Some twisted variants of Chevalley groups

Anja Steinbach

Abstract

We construct a BN-pair for subgroups of Chevalley groups, which are fixed under some distinguished automorphisms. This generalizes the standard approach for the twisted groups of type ${}^2A_\ell$, ${}^2D_\ell$ and ${}^2E_6$ to groups which are not necessarily quasi-split.

Introduction

For root systems and Weyl groups, we refer to Bourbaki [3], Carter [4] and Steinberg [8]. Let Φ be a root system of type $A_\ell, \dots, G_2$ with fundamental system Π and underlying Euclidean space V. The associated Weyl group, W say, is the group generated by the fundamental reflections w_α, $\alpha \in \Pi$ (in the hyperplane orthogonal to α). For Φ of type A_ℓ, D_ℓ or E_6, let τ be the diagram symmetry of order 2. For $J \subseteq \Pi$, we denote by W_J the subgroup of W generated by the w_α, $\alpha \in J$, and by w_0^J the longest element in W_J.

We fix a subset J of Π, $J \neq \Pi$, and set $\Phi_J = \Phi \cap \langle J \rangle$. Let σ be one of the following permutations of Φ: either w_0^J or τw_0^J, provided that $\tau(J) = J$ in the second case. For $\alpha \in \Pi \setminus J$, we set $\mathcal{O}_\alpha := \{\alpha\}$, when $\sigma = w_0^J$, and $\mathcal{O}_\alpha := \{\alpha, \tau(\alpha)\}$, when $\sigma = \tau w_0^J$. This partitions $\Pi \setminus J$ into subsets of size at most 2. We define $\widetilde{V} := C_V(\sigma) \cap J^\perp$. For $v \in V$, $\widetilde{v}$ denotes the orthogonal projection of v onto $\widetilde{V}$.

Received by the editors December 2002.
Communicated by H. Van Maldeghem.
1991 *Mathematics Subject Classification :* 20E42, 20G15.
Key words and phrases : Chevalley group, Steinberg variation, BN-pair, root system.

Proposition. *In the above notation, we assume that $w_0^{J\cup\mathcal{O}_\alpha}(\mathcal{O}_\alpha) = -\mathcal{O}_\alpha$, for all $\alpha \in \Pi \setminus J$. Then $\widetilde{\Phi} := \{\widetilde{r} \mid r \in \Phi \setminus \Phi_J\}$ is a (possibly non-reduced) root system on $\widetilde{V}$.*

We refer to Lemma 1.5 for equivalent formulations of the assumption in the proposition. For Φ of type A_3, for example, the assumption is only satisfied for $J = \emptyset$ and for $J = \{\alpha_1, \alpha_3\}$ (in the notation of Bourbaki [3]) in either case for both choices of σ.

For the definition and properties of Chevalley groups as well as for the standard facts on groups with a BN-pair and their parabolic subgroups, we refer to Carter [4], [5], Steinberg [8] and Bourbaki [3].

For a field K, we denote by $\Phi(K)$ the corresponding universal Chevalley group, defined by the Steinberg generators and relations; see Carter [4, (12.1.1)]. The associated standard root subgroups are $X_r \simeq (K, +)$, $r \in \Phi$. The group $\Phi(K)$ has a BN-pair (B, N) with $N/H = W = \langle w_\alpha \mid \alpha \in \Pi\rangle$, where $H = B \cap N$. For $J \subseteq \Pi$, we denote by U_J the subgroup generated by all X_r, where r is a positive root not contained in Φ_J. We also define L_J to be the subgroup generated by H and all X_r, $r \in \Phi_J$, and $P_J := U_J L_J$.

Next, we assume that J and σ satisfy the assumption of the proposition. Let η_σ be an automorphism of $G := \Phi(K)$ such that the following holds (we give an example after the statement of the main theorem):

(0) We have $\eta_\sigma(X_r) = X_{\sigma(r)}$, for $r \in \Phi$. Furthermore, N is invariant under η_σ. If $nH = w$, then $\eta_\sigma(n)H = \sigma w \sigma^{-1}$.
(1) If P is a parabolic subgroup of L_J, which is η_σ-invariant, then $P = L_J$.
(2) For $\alpha \in \Pi \setminus J$, we have $\langle X_r \mid r \in \Phi^-_{J\cup\mathcal{O}_\alpha} \setminus \Phi_J\rangle \cap \mathrm{Fix}(\eta_\sigma) \neq 1$.

We define $G^1 := \langle U_J \cap \mathrm{Fix}(\eta_\sigma), U_J^- \cap \mathrm{Fix}(\eta_\sigma)\rangle$, as well as $B^1 := P_J \cap G^1$ and $N^1 := \langle n_0^{J\cup\mathcal{O}_\alpha}, L_J \mid \alpha \in \Pi \setminus J\rangle \cap G^1$, $H^1 := L_J \cap G^1$. Here $n_0^{J\cup\mathcal{O}_\alpha} \in N$ with $n_0^{J\cup\mathcal{O}_\alpha} H = w_0^{J\cup\mathcal{O}_\alpha}$, for $\alpha \in \Pi \setminus J$.

Main Theorem. *In the above notation, we assume that the assumption of the proposition and* (0), (1) *and* (2) *holds. Furthermore, we suppose that the root system $\widetilde{\Phi}$ is of type $A_\ell, \ldots, G_2$ or BC_ℓ.*

Then (B^1, N^1) is a BN-pair for G^1 with Weyl group $N^1/H^1 = \langle w_{\widetilde{\alpha}} \mid \alpha \in \Pi\setminus J\rangle \leq O(\widetilde{V})$. Moreover, when G^1 is perfect, then it is quasi-simple.

We call the group G^1 a twisted variant of the Chevalley group G. The present paper is inspired by the construction of a BN-pair for the 'ordinary' twisted groups of type ${}^2A_\ell$, ${}^2D_\ell$ and ${}^2E_6$ (Steinberg variations) in Carter [4] or Steinberg [8]. These are the special cases where $J = \emptyset$ and $\sigma = \tau$.

We remark that Borel and Tits [2], see also Borel [1] or Springer [7], construct a BN-pair for the group G_k of k-rational points of a connected reductive algebraic group G defined over an arbitrary field k.

Next, we give an example satisfying the assumption of the main theorem. Let Φ be the root system of type A_5 and choose $J := \Pi \setminus \{\alpha_1, \alpha_5\}$, $\sigma := \tau w_0^J$ (in the notation of Bourbaki [3]). Then the assumption of the proposition is satisfied and the root system $\widetilde{\Phi}$ is of type BC_1. Let K be the field $\mathbb{C}$ of complex numbers with complex conjugation $c \mapsto \overline{c}$. The associated Chevalley group $G := \Phi(K)$ is $\mathrm{SL}_6(\mathbb{C})$, the group of 6×6-matrices with entries from $\mathbb{C}$ and determinant 1, and has the well known BN-pair. The root elements are elementary matrices (i.e. with main diagonal 1 and one further non-zero entry). By I we denote the 4×4-identity matrix and by M the 6×6 matrix with entries $1, I, 1$ in the (block) diagonal from lower left to upper right and all other entries zero. The automorphism

$$\eta_\sigma : \mathrm{SL}_6(\mathbb{C}) \to \mathrm{SL}_6(\mathbb{C}) \quad \text{with} \quad g \mapsto M^{-1}(\overline{g}^T)^{-1}M$$

satisfies the assumption of the main theorem. Note that the group of elements fixed under η_σ is a unitary group of rank 1 (as the standard hermitian form is anisotropic over $\mathbb{C}$).

