphi_k - \lambda_k\varphi_k = 0 \quad \text{in } \Omega.$$

Applying standard regularity arguments, see e.g. [12], we conclude that each $\varphi_k \in C^\infty(\Omega)$ and

$$-\Delta\varphi_k = \lambda_k\varphi_k \quad \text{in } \Omega.$$

By replacing $\lambda_k\varphi_k$ in (10), we obtain

$$\forall v \in H^1(\Omega): \quad \int_\Omega (\nabla\varphi_k, \overline{\nabla v})_{\mathbb{C}^n}\, dx + \int_\Omega (\Delta\varphi_k)\overline{v}\, dx = \int_{\partial\Omega} \sigma\lambda_k\gamma_0\varphi_k\overline{\gamma_0 v}\, ds. \tag{11}$$

Let us suppose that $\partial\Omega$ is of class C^2. In this case, since φ_k is a weak solution of the problem

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \text{in } \Omega \\ \partial_\nu\varphi_k = \sigma\lambda_k\varphi_k & \text{on } \partial\Omega, \end{cases} \tag{12}$$

standard regularity arguments for elliptic equations, see e.g. [12], show that each $\varphi_k \in C^2(\overline{\Omega})$.

4 Asymptotic upper and lower bounds

In this section we analyse the growth order of the sequence $(\lambda_k)_{k\in\mathbb{N}}$. Let $\mathcal{H}_k$ denote the class of the k–dimensional subspaces of H, and let H_0 be the subspace of H defined by

$$H_0 = \{U \in H \,|\, U = (u, \gamma_0 u),\ \gamma_0 u = 0\}.$$

Moreover, let $\mathcal{H}^0_k$ denote the class of the k–dimensional subspaces of H_0. In order to obtain a first asymptotic result, we show the basic min–max property of the eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$.

Lemma 4.1. *The eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ of Problem (2) share the property*

$$\forall k \in \mathbb{N}: \quad \lambda_k = \min_{E\in\mathcal{H}_{k+1}} \max_{(u,\gamma_0 u)\in E\setminus\{0\}} \frac{\displaystyle\int_\Omega \|\nabla u\|_2^2\, dx}{\displaystyle\int_\Omega |u|^2\, dx + \int_{\partial\Omega} \sigma|\gamma_0 u|^2\, ds}.$$

Proof. The operator T_2 is a compact selfadjoint operator from H to itself. Therefore, we can apply the Courant-Fischer formula

$$\mu_k = \max_{E\in\mathcal{H}_{k+1}} \min_{\underbrace{(u,\gamma_0 u)}_{=U}\in E\setminus\{0\}} \frac{(T_2U,U)_1}{\|U\|_1^2} = \max_{E\in\mathcal{H}_{k+1}} \min_{\underbrace{(u,\gamma_0 u)}_{=U}\in E\setminus\{0\}} \frac{\|U\|_0^2}{\|U\|_1^2}.$$

Accordingly,

$$\frac{1}{\mu_k} = \min_{E\in\mathcal{H}_{k+1}} \max_{(u,\gamma_0 u)\in E\setminus\{0\}} \frac{\displaystyle\int_\Omega \|\nabla u\|_2^2\,dx}{\displaystyle\int_\Omega |u|^2\,dx + \int_{\partial\Omega} \sigma|\gamma_0 u|^2\,ds} + 1.$$

Since $\lambda_k = (1/\mu_k) - 1$, we obtain the desired result. ∎

As a direct consequence, we obtain the

Theorem 4.1. *The eigenvalue sequence* $(\lambda_k)_{k\in\mathbb{N}}$ *of Problem* (2) *satisfies*

$$\limsup_{k\to\infty} \frac{\lambda_k}{k^{2/n}} \leqslant C(\Omega).$$

Proof. Let $(\omega_k)_{k\in\mathbb{N}}$ be the eigenvalue sequence of $-\Delta$ in $H_0^1(\Omega)$ as introduced in Section 1. Then, by Lemma 4.1 and by restriction to the elements in $\mathcal{H}_k^0$, we obtain

$$\lambda_k \leqslant \min_{E\in\mathcal{H}^0_{k+1}} \max_{(u,\gamma_0 u)\in E\setminus\{0\}} \frac{\displaystyle\int_\Omega \|\nabla u\|_2^2\,dx}{\displaystyle\int_\Omega |u|^2\,dx + \int_{\partial\Omega} \sigma|\gamma_0 u|^2\,ds} = \omega_k.$$

