A reduction principle for obtaining Tauberian theorems for statistical convergence in metric spaces

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Abstract

We show a general method to translate Tauberian theorems for summability methods in $\mathbf{R}$ into Tauberian theorems for the corresponding forms of statistical convergence in metric spaces. The main tools (distance functions and the Hausdorff metric) come from set-valued analysis.

1 Introduction

Statistical convergence [5] provides a general framework for summability in metric spaces. The original definition of this notion is in terms of the *natural density* of a subset A of $\mathbf{N}$, i.e. the quantity $\lim_n n^{-1} \cdot \mathrm{card}\{k \leq n : k \in A\}$. A sequence $\{x_n\}_n$ of real numbers converges statistically to $\ell \in \mathbf{R}$ if the natural density of $\{n \in \mathbf{N} : |x_n - \ell| \geq \varepsilon\}$ is 0 for every $\varepsilon > 0$.

However, this notion is easy to generalize in several ways. First, spaces other than $\mathbf{R}$ have been considered; some examples are locally convex spaces [11], Banach spaces [4] and metric spaces [7]. Second, the factor n^{-1} in the definition can be replaced by the coefficients of some summability method, e.g. [7, 8] (n^{-1} corresponding to the Cesàro method).

Let $\{c_n(\lambda)\}_{n,\lambda}$ be the coefficients of a summability method (C), where λ is a discrete or continuous parameter, $c_n(\lambda) \geq 0$ and $\sum_{n=1}^{\infty} c_n(\lambda) = 1$ for each λ (in order

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to preserve the probabilistic interpretation). We will say that a sequence $\{x_n\}_n$ in a metric space (X, ρ) is (C)-*statistically convergent* to $\ell \in X$ if

$$\sum_{n:\, \rho(x_n,\ell)\geq\varepsilon} c_n(\lambda) \to 0$$

for every $\varepsilon > 0$.

One can define random variables ξ_λ in $[0, 1)$ (endowed with its Lebesgue measurable structure) such that

$$\xi_\lambda(\omega) = x_k \text{ if } \omega \in [\sum_{n=1}^{k-1} c_n(\lambda), \sum_{n=1}^{k} c_n(\lambda)).$$

Then $\rho(x_n, \ell)$ is (C)-summable to 0 if, and only if $\xi_\lambda \to \ell$ in $L_1[0, 1)$, whereas $x_n \to \ell$ (C)-statistically if, and only if $\xi_\lambda \to \ell$ in probability. Accordingly, (C)-statistical convergence is weaker than ordinary convergence as soon as the method (C) is *regular*, i.e. if ordinary convergence entails (C)-summability. The reader is referred to the papers [3, 9, 10] for more on the interplay between summability and probability.

Fridy [5], and Fridy and Khan [6, 7, 8] have studied Tauberian theorems in relation to statistical convergence. There are two interesting problems in this connection: First, find Tauberian conditions under which (C)-statistical convergence implies ordinary convergence. Then, ascertain whether classical Tauberian theorems can be strengthened by substituting statistical convergence for ordinary convergence in the hypothesis but not in the conclusion. In order to prove that, it suffices to check that summability methods preserve the relevant properties of the original sequence (see [8] for details), then apply the corresponding Tauberian theorem for statistical convergence. However, it is also interesting to remark that Tauberian theorems involving statistical convergence can also depart from their classical counterparts, as shown in [7].

Our aim in this note is to propose a general reduction principle to translate order-type Tauberian theorems for summability methods into Tauberian theorems for statistical convergence. Fridy and Khan [8] have done so for the Cesàro, Abel and Borel methods. However, their approach uses explicitly the coefficients of those methods; in contrast, intuition suggests that the strong link between summability and statistical convergence should allow one to find out a general approach irrespective of the particularities of each method.

2 The reduction principle

For this section, we assume that the method (C) satisfies a Tauberian theorem of the following type: There exists a sequence $\{a_n\}_n \subset \mathbf{R}$ such that for any sequence $\{b_n\}_n \subset \mathbf{R}$, the conditions

i) b_n is (C)-summable to ℓ,

ii) $b_n - b_{n+1} = O(a_n)$,

imply that $b_n \to \ell$. Our aim is to show that the analogous condition $\rho(x_n, x_{n+1}) = O(a_n)$ is Tauberian in the sense that it, together with (C)-statistical convergence, implies ordinary convergence. Since we wish the exposition of the reduction principle to be as clear and transparent as possible, we have chosen this form of the Tauberian theorem by its formal simplicity. Some variants are discussed in the next section.

