Hopf algebroids and Galois extensions

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Abstract

To a finite Hopf-Galois extension $A|B$ we associate dual bialgebroids $S :=$ End $_BA_B$ and $T := (A \otimes_B A)^B$ over the centralizer R using the depth two theory in [18, Kadison-Szlachányi]. First we extend results on the equivalence of certain properties of Hopf-Galois extensions with corresponding properties of the coacting Hopf algebra [21, 8] to depth two extensions using coring theory [3]. Next we show that T^{op} is a Hopf algebroid over the centralizer R via Lu's theorem [23, 5.1] for smash products with special modules over the Drinfel'd double, the Miyashita-Ulbrich action, the fact that R is a commutative algebra in the pre-braided category of Yetter-Drinfel'd modules [28] and the equivalence of Yetter-Drinfel'd modules with modules over Drinfel'd double [24]. In our last section, an exposition of results of Sugano [29, 30] leads us to a Galois correspondence between sub-Hopf algebroids of S over simple subalgebras of the centralizer with finite projective intermediate simple subrings of a finite projective H-separable extension of simple rings $A \supseteq B$.

1 Introduction

The notion of a Hopf-Galois extension was introduced by Kreimer and Takeuchi in 1981 [21] as a generalization of Galois extensions of fields, commutative rings and noncommutative rings, and studied in connection with affineness theorems for

algebraic groups, non-normal separable field extensions and Takesaki duality in operator algebras by Schneider, Greither-Pareigis, Blattner-Montgomery and others. Finite Hopf-Galois extensions have a theory similar to that of depth two finite index subfactors in the von Neumann algebra theory of "continuous geometry," the explanation being that both are depth two ring extensions [17, 18].

Hopf algebroids over noncommutative rings were introduced by Lu [23] in connection with quantization of Poisson groupoids in Poisson geometry. Examples of Hopf algebroids are first and foremost Hopf algebras and groupoid algebras but more significantly come from solutions to dynamical Yang-Baxter equations [11], weak Hopf algebras [2, 10], finite index subfactors [11] and in the study of the non-flat case of index theory for transversally elliptic operators [7, 1].

A bialgebroid S, i.e., a Hopf algebroid without antipode, and its R-dual T has been associated with a depth two ring extension $A|B$ with centralizer R in Kadison-Szlachányi [18]. S acts from the left on the over-ring A such that the right endomorphism ring is isomorphic to a smash product $A \rtimes S$ [18]. Moreover, T acts from the right on the left endomorphism ring $\mathcal{E}$ [18] such that the endomorphism ring $\text{End} \, {}_A A \otimes_B A$ is similarly isomorphic to a smash product $T \ltimes \mathcal{E}$, which leads to a Blattner-Montgomery duality result if the extension $A|B$ is also Frobenius [16].

In this paper we show via Lu's theorem [23, 5.1] that the bialgebroid T^{op} of an H-Galois extension A with subring of invariants B has Hopf algebroid structure over R. In order to frame it in terms of Lu's hypotheses, the proof makes use of Miyashita-Ulbrich action, Yetter-Drinfel'd modules and Drinfel'd doubles. It is perhaps interesting to mention that Lu's theorem is a quantization of another theorem by Lu in Poisson geometry [22, 23, 1.2] via a dictionary between Poisson geometry and noncommutative algebra [23]. In section 3 we establish some theorems that inform us when depth two extension A/B are separable or Frobenius judging from the dual properties of the underlying R-corings of the acting bialgebroids S or T. In a final expository section of this paper, we show that a one-sided f.g. projective H-separable extension of simple rings, such as special finite Jones index subfactors with simple relative commutant, enjoys a Galois correspondence between intermediate simple rings forming f.g. projectives with the overring, and Hopf subalgebroids over the simple subalgebras of the centralizer. This depends on Sugano's one-to-one correspondence between the intermediate simple subrings and simple subalgebras of the centralizer of the full H-separable extension [29, 30], with its roots in work on certain classical inner Galois theories of simple artinian rings and division rings by Jacobson, Bourbaki, Tominaga and others. We hope that this exposition will be a first step toward an algebraic generalization of the Galois correspondence by Nikshych and Vainerman between finite depth and index intermediate subfactors and coideal subalgebras of a weak *-Hopf algebra [26].

2 Dual bialgebroids over the centralizer

In this section we review the basics of the dual bialgebroid constructions in [18], while computing the bialgebroids of a finite Hopf-Galois extension as a running example.