Finally, (3) permits to conclude. ∎

In fact, we also have shown the

Theorem 4.2. *The eigenvalue sequence* $(\lambda_k)_{k\in\mathbb{N}}$ *of Problem* (2) *satisfies*

$$\lambda_k \leqslant \alpha_k \text{ and } \lambda_k \leqslant \omega_k \qquad \text{for all } k\in\mathbb{N},$$

where $(\alpha_k)_{k\in\mathbb{N}}$ *denotes the eigenvalue sequence of the Laplacian in* $H^1(\Omega)$ *under the Neumann boundary condition and* $(\omega_k)_{k\in\mathbb{N}}$ *the eigenvalue sequence of the Laplacian in* $H_0^1(\Omega)$.

Moreover, as a consequence of Lemma 4.1, we note that the eigenvalues of Problem (2) are decreasing with respect to the parameter σ.

Now, let us suppose that σ is a positive constant and that $n \geqslant 2$. Then we can show that the eigenvalues of Problem (2) grow like $k^{1/(n-1)}$. Let $H \ominus 1\mathbb{R}$ denote the non constant functions of H. The quadratic form q defined on $H \ominus 1\mathbb{R}$ by the inverse of the Rayleigh quotient of Problem (2), *i.e.*

$$q(u) = \frac{\displaystyle\int_\Omega |u|^2\,dx}{\displaystyle\int_\Omega \|\nabla u\|_2^2\,dx} + \frac{\displaystyle\int_{\partial\Omega} \sigma|\gamma_0 u|^2\,ds}{\displaystyle\int_\Omega \|\nabla u\|_2^2\,dx} \tag{13}$$

is decomposed into two terms, the first one being the inverse Rayleigh quotient for the Laplacian under the Neumann boundary condition, whereas the second one is the inverse Rayleigh quotient for the modified Steklov problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u = \mu\sigma u & \text{on } \partial\Omega. \end{cases} \tag{14}$$

For $\sigma = 1$, it is well known that the eigenvalue sequence $(\mu_k)_{k\in\mathbb{N}}$ of Problem (14) behaves asymptotically like

$$\mu_k = C_{\text{Stek}}(\Omega)k^{1/(n-1)} + o(k^{1/(n-1)})$$

with a positive constant $C_{\text{Stek}}(\Omega)$, see [14], [15]. For arbitrary $\sigma > 0$, the eigenvalue sequence of Problem (14) is $(\sigma^{-1}\mu_k)_{k\in\mathbb{N}}$ and, by Lemma 4.1, it follows

Theorem 4.3. *If σ is a positive constant, then the eigenvalue sequence $(\lambda_k)_{k\in\mathbb{N}}$ of Problem* (2) *satisfies*

$$\lambda_k \leqslant \sigma^{-1}\mu_k \qquad \text{for all } k \in \mathbb{N},$$

where $(\mu_k)_{k\in\mathbb{N}}$ is the eigenvalue sequence of the Steklov problem (14) *with $\sigma = 1$.*

Using the preceding results, we obtain the main results about the spectral growth order for constant positive σ.

Theorem 4.4. *Let $n \geqslant 3$ and σ be a positive constant. Then the eigenvalue sequence $(\lambda_k)_{k\in\mathbb{N}}$ of Problem* (2) *satisfies*

$$\frac{1}{2^{1/(n-1)}}\frac{C_{\text{Stek}}(\Omega)}{\sigma} \leqslant \liminf_{k\to\infty}\frac{\lambda_k}{k^{1/(n-1)}} \leqslant \limsup_{k\to\infty}\frac{\lambda_k}{k^{1/(n-1)}} \leqslant \frac{C_{\text{Stek}}(\Omega)}{\sigma}.$$

Proof. The right inequality follows by Theorem 4.3. Thus, it remains to show the left inequality. We first note that the variation of the corresponding Rayleigh quotients for Problem (2), Problem (14) and the Neumann problem takes place in the same space $H^1(\Omega)$. Let A and B denote the operators associated to the first and the second term respectively in the *r.h.s.* of (13) in $H \ominus 1\mathbb{R}$. Then $A + B$ corresponds to q and, by a well known spectral estimate for sums of compact operators, see e.g. [9], it follows that

$$\forall k, m \in \mathbb{N}\backslash\{0\} : \qquad \frac{1}{\lambda_{k+m}} \leqslant \frac{1}{\alpha_k} + \frac{\sigma}{\mu_m}.$$