We will use some elementary notions from set-valued analysis. The reader is referred to [1, 2] for more details. The *ε-envelope* of $K \subset X$ is defined to be the set $K^\varepsilon = \{x \in X : \rho(x, K) < \varepsilon\}$, where $\rho(x, K) = \inf_{y \in K} \rho(x, y)$ is called the *distance function* associated to K. The space $\mathcal{C}$ of all non-empty closed subsets of X can be endowed with the Wijsman topology, i.e. the weak topology generated by all distance functionals $\rho(x, \cdot)$. The space $\mathcal{K}$ of all non-empty closed bounded subsets of X is endowed with the Hausdorff metric

$$d_H(K, L) = \inf\{\varepsilon > 0 : \, K \subset L^\varepsilon, L \subset K^\varepsilon\}.$$

It is well known that

$$d_H(K, L) = \sup_{x \in X} |\rho(x, K) - \rho(x, L)|,$$

therefore the topology of the Hausdorff metric is finer than the Wijsman topology. Also,

$$d_H(K^\varepsilon, L^\varepsilon) \leq d_H(K, L)$$

for every $\varepsilon > 0$.

The reduction principle has two steps: (a) Obtain a version of the above Tauberian theorem for the distance functions associated to a sequence K_n of closed subsets of X (Lemma 1, which can be seen as a Tauberian theorem for the Wijsman topology). (b) Apply it taking K_n to be the $(\varepsilon/2)$-envelope of the singleton $\{x_n\}$ (Theorem 2).

Lemma 1. *Let $\{x_n\}_n \subset X, \ell \in X$ and $\{K_n\}_n \subset \mathcal{C}$. Then, the conditions*

i) $\sum_{n:\, \ell \notin K_n} c_n(\lambda) \to 0$,

ii) $\rho(\ell, K_n) - \rho(\ell, K_{n+1}) = O(a_n)$,

imply that $\rho(\ell, K_n) \to 0$.

Proof. Define the bounded metric $\rho' = \rho/(1+\rho)$, which is equivalent to ρ. Moreover,

$$\rho'(\ell, K_n) - \rho'(\ell, K_{n+1}) = \frac{\rho(\ell, K_n) - \rho(\ell, K_{n+1})}{(1 + \rho(\ell, K_n))(1 + \rho(\ell, K_{n+1}))} = O(a_n).$$

Since $\rho' \leq 1$ and $\rho'(\ell, K_n) = 0$ if, and only if $\ell \in K_n$,

$$\sum_{n=1}^{\infty} c_n(\lambda)\rho'(\ell, K_n) = \sum_{n:\, \ell \notin K_n} c_n(\lambda)\rho'(\ell, K_n) \leq \sum_{n:\, \ell \notin K_n} c_n(\lambda) \to 0.$$

By the Tauberian theorem assumed to hold, it follows that $\rho'(\ell, K_n) \to 0$. Thus, also $\rho(\ell, K_n) \to 0$. ∎

Theorem 2. *Let $\{x_n\}_n \subset X$ and $\ell \in X$. Then, the conditions*

i) $x_n \to \ell$ (C)*-statistically*

ii) $\rho(x_n, x_{n+1}) = O(a_n)$,

imply that $x_n \to \ell$.

Proof. Let $\varepsilon > 0$ and set $K_n = K_n(\varepsilon) = \{x_n\}^{\varepsilon/2}$. Let us check that Lemma 1 can be applied. Actually,

$$\sum_{n:\, \ell \notin K_n} c_n(\lambda) = \sum_{n:\, \rho(x_n,\ell) \geq \varepsilon/2} c_n(\lambda) \to 0$$

according to (i) and

$$\begin{aligned}|\rho(\ell, K_n) - \rho(\ell, K_{n+1})| &\leq d_H(K_n, K_{n+1}) = d_H(\{x_n\}^{\varepsilon/2}, \{x_{n+1}\}^{\varepsilon/2}) \\ &\leq d_H(\{x_n\}, \{x_{n+1}\}) = \rho(x_n, x_{n+1}) = O(a_n).\end{aligned}$$

by (ii). Accordingly, $\rho(\ell, K_n) \to 0$. This implies that there exists $n_0 \in \mathbf{N}$ such that

$$\rho(\ell, \{x_n\}^{\varepsilon/2}) < \varepsilon/2$$

for all $n \geq n_0$. It follows easily now that $\rho(x_n, \ell) < \varepsilon$. Therefore, $x_n \to \ell$. ∎

3 Concluding remarks

Let us remark that also one-sided conditions can be dealt with. For instance, take the condition

$$a_n(x_n - x_{n+1}) \leq C$$

with $X = \mathbf{R}$. One just has to modify the proof by replacing $\rho(\ell, K_n) - \rho(\ell, K_{n+1})$ by its positive part $[\rho(\ell, K_n) - \rho(\ell, K_{n+1})]_+$ and the Hausdorff metric d_H by the *excess functional*

$$e(K, L) = \inf\{\varepsilon > 0 : K \subset L^\varepsilon\} = \sup_{x \in X} [\rho(x, K) - \rho(x, L)]_+.$$

In fact, similar conditions can be established in other metric spaces, e. g. $\mathbf{R}^k$ with the coordinatewise order or $\mathcal{K}$ with the excess functional (taken as a one-sided version of the Hausdorff metric).

On the other hand, more sophisticated conditions can be analogously dealt with, e.g. the condition

$$\lim_{\delta \searrow 0} \limsup_n \max_{n \leq m < n + \delta \cdot a_n} \rho(x_m, x_n) = 0,$$

where $\{a_n\}_n$ is a sequence tending to ∞.

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