The main theorem stated above contributes to the study of the subgroups of Chevalley groups which are fixed under some group of automorphisms. My aim is to investigate the groups with a Tits diagram (we refer to (1.6) below) as subgroups of Chevalley groups, fixed under suitable automorphisms which permute the root subgroups X_r, $r \in \Phi$. (In Mühlherr and Van Maldeghem [6], the corresponding result has been achieved for the Moufang quadrangles of type F_4.) The automorphisms in question do not necessarily fix the unipotent subgroup of the Chevalley group, but a proper parabolic subgroup. The intended strategy is to give explicit automorphisms for which the main theorem above applies. In addition to the various classical groups also forms of exceptional groups will arise. In particular for these groups, a description and complete understanding in the framework of Chevalley groups and groups with BN-pairs seems worth-wile.

The content of the present paper is as follows. Section 1 is devoted to the proof of the proposition. From the assumption of the proposition we deduce in (1.14) below that the restriction of $w_0^{J\cup\mathcal{O}_\alpha}$ to $\widetilde{V}$ is $w_{\widetilde{\alpha}}$, for $\alpha \in \Pi \setminus J$. Thus the $w_{\widetilde{\alpha}}$ permute $\widetilde{\Phi}$. The latter holds for all $w_{\widetilde{r}}$, $r \in \Phi \setminus \Phi_J$, as we show that $w_{\widetilde{r}}$ is the restriction to $\widetilde{V}$ of a conjugate under the group generated by the $w_0^{J\cup\mathcal{O}_\beta}$, $\beta \in \Pi \setminus J$, of some $w_0^{J\cup\mathcal{O}_\alpha}$ (see (1.17) below). Furthermore, we prove in (1.16) that for $\alpha \in \Pi \setminus J$ and $r \in \Phi \setminus \Phi_J$, the vector $\widetilde{r}$ is a positive multiple of $\widetilde{\alpha}$, if and only if $r \in \Phi^+_{J\cup\mathcal{O}_\alpha} \setminus \Phi_J$.

In Section 1 we also remark that the Tits diagrams listed in [9] yield examples for our setup and we give further examples.

We construct the BN-pair for G^1 in Section 2. For this the BN-pair (B, N) for G and Assumption (0) for the automorphism η_σ are indispensable. We use the unique BNB-decomposition in G as well as some unique $P_J N P_J$-decomposition deduced from it (see (2.2) below). Furthermore the distinguished (double) coset representatives defined in (1.2) are an important tool in the proof.

In our proof in (2.9) below that N^1/H^1 is isomorphic to the subgroup of $O(\widetilde{V})$ generated by the $w_{\widetilde{\alpha}}$, $\alpha \in \Pi \setminus J$, Assumption (2) is used via the definition of suitable elements $n_{\widetilde{\alpha}}$. Assumption (1) (on the fixed point free action of η_σ on the L_J-building) is used in (2.5), where we investigate the double coset $P_J g P_J$ for a parabolic subgroup ${}^gP_J$ invariant under η_σ. From the above and the assumption that $\widetilde{\Phi}$ is of type $A_\ell, \ldots, G_2$ or BC_ℓ, we deduce in (2.10) that any element in G^1 may we written as a product with factors in B^1, N^1 and B^1. Next we verify the BN-pair axioms and we finally investigate whether G^1 is quasi-simple. This proves the main theorem.

1 The root system $\widetilde{\Phi}$

In this section, we prove the proposition stated in the introduction. For root systems and Weyl groups, we refer to Bourbaki [3], Carter [4] and Steinberg [8].

1.1. Notation. Let Φ be an indecomposable, spherical root system satisfying the cristallographic condition (and whence of type $A_\ell, \ldots, G_2$). Let Π be a fundamental system for Φ, spanning the Euclidean space V with standard scalar product $(,)$. The associated Weyl group is $W := \langle w_\alpha \mid \alpha \in \Pi \rangle \leq O(V)$. Here w_α is a reflection with $w_\alpha : v \mapsto v - 2(v,\alpha)/(\alpha,\alpha) \cdot \alpha$, for $v \in V$.

For $J \subseteq \Pi$, we define W_J, Φ_J, Φ_J^+, Φ_J^- as usual. The longest element in W is denoted by w_0. We have $w_0(\Pi) = -\Pi$ and $w_0^2 = \mathrm{id}$. Similarly, we define w_0^J in W_J.

1.2. Distinguished (double) coset representatives. For the following, we refer to Carter [4, (2.5.9)] and [5, (2.7.3)]. Let W be a (finite) Weyl group with root system Φ and fundamental reflections w_α, $\alpha \in \Pi$. We fix $J \subseteq \Pi$.

Every left coset of W_J in W contains a unique element, w say, of minimal length. This element is characterized by $w(J) \subseteq \Phi^+$.

Similarly, every double coset of W_J in W contains a unique element, w say, of minimal length. This element is characterized by $w(J) \subseteq \Phi^+$ and $w^{-1}(J) \subseteq \Phi^+$.

1.3. Diagram symmetries. For Φ of type A_ℓ, D_ℓ or E_6, we denote by τ the diagram symmetry of order 2. Then τ gives rise to an isometry of V (also denoted by τ) which permutes Φ and preserves Φ^+ and Φ^-.

For $J \subseteq \Pi$ with $\tau(J) = J$, we have $\tau W_J \tau^{-1} = W_J$ and $\tau w_0^J \tau^{-1} = w_0^J$. In particular, $\tau w_0^J = w_0^J \tau$.

1.4. Notation. Let Φ, Π, V, τ, J, W, W_J, σ and $\mathcal{O}_\alpha$ be as defined in the introduction. We consider σ as isometry of V, which permutes Φ. We set

$$C_V(\sigma) = \{v \in V \mid \sigma(v) = v\}, \qquad C_W(\sigma) = \{w \in W \mid \sigma w \sigma^{-1} = w\},$$
$$\mathrm{Stab}(\Phi_J) = \{w \in W \mid w(\Phi_J) = \Phi_J\}, \quad W^1 := \langle w_0^{J\cup\mathcal{O}_\alpha} \mid \alpha \in \Pi \setminus J \rangle.$$

We define $\widetilde{V}$ and $\widetilde{v}$ as in the introduction and set $\widetilde{W} := \langle w_{\widetilde{\alpha}} \mid \alpha \in \Pi \setminus J \rangle \leq O(\widetilde{V})$, $\widetilde{\Phi} := \{\widetilde{r} \mid r \in \Phi \setminus \Phi_J\}$ and $\widetilde{\Pi} := \{\widetilde{\alpha} \mid \alpha \in \Pi \setminus J\}$.

We recall the assumption of the proposition stated in the introduction:

(H) $w_0^{J\cup\mathcal{O}_\alpha}(\mathcal{O}_\alpha) = -\mathcal{O}_\alpha$, for $\alpha \in \Pi \setminus J$.