This yields for all $k > 0$ that

$$\frac{\lambda_{2k}}{(2k)^{1/(n-1)}} \geqslant \frac{1}{2^{1/(n-1)}}\frac{\mu_k}{k^{1/(n-1)}}\frac{1}{\sigma + \mu_k\alpha_k^{-1}}$$

and

$$\frac{\lambda_{2k+1}}{(2k+1)^{1/(n-1)}} \geqslant \frac{1}{\left(2+\frac{1}{k}\right)^{1/(n-1)}}\frac{\mu_k}{k^{1/(n-1)}}\frac{1}{\sigma + \mu_k\alpha_{k+1}^{-1}}.$$

Both *r.h.s.* have the same limit $\frac{1}{\sigma 2^{1/(n-1)}}C_{\text{Stek}}$ since for $n \geqslant 3$,

$$\lim_{k\to\infty}\mu_k\alpha_k^{-1} = \lim_{k\to\infty}\mu_k\alpha_{k+1}^{-1} = 0. \tag{15}$$

As

$$\liminf_{k\to\infty}\frac{\lambda_{2k}}{(2k)^{1/(n-1)}}\,,\ \liminf_{k\to\infty}\frac{\lambda_{2k+1}}{(2k+1)^{1/(n-1)}} \geqslant \frac{1}{\sigma 2^{1/(n-1)}}C_{\text{Stek}},$$

and as both sequences cover the eigenvalue sequence $(\lambda_k)_{k\in\mathbb{N}}$, the proof is achieved. ■

It has been shown in [11] by means of the Hilbert-Schmidt operator theory, that for a two–dimensional bounded domain

$$\liminf_{k\to\infty} \frac{\lambda_k}{k} > 0.$$

Using the preceding techniques, Theorem 4.1 and Theorem 4.3, we obtain the following more precise result.

Theorem 4.5. *Let $n = 2$ and σ be a positive constant. Then the eigenvalue sequence $(\lambda_k)_{k\in\mathbb{N}}$ of Problem* (2) *satisfies*

$$\frac{1}{2}\,\frac{C(\Omega)\,C_{\mathrm{Stek}}(\Omega)}{\sigma C(\Omega)+C_{\mathrm{Stek}}(\Omega)} \leqslant \liminf_{k\to\infty}\frac{\lambda_k}{k} \leqslant \limsup_{k\to\infty}\frac{\lambda_k}{k} \leqslant \min\left\{C(\Omega), \frac{C_{\mathrm{Stek}}(\Omega)}{\sigma}\right\}.$$

Proof. For $n = 2$, Formula (15) reads

$$\lim_{k\to\infty}\mu_k\alpha_k^{-1} = \lim_{k\to\infty}\mu_k\alpha_{k+1}^{-1} = \frac{C_{\mathrm{Stek}}(\Omega)}{C(\Omega)}.$$

Now we can follow the proof of Theorem 4.4. ■

Mutatis mutandis, the eigenvalues of Problem (2) grow like $k^{1/(n-1)}$ if σ is a positive function and if $n \geqslant 2$. This follows readily as for the constant case above. Omitting the details we are led to the

Corollary 4.1. *Let $n \geqslant 2$ and σ be a continuous positive function. Then the eigenvalue sequence $(\lambda_k)_{k\in\mathbb{N}}$ of Problem* (2) *satisfies*

$$\frac{1}{2^{1/(n-1)}}\,\frac{C_{\mathrm{Stek}}(\Omega)}{\max\limits_{\partial\Omega}\sigma} \leqslant \liminf_{k\to\infty}\frac{\lambda_k}{k^{1/(n-1)}} \leqslant \limsup_{k\to\infty}\frac{\lambda_k}{k^{1/(n-1)}} \leqslant \frac{C_{\mathrm{Stek}}(\Omega)}{\min\limits_{\partial\Omega}\sigma}$$

for $n \geqslant 3$ and

$$\frac{1}{2}\,\frac{C(\Omega)\,C_{\mathrm{Stek}}(\Omega)}{C(\Omega)\max\limits_{\partial\Omega}\sigma+C_{\mathrm{Stek}}(\Omega)} \leqslant \liminf_{k\to\infty}\frac{\lambda_k}{k} \leqslant \limsup_{k\to\infty}\frac{\lambda_k}{k} \leqslant \min\left\{C(\Omega), \frac{C_{\mathrm{Stek}}(\Omega)}{\min\limits_{\partial\Omega}\sigma}\right\}$$

for $n = 2$.