Let B be a unital subring of A, an associative noncommutative ring with unit, or an image of a ring homomorphism $\overline{B} \to A$. Recall that the ring extension $A|B$ is said to be of *depth two* if

$$A \otimes_B A \oplus * \cong \oplus^n A$$

as natural B-A and A-B-bimodules [18]. Equivalently, there are elements $\beta_i \in S := \text{End} \, {_B}A_B$, $t_i \in T := (A \otimes_B A)^B$ (called a *left D2 quasibasis*) such that $(a, a' \in A)$

$$a \otimes a' = \sum_{i=1}^n t_i \beta_i(a) a', \tag{1}$$

and a right D2 quasibasis $\gamma_j \in S$, $u_j \in T$ such that

$$a \otimes a' = \sum_j a \gamma_j(a') u_j. \tag{2}$$

Fix both D2 quasibases for our work in this paper.

Example 2.1. *Consider a Hopf-Galois extension $A|B$ with n-dimensional Hopf k-algebra H [21] with k an arbitrary field. Our convention is that H^* acts from the left on A with subalgebra of invariants B, or equivalently, there is a dual right coaction $A \to A \otimes_k H$, $a \mapsto a_{(0)} \otimes a_{(1)}$: the Galois isomorphism $\beta : A \otimes_B A \xrightarrow{\cong} A \otimes_k H$ given by $\beta(a \otimes a') = a a'_{(0)} \otimes a'_{(1)}$, which is an A-B-bimodule, right H-comodule morphism. It follows that $A \otimes_B A \cong \oplus^n A$ as A-B-bimodules. As B-A-bimodules there is similarly an isomorphism $A \otimes_B A \cong \oplus^n A$ by making use of the opposite Galois isomorphism β' given by $\beta'(a \otimes a') = a_{(0)} a' \otimes a_{(1)}$.*

Now compute a right D2 quasibasis $\{\gamma_i\}$, $\{u_i\}$ for $A|B$. Let $\{h_i\}$, $\{p_i\}$ be dual k-bases in H, H^*, respectively. Define $\gamma_i \in \text{End} \, {_B}A_B$ by $\gamma_i(a) := p_i \cdot a$ $(a \in A, i \in \{1, 2, \ldots, n\})$. Let $u_i := \beta^{-1}(1 \otimes h_i) \in (A \otimes_B A)^B$. We verify this: $(a, a' \in A)$

$$
\begin{aligned}
\sum_i a \gamma_i(a') u_i &= \sum_i a(p_i \cdot a') \beta^{-1}(1 \otimes h_i) \\
&= \sum_i a a'_{(0)} p_i(a'_{(1)}) \rangle \beta^{-1}(1 \otimes h_i) \\
&= \beta^{-1}(a a'_{(0)} \otimes a'_{(1)}) \\
&= a \otimes a'
\end{aligned}
$$

The paper [18] found a bialgebroid with action and smash product structure within the Jones construction above a depth two ring extension $A|B$. Namely, if R denotes the centralizer of B in A, a left R-bialgebroid structure on S is given by the composition ring structure on S with source and target mappings corresponding to the left regular representation $\lambda : R \to S$ and right regular representation $\rho : R^{\text{op}} \to S$, respectively. Since these commute $(\lambda(r)\rho(r') = \rho(r')\lambda(r)$ for every $r, r' \in R)$, we may induce an R-bimodule structure on S solely from the left by

$$r \cdot \alpha \cdot r' := \lambda(r)\rho(r')\alpha = r\alpha(?)r'.$$

Now an R-coring structure $\mathcal{S} = (S, \Delta, \varepsilon)$ is given by

$$\Delta(\alpha) := \sum_i \alpha(\,?\,t_i^1)t_i^2 \otimes_R \beta_i \tag{3}$$

for every $\alpha \in S$, denoting $t_i = t_i^1 \otimes t_i^2 \in B$ by suppressing a possible summation, and

$$\varepsilon(\alpha) = \alpha(1) \tag{4}$$

satisfying the additional axioms of a bialgebroid (cf. section 4), such as multiplicativity of Δ and a condition that makes sense of this requirement. We have the equivalent formula for the coproduct [18, Th'm 4.1]:

$$\Delta(\alpha) := \sum_j \gamma_j \otimes_R u_j^1 \alpha(u_j^2\,?\,) \tag{5}$$

Since $S \otimes_R S \cong \operatorname{Hom}_{B-B}(A \otimes_B A, A)$ via $\alpha \otimes \beta \mapsto (a \otimes a' \mapsto \alpha(a)\beta(a'))$, we have the simpler formula via identication,

$$\Delta(\alpha)(a \otimes a') = \alpha(aa'), \tag{6}$$

which clearly shows this bialgebroid structure on S to be a generalization to depth two ring extensions of Lu's bialgebroid $\operatorname{End}_k C$ over a finite dimensional k-algebra C (cf. section 4).