The aim is to show that $\widetilde{\Phi}$ is a root system (with fundamental system $\widetilde{\Pi}$ and Weyl group $\widetilde{W}$). Possibly, $\widetilde{\Phi}$ is non-reduced (and for some root in $\widetilde{\Phi}$ also a proper positive multiple is a root).

Next, we give equivalent formulations of (H).

1.5. Lemma. *Let $\alpha \in \Pi \setminus J$. The following conditions are equivalent:*

(a) w_0^J *and* $w_0^{J\cup\mathcal{O}_\alpha}$ *commute.* (b) $w_0^{J\cup\mathcal{O}_\alpha} \in C_W(\sigma)$
(c) $w_0^{J\cup\mathcal{O}_\alpha}(J) = -J$ (d) $w_0^{J\cup\mathcal{O}_\alpha} \in \mathrm{Stab}(\Phi_J)$
(e) $w_0^{J\cup\mathcal{O}_\alpha}(\mathcal{O}_\alpha) = -\mathcal{O}_\alpha$

Proof. First, we note that (a) and (b) are equivalent. Indeed, either $\sigma = w_0^J$ or $\sigma = \tau w_0^J$. In the latter case, $w_0^{J\cup\mathcal{O}_\alpha}$ commutes with τ by (1.3). Since $w_0^{J\cup\mathcal{O}_\alpha}$ switches $J \cup \mathcal{O}_\alpha$ and $-(J \cup \mathcal{O}_\alpha)$, also (c), (d), (e) are equivalent.

Furthermore, (a) implies (e). For this, we assume that $w_0^{J\cup\mathcal{O}_\alpha}(\alpha) \in -J$. Then $w_0^J(\alpha) = w_0^{J\cup\mathcal{O}_\alpha} w_0^J w_0^{J\cup\mathcal{O}_\alpha}(\alpha)$ is contained in $-(J\cup\mathcal{O}_\alpha)$ and in $\alpha+\langle J\rangle$, a contradiction. Finally, (c) and (e) imply (a). Indeed, $w := w_0^J w_0^{J\cup\mathcal{O}_\alpha} w_0^J$ is in $W_{J\cup\mathcal{O}_\alpha}$ and switches Φ_J^+ and Φ_J^- and also $\Phi^+_{J\cup\mathcal{O}_\alpha} \setminus \Phi_J$ and $\Phi^-_{J\cup\mathcal{O}_\alpha} \setminus \Phi_J$; whence $w = w_0^{J\cup\mathcal{O}_\alpha}$. ∎

1.6. A connection with Tits diagrams. For Tits diagrams, we refer to Tits [9], Van Maldeghem [12]. In [9] there is a list of Dynkin diagrams with some additional information which describes the algebraic semisimple groups over arbitrary fields. Every such Tits diagram yields a choice of J and σ such that the assumption of the proposition is satisfied. Indeed, the subset J is the part of Π that is not encircled, the so-called anisotropic kernel. We choose $\sigma := w_0^J$ and $\sigma := \tau w_0^J$ depending on whether the Tits diagram is straight or bended (so-called inner and outer forms, respectively). The set $\mathcal{O}_\alpha$, $\alpha \in \Pi \setminus J$, is the so-called (distinguished orbit or) isotropic orbit containing α. It is in the encircled part of the Tits diagram. For outer forms, the Tits diagram is invariant under the diagram symmetry.

When $J = \emptyset$ in (1.4), then $w_0^J = \mathrm{id}$ and σ is trivial or a diagram symmetry. This leads to split and quasi-split forms, that is to Chevalley groups or 'ordinary' twisted variants (Steinberg variations).

1.7. Example. We remark that there exists subsets J which do not arise in the list of diagrams in [9] (as explained in (1.6)), but satisfy the hypotheses of the proposition. For example for Φ of type E_7, we could take $J = \Pi \setminus \{\alpha_4, \alpha_6\}$ in the notation of Bourbaki [3]. Then we obtain the fundamental roots $\widetilde{\alpha_4} = \frac{1}{6}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 3\alpha_5)$ and $\widetilde{\alpha_6} = \frac{1}{2}(\alpha_5 + 2\alpha_6 + \alpha_7)$. The angle between the two roots is $5\pi/6$ and $\widetilde{\alpha_6}$ is $\sqrt{3}$ times as long as $\widetilde{\alpha_4}$, as in root systems of type G_2. Inspection of the root system of type E_7 shows that there are 9 positive roots such that α_4 or α_6 have a non-zero coefficient. The possible values for these pairs of coefficients are $(0,1), (3,1), (3,2)$, $(1,0), (1,1), (2,1)$ and $(2,0), (2,2), (4,2)$. Note that the last three pairs are the middle ones multiplied by 2. Thus the non-reduced root system $\widetilde{\Phi}$ consists of the 12 roots of a root system of type G_2 together with doubles of the short roots, the phenomenon which also arises in root systems of type BC_2.

We remark that a similar case is Φ of type E_8 and $J = \Pi \setminus \{\alpha_1, \alpha_6\}$ in the notation of Bourbaki [3].

In the following, we prove the proposition. We will use Hypothesis (H) in (1.14) and below.

1.8. Lemma. *For σ as in* (1.4)*, we have $\sigma^2 = \mathrm{id}$ and $\sigma W \sigma^{-1} = W$. Furthermore, σ switches Φ_J^+ and Φ_J^-, but leaves $\Phi^+ \setminus \Phi_J$ invariant.* ∎

1.9. Lemma. *The kernel of the projection onto $\widetilde{V}$ is $\widetilde{V}^\perp = \langle J, v - \sigma(v) \mid v \in V\rangle = \langle J, \alpha - \tau(\alpha) \mid |\mathcal{O}_\alpha| = 2\rangle$. In particular, $\widetilde{\alpha} = \widetilde{\tau(\alpha)}$, if $|\mathcal{O}_\alpha| = 2$, and $r \in \Phi_J$, when $r \in \Phi$ with $\widetilde{r} = 0$.*

Proof. We have $\widetilde{V}^\perp = \langle J\rangle + C_V(\sigma)^\perp$. Since $C_V(\sigma)^\perp = \langle v - \sigma(v) \mid v \in V\rangle$, the first equality holds. For the proof of the second equality, we first note that

$$(*)\ \ v + \langle J\rangle = w_0^J(v) + \langle J\rangle, \text{ for } v \in V.$$

Using $(*)$ twice, we obtain that the right hand side is contained in the left hand side (as for α with $|\mathcal{O}_\alpha| = 2$, necessarily $\sigma = w_0^J\tau$ and $\alpha - \tau(\alpha) \in w_0^J(\alpha - \tau(\alpha)) + \langle J\rangle = \alpha - \sigma(\alpha) + \langle J\rangle$). For the other inclusion, it suffices to consider the basis Π of V. When $\sigma = w_0^J$, $(*)$ yields that $\alpha - \sigma(\alpha) \in \langle J\rangle$, as desired. When $\sigma = w_0^J\tau$, then $\alpha - \sigma(\alpha) \in \alpha - \tau(\alpha) + \langle J\rangle$ by $(*)$. As $\alpha - \tau(\alpha)$ is in $\langle J\rangle$, for $\alpha \in J$ or $|\mathcal{O}_\alpha| = 1$, the second equality holds.