5 The case of the unit disk

This section is devoted to the eigenvalue sequence $(\lambda_k)_{k\in\mathbb{N}}$ of Problem (2) on the unit disk D in $\mathbb{R}^2$, with constant $\sigma > 0$. Let us first determine some particular eigenfunctions by using the geometrical properties of D. A separation ansatz $u(r,\theta) = f(r)\alpha(\theta)$ in polar coordinates leads to the singular eigenvalue problem

$$\begin{cases} r^2 f''(r) + r f'(r) + (r^2\lambda - n^2) f(r) = 0 \quad \text{in } (0,1), \\ f'(1) = \lambda\sigma f(1), \\ f \in L^\infty(0,1), \end{cases} \tag{16}$$

and to

$$\alpha''(\theta) = -n^2\alpha(\theta)$$

with $n \in \mathbb{Z}$. The bounded solutions of the differential equation in (16) are the scalar multiples of the modified Bessel function $f_{n,\lambda}(r) = J_n(r\sqrt{\lambda})$, where J_n is the Bessel function of order n, see e.g. [13]. Therefore, using the boundary condition in (16), we are led to the following two lemmata.

Lemma 5.1. *Let n be an integer and $\lambda \in \mathbb{R}_+$ be a solution of the equation*

$$f'_{n,\lambda}(1) - \lambda\sigma f_{n,\lambda}(1) = \sqrt{\lambda}J'_n(\sqrt{\lambda}) - \lambda\sigma J_n(\sqrt{\lambda}) = 0.$$

Then λ is an eigenvalue of Problem (2) *on the unit disk.*

Lemma 5.2. *All non-zero eigenvalues of Problem* (2) *on the unit disk are given by the set*

$$Z = \bigcup_{n\in\mathbb{Z}} \left\{ \mu^2 \;\middle|\; \underbrace{J'_n(\mu) - \sigma\mu J_n(\mu)}_{=-\frac{F_n(\mu)}{2}} = 0 \right\}.$$

Proof. Let us assume that there exists an eigenvalue λ of (2) not belonging to Z. Let $(r,\theta) \mapsto \mathfrak{u}(r,\theta)$ be an eigenfunction (in polar coordinates) associated to λ. Since $\mathfrak{u}$ belongs to the space $C^\infty((0,1] \times \mathbb{R})$ and is 2π–periodic according to the second variable, the following expansion holds

$$\mathfrak{u}(r,\theta) = \sum_{n=-\infty}^{+\infty} g_n(r)e^{in\theta},$$

where $g_n \in C^\infty(0,1) \cap L^\infty([0,1))$ can be written as

$$\forall r \in (0,1): \quad g_n(r) = \frac{1}{2\pi}\int_0^{2\pi} \mathfrak{u}(r,\theta)e^{-in\theta}\, d\theta.$$

The function $\mathfrak{u}$ satisfies the following equation

$$\frac{\partial^2\mathfrak{u}}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial\mathfrak{u}}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2\mathfrak{u}}{\partial\theta^2}(r,\theta) = -\lambda\mathfrak{u}(r,\theta),$$

and, therefore,

$$\sum_{n=-\infty}^{+\infty} \Big(r^2 g''_n(r) + r g'_n(r) + (r^2\lambda - n^2)g_n(r)\Big)e^{in\theta} = 0.$$

The L^2–orthogonality of the sequence $(e^{in\theta})_{n\in\mathbb{Z}}$ and the last identity yield

$$\forall n \in \mathbb{Z}: \quad r^2 g''_n(r) + r g'_n(r) + (r^2\lambda - n^2)g_n(r) = 0 \quad \text{in } (0,1). \tag{17}$$