Example 2.2. *We determine the R-bialgebroid S for the Hopf-Galois extension $A|B$ introduced above. It is well-known (see for example [25]) that the right endomorphism ring is a smash product:*

$$A \rtimes H^* \cong \operatorname{End} A_B \tag{7}$$

via $a \rtimes p \mapsto \lambda(a) \circ (p \cdot\,?\,)$. This is an A-B-isomorphism (where $a'(a \otimes p)b := a'ab \otimes p$ and $\operatorname{End} A_B$ is the natural A-A-bimodule). The B-centralizer in $\operatorname{End} A_B$ is of course $(\operatorname{End} A_B)^B = S$, whence

$$\Phi : S \xrightarrow{\;\cong\;} R \rtimes H^* \tag{8}$$

with multiplication given by the smash product:

$$(r \rtimes p)(r' \rtimes p') = r(p_{(1)} \cdot r') \rtimes p_{(2)}p'. \tag{9}$$

If $t \in H, T \in H^*$ denote a dual pair of left integrals (where $T(t) = 1$), and $\sum_i x_i \otimes y_i = \beta^{-1}(1 \otimes t)$, $\Phi(\alpha) = \sum_i (\alpha(x_i) \rtimes T)(y_i \rtimes 1)$ for $\alpha \in S$ (cf. [25]).

The induced R-coring structure is (the trivial structure except for the more complex right R-module action) given by $\tilde{s}(r) = r \rtimes 1$,

$$\tilde{t}(r) = \Phi(\rho(r)) = \sum_i x_i r(T_{(1)} \cdot y_i) \rtimes T_{(2)},$$

with coproduct

$$\begin{aligned}
\Delta(r \rtimes p) &= (\Phi \otimes \Phi)(\sum_i \gamma_i(?) \otimes_R u_i^1 r(p \cdot (u_i^2\,?\,))) \\
&= (\Phi \otimes \Phi)(\sum_i \gamma_i(?)u_i^1 r(p_{(1)} \cdot u_i^2) \otimes_R (p_{(2)} \cdot\,?\,)) \tag{10} \\
&= r \rtimes p_{(1)} \otimes p_{(2)}, \tag{11}
\end{aligned}$$

and counit

$$\varepsilon(r \rtimes p) = r(p \cdot 1_A) = r\varepsilon(p).$$

The formula for Δ makes use of the depth two eq. (2).

The left action of S on A is very simply given by evaluation,

$$\alpha \rhd a = \alpha(a). \tag{12}$$

This action has invariant subring (of elements $a \in A$ such that $\alpha \rhd a = \varepsilon(\alpha)a$) equal precisely to B if the natural module A_B is balanced [18]. This action is measuring because $\alpha_{(1)}(a)\alpha_{(2)}(a') = \alpha(aa')$ by eq. (6).

The smash product $A \rtimes' S$, which is $A \otimes_R S$ as abelian groups with associative multiplication given by eq. (9), is isomorphic as rings to $\operatorname{End} A_B$ via $a \otimes_R \alpha \mapsto \lambda_a \alpha$ [18].

Example 2.3. *For the H-Galois extension $A|B$ just introduced, the action of S on A under the isomorphism $S \cong R \rtimes H^*$ is just given by $(r \rtimes p) \cdot a = r(p \cdot a)$. The smash product of A with the bialgebroid $R \rtimes H^*$ just recovers the ordinary smash product of A with H^*:*

$$A \rtimes' (R \rtimes H^*) \cong A \rtimes H^*$$

as ring isomorphism by an easy exercise.

For any subring B in ring A, the construct $T = (A \otimes_B A)^B$ ("the B-central tensor-square of A over B") has a unital ring structure induced from $T \cong \operatorname{End}_A(A \otimes_B A)_A$ via $F \mapsto F(1 \otimes 1)$, which is given by

$$tt' = t'^1 t^1 \otimes t^2 t'^2 \tag{13}$$

for each $t, t' \in T$. There are obvious commuting homomorphisms of R and R^{op} into T given by $r \mapsto 1 \otimes r$ and $r' \mapsto r' \otimes 1$, respectively. From the right, these two "source" and "target" mappings induce the R-R-bimodule structure $_R T_R$ given by

$$r \cdot t \cdot r' = (t^1 \otimes t^2)(r \otimes r') = rtr',$$

the ordinary bimodule structure on a tensor product.