The next statement follows, as $\alpha - \tau(\alpha)$ is contained in the kernel of the projection on $\widetilde{V}$, for $|\mathcal{O}_\alpha| = 2$. Finally, let $r \in \Phi$ with $\widetilde{r} = 0$. By the above we may express r as a linear combination of vectors in J and vectors $\alpha - \tau(\alpha)$ with $\alpha \in \Pi \setminus J$. As every root is positive or negative, we deduce that $r \in \Phi_J$. ■

We remark that $\widetilde{v} = \widetilde{w(v)}$ for all $v \in V$ and $w \in W_J$, as $v - w(v)$ is a linear combination of vectors from J. Hence W_J acts trivially on $\widetilde{V}$ (even on $J^\perp$).

1.10. Lemma. *We denote by $\mathcal{O}_{\alpha_1}, \ldots, \mathcal{O}_{\alpha_n}$ the distinct $\mathcal{O}_\alpha$. Then $\widetilde{\alpha_1}, \ldots, \widetilde{\alpha_n}$ is a basis of $\widetilde{V}$.*

Proof. By (1.9) $\widetilde{\alpha_1}, \ldots, \widetilde{\alpha_n}$ span $\widetilde{V}$ and are linearly independent. ■

The following characterization of the projection onto $\widetilde{V}$ is sometimes useful for explicit calculations.

1.11. Lemma. *For $v \in V$, the vector $\widetilde{v}$ is the projection of $\frac{1}{2}(v + \sigma(v))$ onto $J^\perp$.*

Proof. We write $\frac{1}{2}(v + \sigma(v)) = x + y$ with $x \in \langle J\rangle$, $y \in J^\perp$. Then $v - y = x + \frac{1}{2}(v - \sigma(v))$. Whence $v - y$ is contained in $\widetilde{V}^\perp$ by (1.9), as desired. ■

1.12. Lemma. *The space $\widetilde{V}$ is invariant under $C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$. In particular, $\widetilde{w(r)} = w(\widetilde{r})$, for $w \in C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$ and $r \in \Phi$.* ■

1.13. Lemma. *The kernel of the action of $C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$ on $\widetilde{V}$ is $C_{W_J}(\sigma)$.*

Proof. By (1.12), $C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$ acts on $\widetilde{V}$. By the remark after (1.9), $C_{W_J}(\sigma)$ is contained in the kernel of the action. For the converse, let $w_1 \in C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$, $w_1 \notin W_J$. We write $w_1 = wg$, where w is the shortest element in wW_J and $g \in W_J$. Then $w(J) \subseteq \Phi^+$, $w \neq 1$. Furthermore, w stabilizes $\langle J\rangle$ (as the latter holds for w_1 and g). Whence $\widetilde{w(x)} = 0$ by (1.9), for $x \in \langle J\rangle$.

As $w \neq 1$, there exists $\beta \in \Pi \setminus J$ with $w(\beta) \in \Phi^-$. Then $\widetilde{\beta} \neq 0$ by (1.9). As $g \in W_J$, we have $g(\beta) = \beta + x$ with $x \in \langle J\rangle$. We obtain $w_1(\widetilde{\beta}) = \widetilde{w_1(\beta)} = \widetilde{w(\beta)} + \widetilde{w(x)} = \widetilde{w(\beta)} \neq \widetilde{\beta}$ and w_1 is not in the kernel of the action. ■

1.14. Lemma. *We have* $w_0^{J\cup\mathcal{O}_\alpha} \mid_{\widetilde{V}} = w_{\widetilde{\alpha}}$, $\alpha \in \Pi \setminus J$. *In particular,* $w_{\widetilde{\alpha}}(\widetilde{r}) \in \widetilde{\Phi}$, *for* $r \in \Phi \setminus \Phi_J$.

Proof. Let $\alpha \in \Pi \setminus J$. We have $w_0^{J\cup\mathcal{O}_\alpha}(\widetilde{\alpha}) = -\widetilde{\alpha}$ by (H) and (1.12). If $v \in \widetilde{V}$ with $(v, \widetilde{\alpha}) = 0$, then $(v, \alpha) = 0$ and also $(v, \tau(\alpha)) = 0$ (as $\widetilde{\alpha} - \alpha, \widetilde{\alpha} - \tau(\alpha) \in \widetilde{V}^\perp$). Thus $v \in (J \cup \mathcal{O}_\alpha)^\perp$ and v is centralized by $W_{J\cup\mathcal{O}_\alpha}$. This yields the first claim and (1.14) follows. ∎

1.15. Notation. For $\alpha \in \Pi \setminus J$ and $w \in C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$, we define

$$S(w, \alpha) \ := \ w(\Phi^+_{J\cup\mathcal{O}_\alpha} \setminus \Phi_J) \subseteq \Phi.$$

Then $-S(w, \alpha) = S(ww_0^{J\cup\mathcal{O}_\alpha}, \alpha)$. We call any such $S(w, \alpha)$ a part, compare Steinberg [8, p. 174].

1.16. Lemma. *For* $\alpha \in \Pi \setminus J$ *and* $r \in \Phi \setminus \Phi_J$, *we have* $\widetilde{r} = \mu\widetilde{\alpha}$ *with* $\mu > 0$, *if and only if* $r \in S(\mathrm{id}, \alpha)$.

Proof. Let $r \in S(\mathrm{id}, \alpha)$. If $\mathcal{O}_\alpha = \{\alpha\}$, then $r \in \mu\alpha + \langle J\rangle$ and $\widetilde{r} = \mu\widetilde{\alpha}$ with $\mu > 0$. If $|\mathcal{O}_\alpha| = 2$, then $r \in \mu_1\alpha + \mu_2\tau(\alpha) + \langle J\rangle$ and $\widetilde{r} = (\mu_1 + \mu_2)\widetilde{\alpha}$ with $\mu := \mu_1 + \mu_2 > 0$.

Conversely, let $\alpha \in \Pi \setminus J$, $r \in \Phi \setminus \Phi_J$ such that $\widetilde{r} = \mu\widetilde{\alpha}$ with $\mu > 0$. Then $r - \mu\alpha$ is contained in the kernel of the projection onto $\widetilde{V}$. With (1.9) we deduce $r \in \Phi^+_{J\cup\mathcal{O}_\alpha} \setminus \Phi_J$, as desired. ∎

We remark that in the examples of Tits [9], we have $\mu = 1$ or $\mu = 2$. In our setup we could take Φ of type E_6, $J = \Pi \setminus \{\alpha_4\}$ and $\sigma := w_0^J$ in the notation of Bourbaki [3], for example. Then the assumption of the proposition is satisfied and $\widetilde{\Pi} = \{\widetilde{\alpha_4}\}$. For the roots $r := \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$ and $s := \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, we have $\widetilde{r} = 2\widetilde{\alpha_4}$ and $\widetilde{s} = 3\widetilde{\alpha_4}$.

For the following lemma, compare Carter [4, (13.2.1), (2.1.8)]. The group W^1 was defined in (1.4).

1.17. Lemma. *For* $r \in \Phi \setminus \Phi_J$, *there exist* $w \in W^1$ *and* $\alpha \in \Pi \setminus J$ *such that* $r \in S(w, \alpha)$. *In particular,* $ww_0^{J\cup\mathcal{O}_\alpha}w^{-1} \mid_{\widetilde{V}} = w_{\widetilde{r}}$ *and* $w_{\widetilde{r}}(\widetilde{s}) \in \widetilde{\Phi}$, *for* $s \in \Phi - \Phi_J$.