Moreover, as an eigenfunction, $\mathfrak{u}$ has to fulfill

$$\sum_{n=-\infty}^{+\infty} \Big(g'_n(1) - \sigma\lambda g_n(1)\Big)e^{in\theta} = 0,$$

from which we deduce $g'_n(1) = \sigma\lambda g_n(1)$ for $n \in \mathbb{Z}$. Accordingly, the function g_n is of the form $g_n(r) = C_n J_n(r\sqrt{\lambda})$ with an arbitrary real constant C_n. Since the $\mathfrak{u}$ is nontrivial, there exists an integer $n_0 \in \mathbb{Z}$ such that $C_{n_0} \neq 0$. Finally, $g'_{n_0}(1) - \sigma\lambda g_{n_0}(1) = 0$ shows that $\lambda \in Z$, which is a contradiction. ∎

Asymptotically the large roots of F_n become close to those of the homogeneous Dirichlet problem, since

$$F_n(\mu) = 2\sigma\sqrt{\frac{2\mu}{\pi}}\left(\cos\Big(\mu - \frac{n\pi}{2} - \frac{\pi}{4}\Big) + O\left(\frac{1}{\mu}\right)\right), \quad \mu \to +\infty, \tag{18}$$

see [11]. It is well known [8] that the eigenvalue sequence $(\omega_k)_{k\in\mathbb{N}}$ of the homogeneous Dirichlet problem satisfies

$$\lim_{k\to\infty} \frac{\omega_k}{k} = C(D) = 4.$$

This is essentially due to the invariance under rotations of the homogeneous Dirichlet problem, that remains also valid for (2) on D. But, while the non-zero eigenvalues of the Dirichlet problem are double, those of Problem (2) on the unit disk with $\sigma > 1/8$ must have higher multiplicities by Theorems 4.4, 4.5 and by Lemma 5.2. More precisely: The eigenvalue sequence $(\mu_k)_{k\in\mathbb{N}}$ of the Steklov problem

$$(S)\ \begin{cases} \Delta u = 0 & \text{in } D, \\ \partial_\nu u = \mu u & \text{on } \partial D, \end{cases}$$

can be determined via the separation ansatz $u(r,\theta) = g(r)\alpha(\theta)$ in polar coordinates that leads to the equations

$$\begin{cases} \alpha'' + n^2\alpha(\theta) = 0 & \theta \in [0, 2\pi], n \in \mathbb{Z} \\ r^2 g''(r) + r g'(r) - n^2 g(r) = 0 & r \in (0,1), \\ g'(1) = \mu g(1). \end{cases}$$

The second equation is an Euler equation admitting solutions only of the form

$$g(r) = Ar^n + \frac{B}{r^n}, \qquad A, B \in \mathbb{R}.$$

Moreover, the boundedness condition $u \in L^\infty(D)$ leads to $B = 0$ (for $n \in \mathbb{N}$) and $g(r) = Ar^n$ so that the boundary condition $g'(1) = \mu g(1)$ reads $nA = \mu A$, *i.e.* $\mu = n$. In fact, this ansatz yields already all eigenvalues of Problem (S). An eigenfunction u associated to an eigenvalue μ not belonging to $\mathbb{Z}$ would satisfy

$$\int_0^{2\pi} u(1,\theta)e^{in\theta}\, d\theta = 0, \qquad \text{for all } n \in \mathbb{Z},$$

that, in turn, would imply that u vanishes on ∂D and, thereby, also in D as a harmonic function. Application of Theorem 4.3 yields

$$\lambda_k \leqslant \sigma^{-1}\left\lceil \frac{k+1}{2} \right\rceil \qquad \text{for all } k \in \mathbb{N}$$

on D. Moreover, $C_{\text{Stek}}(D) \leqslant \frac{1}{2}$. In fact, it has been shown by Eastham [10] that $C_{\text{Stek}}(D)$ amounts to $\frac{1}{2}$ making use of a Green function calculus and Tauberian theorems. Resuming all this we conclude

Theorem 5.1. *Let σ be a positive constant. Then the eigenvalue sequence $(\lambda_k)_{k\in\mathbb{N}}$ of Problem* (2) *on the unit disk D satisfies*

$$\frac{2}{8\sigma+1} \leqslant \liminf_{k\to\infty} \frac{\lambda_k}{k} \leqslant \limsup_{k\to\infty} \frac{\lambda_k}{k} \leqslant \min\left\{4, \frac{1}{2\sigma}\right\}.$$

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