For a D2 extension A/B, there is a right R-bialgebroid structure on T with coring structure $\mathcal{T} = (T, \Delta, \varepsilon)$ given by the two equivalent formulas:

$$\Delta(t) = \sum_i t_i \otimes_R (\beta_i(t^1) \otimes_B t^2) = \sum_j (t^1 \otimes_B \gamma_j(t^2)) \otimes_R u_j \tag{14}$$

$$\varepsilon(t) = t^1 t^2 \tag{15}$$

By [18, Th'm 5.2] Δ is multiplicative and the other axioms of a right bialgebroid are satisfied. Since the D2 conditions yield $T \otimes_R T \cong (A \otimes_B A \otimes_B A)^B$, the coproduct enjoys a Lu generalized formula,

$$\Delta(t) = t^1 \otimes 1 \otimes t^2 \quad (t \in T). \tag{16}$$

Indeed, T is a right-handed generalization of Lu's bialgebroid $C^e = C \otimes_k C^{\operatorname{op}}$ for a finite dimensional k-algebra C, although T, unlike C^e, has in general no antipode.

Example 2.4. *We return to the example of $A|B$ an H-Galois extension, to compute the R-bialgebroid T. Since $\beta : A \otimes_B A \to A \otimes H$ is an A-B-bimodule isomorphism, it follows that $T = A^B \otimes H \cong R \otimes H$ via β. We next study the multiplication $\star$ imposed on $R \otimes H$ by β and the multiplication (13) on T. Let $h, h' \in H$ and $t, t' \in T$ such that $\beta(t) = 1 \rtimes h$ and $\beta(t') = 1 \rtimes h'$. We compute using the fact that β is an H-comodule homomorphism in the last step: $(r, r' \in R, h, h' \in H)$*

$$
\begin{aligned}
(r \otimes h) \star (r' \otimes h') &= \beta(rt)\beta(r't') \\
&= \beta(r't'^1 rt^1 \otimes t^2 t'^2) \\
&= r't'^1 r(t^1 t^2{}_{(0)})t'^2{}_{(0)} \otimes t^2{}_{(1)} t'^2{}_{(1)} \\
&= r'b'^1 rb'^2{}_{(0)} \otimes hb'_{(2)} \\
&= r'(r \lhd h'_{(1)}) \otimes hh'_{(2)} \qquad\qquad (17)
\end{aligned}
$$

where $\lhd$ denotes the Miyashita-Ulbrich action of H on R from the right [33, 9, 28, 15]. (Recall that if $\beta(t) = 1 \otimes h$ then $r \lhd h := t^1 r t^2$.) From this formula for $\star$, we see that β induces an algebra isomorphism,

$$
T^{\mathrm{op}} \cong R \rtimes H^{\mathrm{op}} \qquad\qquad (18)
$$

where the right action by H is equivalent to a left action by H^{op}.

The R-coring structure on $R \rtimes H^{\mathrm{op}}$ induced from T^{op} is (the trivial structure) given by $(b := \beta^{-1}(1 \otimes h))$

$$
\begin{aligned}
\tilde{s}(r) &= r \rtimes 1 \quad (r \in R) & (19) \\
\tilde{t}(r) &= r_{(0)} \rtimes r_{(1)} & (20) \\
\Delta(r \rtimes h) &= r\beta \otimes \beta \Delta_T(\beta^{-1}(1 \otimes h) \\
&= r\beta(b^1 \otimes b^2{}_{(0)} p_i(b^2{}_{(1)})) \otimes \beta(u_i) \\
&= r\beta(b^1 \otimes b^2{}_{(0)}) \otimes b^2{}_{(1)} \\
&= r \rtimes h_{(1)} \otimes h_{(2)} & (21) \\
\varepsilon(r \rtimes h) &= \varepsilon_T(\beta^{-1}(r \otimes h) \\
&= r\varepsilon(h) & (22)
\end{aligned}
$$

The formula for Δ again uses the right H-comodule property of β, while the formula uses the counitality of the H-comodule A with Eq. (15).

Example 2.5. *The first of two special cases of finite Hopf-Galois extensions with normal basis property is naturally a finite dimensional Hopf algebra H coacting on itself via its comultiplication Δ. The coinvariant subalgebra B is the unit subalgebra $k1_H$, $R = H$, the R-bialgebroid S is $\mathrm{End}_k H \cong H \rtimes H^*$ by example 2.2, and the bialgebroid structure is the same as the "Heisenberg double" of H in Lu's [23, section 6], for which Lu finds an antipode and Hopf algebroid structure.*

The Miyashita-Ulbrich action of H on itself from the right is given by ordinary conjugation, $h \lhd a = S(a_{(1)})ha_{(2)}$. Thus the H-bialgebroid structure on T^{op} is given above in example 2.4 — with antipode and Hopf algebroid structure in section 5 below.