Proof. Let $r \in \Phi^+ \setminus \Phi_J$. By (1.10), there exist integers $c_i \geq 0$ with $\widetilde{r} = \sum_{i=1}^n c_i\widetilde{\alpha_i}$. We proceed by induction on $H(r) := \sum_{i=1}^n c_i > 0$. If $H(r) = 1$, then $\widetilde{r} = \widetilde{\alpha}$ with $\alpha \in \Pi \setminus J$; whence $r \in S(\mathrm{id}, \alpha)$ by (1.16). Next, let $H(r) > 1$. Since $\widetilde{r} \neq 0$, there exists $\alpha \in \Pi \setminus J$ with $(\widetilde{r}, \widetilde{\alpha}) > 0$. (Otherwise $(\widetilde{r}, \widetilde{\alpha}) \leq 0$, for all $\alpha \in \Pi \setminus J$. Whence $(\widetilde{r}, \widetilde{r}) = \sum_{i=1}^n (\widetilde{r}, \widetilde{\alpha_i}) \leq 0$, a contradiction.) We may assume that $r \notin S(\mathrm{id}, \alpha)$. With (1.12) and (1.14) we obtain $s := w_0^{J\cup\mathcal{O}_\alpha}(r) \in \Phi^+ \setminus \Phi_J$ and $\widetilde{s} = w_0^{J\cup\mathcal{O}_\alpha}(\widetilde{r}) = w_{\widetilde{\alpha}}(\widetilde{r}) = \widetilde{r} - \mu\widetilde{\alpha}$ with $\mu := 2(\widetilde{r}, \widetilde{\alpha})/(\widetilde{\alpha}, \widetilde{\alpha}) > 0$. Thus $H(s) = H(r) - \mu < H(r)$. By induction, there exist $w \in W^1$ and $\beta \in \Pi \setminus J$ with $s \in S(w, \beta)$. This yields $r = w_0^{J\cup\mathcal{O}_\alpha}(s) \in S(w_0^{J\cup\mathcal{O}_\alpha}w, \beta)$, as desired.

We have shown that any $r \in \Phi^+ \setminus \Phi_J$ is contained in an $S(w, \alpha)$. In this case, $-r \in S(ww_0^{J\cup\mathcal{O}_\alpha}, \alpha)$. This proves the first claim and (1.17) follows. ∎

1.18. We define $r \approx s$, if and only if $\widetilde{r} = \mu\widetilde{s}$ with $0 < \mu \in \mathbb{R}$. Then $\approx$ is an equivalence relation on $\Phi \setminus \Phi_J$. By (1.16) and (1.17), the parts $S(w,\alpha)$ are the equivalence classes of $\approx$. This yields that distinct 'parts' are disjoint.

Proof of the proposition. By (1.9) $\widetilde{\Phi}$ is a finite set of non-zero vectors which generates $\widetilde{V}$. For $s \in \Phi - \Phi_J$, we have $w_{\widetilde{r}}(\widetilde{s}) \in \widetilde{\Phi}$ by (1.17). ∎

1.19. Remark. The groups W^1 generated by the $w_0^{J\cup\mathcal{O}_\alpha}$, $\alpha \in \Pi \setminus J$, and $C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$ are not necessarily equal. This is easily verified for Φ of type A_3, $J = \Pi \setminus \{\alpha_2\}$ and $\sigma := w_0^J$ in the notation of Bourbaki [3].

The restriction to $\widetilde{V}$ is a surjective homomorphism from W^1 to $\widetilde{W}$, the subgroup of $O(\widetilde{V})$ generated by the $w_{\widetilde{\alpha}}$, $\alpha \in \Pi \setminus J$. Thus the Weyl group of $\widetilde{\Phi}$ is finite.

2 Construction of a BN-pair

For the definition and properties of Chevalley groups as well as for the standard facts on groups with a BN-pair and their parabolic subgroups, we refer to Carter [4], [5], Steinberg [8] and Bourbaki [3]. We continue with the setting of Section 1. In addition, we use the assumption that the root system $\widetilde{\Phi}$ is of type $A_\ell, \ldots, G_2$ or BC_ℓ in (2.10) below.

2.1. Universal Chevalley groups. Let K be a field and Φ a root system as in (1.1). By $\Phi(K)$ we denote the corresponding universal Chevalley group, defined by the Steinberg generators and relations; see Carter [4, (12.1.1)]. This group has a BN-pair (B,N) with $H := B \cap N$ and $H/N = W$.

For $S \subseteq \Phi$, we define $X_S := \langle X_r \mid r \in S\rangle$. Here $X_r \simeq (K,+)$ is the root subgroup corresponding to the root r.

2.2. Unique P_JNP_J-decomposition. Let $x \in G = \Phi(K)$. We assume that $P_JxP_J = P_JnP_J$, where $n \in N$ with $nH = w \in \mathrm{Stab}(\Phi_J)$.

Then x may be expressed as a product $x = ulnu'$ with $u \in U_J$, $l \in L_J$ and $u' \in U^-_{w,J} := \langle X_r \mid r \in \Phi^+ \setminus \Phi_J,\ w(r) \in \Phi^-\rangle$. Indeed, the other factors in $u' \in U_J$ switch to the left. (We remark, that $U^-_{w,J}$ is not a standard notation.) This decomposition with a given n is unique. Indeed, let $p_1nu_1 = p_2nu_2$ with $p_1, p_2 \in P_J$ and $u_1, u_2 \in U^-_{w,J}$. Then $nu_1u_2^{-1}n^{-1} = p_1^{-1}p_2 \in U_J^- \cap P_J = 1$. Whence $u_1 = u_2$ and $p_1 = p_2$.

2.3. Notation. Let Φ, Π, J and $\sigma : \Phi \to \Phi$ be as in (1.4). We consider the group $G := \Phi(K)$ as defined in (2.1). Let $\eta_\sigma : G \to G$ be an automorphism of G satisfying (0), (1) and (2) of the introduction.

The groups U_J, U_J^- are invariant under η_σ, as well as the parabolic subgroups $P_J = N(U_J)$ and $P_J^- = N(U_J^-)$ and the Levi complement $L_J = P_J \cap P_J^-$. The action of η_σ on the building of L_J is fixed point free.

Assumption (0) is satisfied for any automorphism η_σ of $G = \Phi(K)$ with $\eta_\sigma : x_r(t) \mapsto x_{\sigma(r)}(c_r\bar{t})$, for $r \in \Phi$, $t \in K$. Here $0 \neq c_r \in K$, $r \in \Phi$, and the map $t \mapsto \bar{t}$, is an automorphism of K. Such automorphisms exist for an arbitrary choice of fundamental coefficients c_α, $\alpha \in \Pi$.

We define

$$U^1 := U_J \cap \mathrm{Fix}(\eta_\sigma), \qquad V^1 := U_J^- \cap \mathrm{Fix}(\eta_\sigma),$$
$$G^1 := \langle U^1, V^1 \rangle, \qquad B^1 := P_J \cap G^1, \qquad H^1 := L_J \cap G^1.$$

Note that $B^1 = U^1 H^1$ with nilpotent normal subgroup U^1.

The aim is to construct a BN-pair for G^1. For this, we exhibit a suitable group N^1 first. We use the distinguished representatives of double cosets $W_J w W_J$, see (1.2). We recall from (2.3) that P_J is invariant under η_σ.

2.4. Lemma. *Let $g \in G$ with $\eta_\sigma(g) \in P_J g P_J$. We write $P_J g P_J = P_J n P_J$, where $n \in N$ with $nH = w$, w the shortest element in $W_J w W_J$. Then $w \in C_W(\sigma)$.*

Proof. We have $P_J n P_J = BN_J n N_J B$ by Carter [5, (2.8.1)]. Since $P_J g P_J$ is invariant under η_σ, we obtain $BN_J n N_J B = BN_J \eta_\sigma(n) N_J B$ with $\eta_\sigma(n) \in N$. By Assumption (0), $\eta_\sigma(n)H = \sigma w \sigma^{-1}$. Thus $\sigma w \sigma^{-1} \in W_J w W_J$ with the same (minimal) length as w, whence $\sigma w \sigma^{-1} = w$. ∎

In the following lemma, we use Assumption (1) on the fixed point free action on the L_J-building. We write a left conjugate by an upper index to the left, i. e. ${}^g P_J$ means $g P_J g^{-1}$ and so on.

2.5. Lemma. *Let $g \in G$ with ${}^g P_J$ invariant under η_σ. We write $P_J g P_J = P_J n P_J$, where $n \in N$ with $nH = w$, w the shortest element in $W_J w W_J$. Then $w(J) = J$. In particular, $w \in C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$.*

Proof. We write $g = pnp'$ with $p, p' \in P_J$. For $K := J \cap w(J)$, we have $P_K = U_J(P_J \cap {}^n P_J)$ by Carter [5, (2.8.4)]. Conjugation by p yields that ${}^p P_K = U_J(P_J \cap {}^g P_J)$ is invariant under σ. We write $p = lu$ with $l \in L_J$ and $u \in U_J$. By Carter [5, (2.6.6)], ${}^l(L_J \cap P_K)$ is a parabolic subgroup of L_J. With $u \in U_J \subseteq U_K \subseteq P_K$, we see that ${}^l(L_J \cap P_K) = L_J \cap {}^p P_K$ is invariant under σ. Thus ${}^l(L_J \cap P_K) = L_J$ by Assumption (1) and $P_J = U_J L_J \subseteq P_K$. We obtain $J \subseteq K \subseteq w(J)$, whence $w(J) = J$.

For the last assertion, we note that $g^{-1}\eta_\sigma(g) \in N(P_J) = P_J$, whence $\eta_\sigma(g) \in g P_J$. Now (2.4) applies. ∎

2.6. Lemma. *Let $1 \neq w \in W_{J \cup \mathcal{O}_\alpha}$ with $w \in C_W(\sigma)$ and $w(\Phi_J^+) = \Phi_J^+$. Then $w_0^J w = w_0^{J \cup \mathcal{O}_\alpha}$.*

Proof. The element $w_0^J w \in W_{J \cup \mathcal{O}_\alpha}$ maps Φ_J^+ to Φ_J^-. Since $w \neq 1$, we have $w(\alpha) \in -S(\mathrm{id}, \alpha)$ (as defined in (1.15)). Thus $w_0^J w \in W_{J \cup \mathcal{O}_\alpha}$ maps $S(\mathrm{id}, \alpha)$ to $-S(\mathrm{id}, \alpha)$. We deduce that $w_0^J w = w_0^{J \cup \mathcal{O}_\alpha}$. ∎

2.7. Lemma. *Let $\alpha \in \Pi \setminus J$ and $n_0^{J \cup \mathcal{O}_\alpha} \in N$ with $n_0^{J \cup \mathcal{O}_\alpha} H = w_0^{J \cup \mathcal{O}_\alpha}$. Then there exists $n_{\tilde{\alpha}} \in (n_0^{J \cup \mathcal{O}_\alpha} L_J) \cap G^1$.*

Proof. By Assumption (2) there is $1 \neq x \in X_{-S(\mathrm{id},\alpha)} \cap G^1$. Then $x \in {}^{n_0^{J\cup\mathcal{O}_\alpha}}X_{S(\mathrm{id},\alpha)} \subseteq {}^{n_0^{J\cup\mathcal{O}_\alpha}}B \subseteq BN_{J\cup\mathcal{O}_\alpha}B$ (with the BN-pair axioms in G).

We write $P_JxP_J = P_JnP_J$ where $n \in N$ with $nH = w$, w the shortest element in W_JwW_J. Because of $BN_JnN_JB = P_JnP_J = P_JxP_J \subseteq BN_{J\cup\mathcal{O}_\alpha}B$, we obtain $w \in W_{J\cup\mathcal{O}_\alpha}$. Furthermore $w \neq 1$, since otherwise $x \in P_J \cap U_J^- = 1$.

Since $\eta_\sigma(x) = x$, we may apply (2.5). This yields that $w(J) = J$ and $w \in C_W(\sigma)$. With (2.6) we deduce that $w_0^J w = w_0^{J\cup\mathcal{O}_\alpha}$.

Thus $P_JxP_J = P_JnP_J = P_Jn_0^{J\cup\mathcal{O}_\alpha}P_J$. There is a unique expression $x = uln_0^{J\cup\mathcal{O}_\alpha}u'$ with $u \in U_J$, $l \in L_J$ and $u' \in U^-_{w_0^{J\cup\mathcal{O}_\alpha},J}$ by (2.2).

Since $\eta_\sigma(x) = x$, comparing factors yields that $\eta_\sigma(u) = u$ and $\eta_\sigma(u') = u'$. Thus $u^{-1}x(u')^{-1} = ln_0^{J\cup\mathcal{O}_\alpha} \in G^1$, as desired (as $L_Jn_0^{J\cup\mathcal{O}_\alpha} = n_0^{J\cup\mathcal{O}_\alpha}L_J$). ∎

2.8. The subgroup N^1. We define $n_{\widetilde{\alpha}}$ as in (2.7). The existence of $n_{\widetilde{\alpha}}$ is independent of the particular choice of the preimage $n_0^{J\cup\mathcal{O}_\alpha} \in N$. Notice that $n_{\widetilde{\alpha}}$ is unique modulo H^1.

We set $N^1 := \langle n_0^{J\cup\mathcal{O}_\alpha}, L_J \mid \alpha \in \Pi \setminus J\rangle \cap G^1$. Every $n_{\widetilde{\alpha}}$ normalizes L_J. Whence $N^1 = \langle n_{\widetilde{\alpha}} \mid \alpha \in \Pi \setminus J\rangle H^1$ and H^1 is a normal subgroup of N^1.

Next, we verify that $N^1/H^1 \simeq \widetilde{W}$, where $\widetilde{W}$ was defined in (1.4).

2.9. Lemma. *Let $n^1 \in N^1$. We write $n^1L_J = nL_J$, where $n \in N$ with $nH = w$, w the shortest element in wW_J. Then $w \in C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$ and $w\,|_{\widetilde{V}} \in \widetilde{W}$.*

Thus $\varphi : N^1/H^1 \to \widetilde{W}$ with $\varphi : n^1H^1 \mapsto w\,|_{\widetilde{V}}$ is well-defined and yields an isomorphism.

Proof. Because of $\eta_\sigma(n^1) = n^1$, we have $nL_J = \eta_\sigma(n)L_J$, where $\eta_\sigma(n) \in N$ with $\eta_\sigma(n)H = \sigma w\sigma^{-1}$. Thus $\sigma w\sigma^{-1} \in wW_J$ with same length as w; i.e. $\sigma w\sigma^{-1} = w$. Furthermore, since $n^1 \in N^1$, we deduce that $w \in \langle w_0^{J\cup\mathcal{O}_\alpha}, W_J \mid \alpha \in \Pi \setminus J\rangle \subseteq \mathrm{Stab}(\Phi_J)$. With (1.14) this yields the first claim.

We have shown that φ is a mapping. When w_i is the shortest element in w_iW_J and $w_i \in \mathrm{Stab}(\Phi_J)$ $(i = 1, 2)$, then w_1w_2 is the shortest element in $w_1w_2W_J$ by (1.2). Thus φ is a homomorphism.

By definition we have $n_{\widetilde{\alpha}} \in N^1$ with $n_{\widetilde{\alpha}}L_J = n_0^{J\cup\mathcal{O}_\alpha}L_J$. The shortest element in $w_0^{J\cup\mathcal{O}_\alpha}W_J$ is $w_0^{J\cup\mathcal{O}_\alpha}w_0^J$. Note that w_0^J is contained in $C_{W_J}(\sigma)$, the kernel of the action on $\widetilde{V}$. With (1.14), we deduce that $\varphi(n_{\widetilde{\alpha}}H^1) = w_0^{J\cup\mathcal{O}_\alpha}\,|_{\widetilde{V}} = w_{\widetilde{\alpha}}$. Whence φ is surjective. Finally, φ is injective. Indeed, if n^1H^1 is in the kernel of φ, then the associate w is contained in the kernel of the action on $\widetilde{V}$. Now (1.13) yields that $w \in W_J$ and $n^1 \in L_J \cap G^1 = H^1$. ∎

2.10. $\mathbf{B^1N^1B^1}$-decomposition. *Any $g \in G^1$ may be written in the form $g = bnb'$ with $b, b' \in B^1$ and $n \in N^1$.*

Proof. Let $g \in G^1$, whence $\eta_\sigma(g) = g$. We write $P_J g P_J = P_J n P_J$, where $n \in N$ with $nH = w$, w the shortest element in $W_J w W_J$. By (2.5), we have $w \in C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$. Thus $w\mid_{\widetilde{V}}$ permutes $\widetilde{\Phi}$. From this we deduce as follows that $w\mid_{\widetilde{V}} \in \widetilde{W}$. As $\widetilde{\Phi}$ is of type $A_\ell, \ldots, G_2$ or BC_ℓ, we may write $w\mid_{\widetilde{V}} = \widetilde{w}d$ with $\widetilde{w} \in \widetilde{W}$ and d a diagram symmetry which preserves length. We suppose that $d \neq 1$. We write $\widetilde{w} = w^1\mid_{\widetilde{V}}$ with $w^1 \in W^1$. Then $x := (w^1)^{-1}w \in C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$ and $x\mid_{\widetilde{V}} = d \neq 1$. Thus x is not contained in the kernel of the action on $\widetilde{V}$. As in the proof of (1.13) there exists $\beta \in \Pi \setminus J$ such that $\widetilde{\beta}$ is made negative by x. But x acts as a diagram symmetry, a contradiction.

By (2.9) there exists $n^1 \in N^1$ with $\varphi(n^1H^1) = w\mid_{\widetilde{V}}$. We write $n^1L_J = \overline{n}L_J$, where $\overline{n} \in N$ with $\overline{n}H = \overline{w}$, the shortest element in $\overline{w}W_J$. Then $w^{-1}\overline{w} \in C_{W_J}(\sigma)$, the kernel of the action on $\widetilde{V}$.

Thus $n^1L_J = \overline{n}L_J = nL_J$ and $P_J g P_J = P_J n P_J = P_J n^1 P_J$ with $n^1 \in N^1$. By (2.2) there is a unique expression $x = uln^1u'$ with $u \in U_J$, $l \in L_J$ and $u' \in U^-_{w,J}$.

Since $\eta_\sigma(x) = x$, $\eta_\sigma(n^1) = n^1$ and U_J, L_J and $U^-_{w,J}$ are invariant under η_σ, comparing factors yields that $\eta_\sigma(u) = u$, $\eta_\sigma(u') = u'$ and $\eta_\sigma(l) = l$. Thus $g \in B^1N^1B^1$, as desired. ∎

2.11. Theorem. *The subgroups B^1 and N^1 form a BN-pair for G^1. The associated Weyl group is $N^1/H^1 = \widetilde{W} = \langle w_{\widetilde{\alpha}} \mid \alpha \in \Pi \setminus J\rangle$, where $H^1 = B^1 \cap N^1$.*

Proof. We verify the BN-pair axioms.

(BN1) By (2.10) G^1 is generated by B^1 and N^1.

(BN2) We have $B^1 \cap N^1 \subseteq (P_J \cap NL_J) \cap G^1 \subseteq L_J \cap G^1 = H^1$. Thus $B^1 \cap N^1$ coincides with H^1, whence is a normal subgroup of N^1 by (2.8).

(BN3) By (2.9) we have $N^1/H^1 = \widetilde{W} = \langle w_{\widetilde{\alpha}} \mid \alpha \in \Pi \setminus J\rangle$.

Let $\alpha \in \Pi \setminus J$ and $n_{\widetilde{\alpha}} \in N^1$ with $n_{\widetilde{\alpha}}H^1 = w_{\widetilde{\alpha}}$.

(BN4) We show that $(B^1n_{\widetilde{\alpha}}B^1)(B^1n^1B^1) \subseteq B^1n_{\widetilde{\alpha}}n^1B^1 \cup B^1n^1B^1$, for $n^1 \in N^1$.

Indeed, we have $B^1n_{\widetilde{\alpha}}B^1 \subseteq P_J n_0^{J\cup\mathcal{O}_\alpha} P_J$. Thus the left hand side, A say, is contained in $(P_J n_0^{J\cup\mathcal{O}_\alpha} P_J)(P_J n^1 P_J)$. With (BN4) for G, we obtain $A \subseteq P_J N_{J\cup\mathcal{O}_\alpha} n^1 P_J \cap G^1$. We write $n^1L_J = nL_J$, where $n \in N$ with $nH = w$, w the shortest element in wW_J. Then $w(J) \subseteq \Phi^+$ by (1.2). Furthermore, $w \in C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$ by (2.9). Thus $w(\Phi_J^+) = \Phi_J^+$.

Let $x \in P_J N_{J\cup\mathcal{O}_\alpha} n^1 P_J \cap G^1$. We write $P_J x P_J = P_J n' n P_J$, where $n' \in N$ with $n'H = w'$, w' the shortest element in $W_J w'$. Then $w' \in W_{J\cup\mathcal{O}_\alpha} \subseteq \mathrm{Stab}(\Phi_J)$. As $(w')^{-1}$ is the shortest element in $(w')^{-1}W_J$, we have $(w')^{-1}(J) \subseteq \Phi^+$ by (1.2). Since $(w')^{-1}$ stabilizes Φ_J, we obtain $(w')^{-1}(\Phi_J^+) = \Phi_J^+$. This yields $w'(\Phi_J^+) = \Phi_J^+$.

Together we obtain $w'w(\Phi_J^+) = \Phi_J^+$ and $w'w$ is the shortest element in $W_J w'w W_J$ by (1.2). By (2.4) we deduce $w'w \in C_W(\sigma)$. Since $w \in C_W(\sigma)$, this yields $w' \in C_W(\sigma)$. By (2.6) we obtain $w' = 1$ or $w' = w_0^J w_0^{J\cup\mathcal{O}_\alpha}$. Whence $P_J x P_J$ is one of $P_J n^1 P_J$ or $P_J n_0^{J\cup\mathcal{O}_\alpha} n P_J = P_J n_{\widetilde{\alpha}} n^1 P_J$. As at the end of the proof of (2.10), we obtain that (BN4) holds in G^1.

(BN5) We have ${}^{n_{\tilde{\alpha}}}B^1 \neq B^1$, as by Assumption (2) there is $1 \neq x \in X_{-S(\mathrm{id},\alpha)} \cap G^1 = {}^{n_{\tilde{\alpha}}}(X_{S(\mathrm{id},\alpha)} \cap G^1)$. Thus x is contained in ${}^{n_{\tilde{\alpha}}}B^1$, but not in B^1 (since $P_J \cap U_J^- = 1$). ∎

2.12. Theorem. *If G^1 is perfect, then it is quasi-simple.*

Proof. By (2.11) G^1 has a BN-pair (B^1, N^1) with Weyl group $\widetilde{W}$. Furthermore, $\widetilde{\Pi}$ is indecomposable by assumption. Also U^1 is a nilpotent normal subgroup of B^1.

Let $\widetilde{n}_0 \in N^1$ with $\widetilde{n}_0 H^1 = \widetilde{w}_0$, the longest element in $\widetilde{W}$. Then $V^1 = {}^{\widetilde{n}_0}U^1$ and $G^1 = \langle U^1, V^1 \rangle$ is generated by the conjugates of U^1.

Now the criterion for the quasi-simplicity of groups with a BN-pair of Tits [10, p.319], see also Bourbaki [3], applies. Thus any proper normal subgroup F of G^1 is contained in B^1. We obtain $F \leq B^1 \cap {}^{\widetilde{n}_0}B^1 \subseteq P_J \cap P_J^- = L_J$, whence $F \leq H^1$. But then $[U^1, F] \leq [U^1, H^1] \leq U^1$ and $[U^1, F] \leq F \leq H^1$. Therefore $[U^1, F] \leq U^1 \cap H^1 \leq U_J \cap L_J = 1$ and F centralizes U^1. This yields $F \leq \mathrm{Z}(G)$. ∎

Note that (2.11) and (2.12) prove the main theorem.

We remark that by the standard argument the Weyl group $\widetilde{W}$ is defined by the standard generators and relations. From this we deduce that the action of W^1 on $\widetilde{V}$ is faithful. Furthermore, the group $C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$ is the semidirect product of $W^1 \simeq \widetilde{W}$ and the normal subgroup $C_{W_J}(\sigma)$.

2.13. Root subgroups for G^1. For $\alpha \in \Pi \setminus J$, $w \in C_W(\sigma) \cap \mathrm{Stab}(\Phi_J)$, we recall the definition of the parts $S(w, \alpha)$ in (1.15). The parts are the equivalence classes of $\approx$ by (1.18). Any root in $\widetilde{\Phi}$ is of the form $\widetilde{r}$ with $r \in \Phi \setminus \Phi_J$. Let S_r be the unique part in Φ which contains r, see (1.18). This part is independent of the particular choice of the preimage r.

We define the root subgroup of G^1 associated to $\widetilde{r}$ as

$$U_{\widetilde{r}} := X_{S_r} \cap G^1 = \langle X_s \mid \widetilde{s} = \mu \widetilde{r} \text{ with } \mu > 0 \rangle \cap G^1.$$

By Assumption (2) these root subgroups are non-trivial. Conjugation by $n_{\widetilde{\alpha}}$ interchanges $U_{\widetilde{\alpha}}$ and $U_{-\widetilde{\alpha}}$. Let $S := S(w, \alpha)$ be a part. Then S is invariant under σ and W_J. Furthermore, for $s, s' \in S$ and $r \in \Phi_J$ with $s + s', s + r \in \Phi$, we have $s + s', s + r \in S$. We deduce that X_S is invariant under η_σ and under L_J.

As in Carter [4, (13.6.1), (13.6.5)], we obtain $U^1 = \prod_{\widetilde{r} \in \widetilde{\Phi}^+} U_{\widetilde{r}}$ with uniqueness of expression. Furthermore, $G^1 = \langle U_{\widetilde{\alpha}}, U_{-\widetilde{\alpha}} \mid \alpha \in \Pi \setminus J \rangle$. From this it may be deduced that G^1 satisfies the so-called Moufang condition as stated in Tits [11].

To verify that G^1 perfect, it suffices to show that the root subgroups $U_{\widetilde{\alpha}}$ are vector spaces over some ground field with scalar multiplication defined via the action of diagonal elements; see Tits [10, p. 324].

References

[1] Borel, A.: Linear algebraic groups. 2nd enlarged ed. Graduate Texts in Mathematics, 126. Springer, 1991.

[2] Borel, A., Tits, J.: Groupes reductifs. Publ. Math., Inst. Hautes Étud. Sci. **27** (1965), 659-755 .

[3] Bourbaki, N.: Groupes et algèbres de Lie. Chapitres IV, V et VI. Hermann, Paris, 1968.

[4] Carter, R.: Simple groups of Lie type. Pure and Applied Mathematics. Vol. XXVIII. John Wiley & Sons, 1972.

[5] Carter, R.: Finite groups of Lie type. Conjugacy classes and complex characters. Pure and Applied Mathematics. John Wiley & Sons, 1985.

[6] Mühlherr, B., Van Maldeghem, H.: Exceptional Moufang quadrangles of type F_4. Canad. J. Math. **51** (1999), 347-371.

[7] Springer, T.: Linear algebraic groups. 2nd ed. Progress in Mathematics 9. Birkhäuser, 1998.

[8] Steinberg, R.: Lectures on Chevalley groups. Yale University Lecture Notes, 1967.

[9] Tits, J.: Classification of algebraic semisimple groups. In: Algebraic groups and Discontinuous groups, Boulder 1965. Proc. Symp. Pure Math. **9** (1966), 32-62.

[10] Tits, J.: Algebraic and abstract simple groups. Ann. Math. (2) **80** (1964), 313-329.

[11] Tits, J.: Buildings of spherical type and finite BN-pairs. Lecture Notes in Math. **386**. Springer, Berlin, 1974.

[12] Van Maldeghem, H.: Generalized polygons. Monographs Math. **93**. Birkhäuser, Basel, 1998.

Anja Steinbach
Justus-Liebig-Universität Gießen
Mathematisches Institut
Arndtstraße 2
D 35392 Gießen

Anja.Steinbach@math.uni-giessen.de