The second example of an elementary nature is obtained from groups G and N where N is a normal subgroup of G of index n and G/N its factor group. Given any

field k, the group algebra $A = k[G]$ is Galois over $B = k[N]$ with cocommutative Hopf algebra $H = k[G/N]$. The Galois map $\beta : A \otimes_B A \to A \otimes H$ is given by $\beta(g \otimes g') = gg' \otimes g'N$ for every $g, g' \in G$. Given a set of right coset representatives $g_1, \ldots, g_n$, the prescription for finding right D2 quasibases in example 2.1 yields $u_i = g_i^{-1} \otimes g_i$ and $\gamma_i(g) = 0$ if $gN \neq g_iN$ and $\gamma_i(g) = g$ if $gN = g_iN$. Since $\beta^{-1}(1 \otimes gN) = g^{-1} \otimes g$, the action associated to T^{op} above is the Miyashita action given by $x \triangleleft gN = g^{-1}xg$ where $x \in C_A(B)$.

3 When D2 extensions are separable, split or Frobenius

Given a D2 extension A/B, we made the acquaintance in the previous section of the underlying R-corings $\mathcal{S}$ and $\mathcal{T}$ of the R-bialgebroids $S = \mathrm{End}_{B-B} A$ and $T = (A \otimes_B A)^B$, respectively. In this section we show that coring properties of $\mathcal{S}$ or $\mathcal{T}$ such as coseparability determine properties of A/B such as separability, and vice versa.

For the next theorem, recall that any R'-coring $(\mathcal{C}, \Delta, \varepsilon)$ is *coseparable* if there is an R'-R'-homomorphism $\gamma : \mathcal{C} \otimes_{R'} \mathcal{C} \to R'$ (called a *cointegral*) such that $\gamma(c_{(1)} \otimes c_{(2)}) = \varepsilon(c)$ and $c_{(1)}\gamma(c_{(2)} \otimes c') = \gamma(c \otimes c'_{(1)})c'_{(2)}$ for every $c, c' \in \mathcal{C}$ (cf. [3, 5, 6]).

Theorem 3.1. *Let A/B be a right f.g. projective D2 extension. Then A/B is a separable extension if and only if the R-coring $\mathcal{S}$ is coseparable.*

Proof. ($\Rightarrow$ [5, Example 3.6]) Given separability element $e = e^1 \otimes e^2 \in (A \otimes_B A)^A$ for A/B, define cointegral $\gamma : \mathcal{S} \otimes_R \mathcal{S} \to R$ by $\gamma(\alpha \otimes \beta) = \alpha(e^1)\beta(e^2)$. The rest of the proof follows [5, Example 3.6] and does not require A_B to be finite projective.

Suppose a dual basis for the natural module A_B is given by $\{a_k\}, \{f_k\}$.

($\Leftarrow$) Given cointegral $\gamma : \mathcal{S} \otimes \mathcal{S} \to R$, define $e = \sum_i b_i\gamma(\beta_i \otimes I_A)$ where I_A is the identity map on A and $\{b_i\}, \{\beta_i\}$ is the left D2 quasibases introduced above. Of course, $e \in (A \otimes_B A)^B$; also, ($\alpha \in \mathcal{S}$)

$$\alpha(e^1)e^2 = \sum_i \alpha(b_i^1)b_i^2\gamma(\beta_i \otimes I_A) = \gamma(\sum_i \alpha(b_i^1)b_i^2\beta_i \otimes I_A) = \gamma(\alpha \otimes I_A),$$

whence if $\alpha = I_A$, $e^1e^2 = \gamma(I_A \otimes I_A) = \varepsilon(I_A) = 1_A$ since $\Delta(I_A) = I_A \otimes I_A$.

It follows that $\alpha(e^1)e^2a = \gamma(\alpha \otimes I_A)a$ for $a \in A$, but

$$\alpha(ae^1)e^2 = \alpha_{(1)}(a)\gamma(\alpha_{(2)} \otimes I_A) = \gamma(\alpha \otimes I_A)I_A(a) = \alpha(e^1)e^2a.$$

Since $A \rtimes S \cong \mathrm{End}\, A_B$ via $a \rtimes \alpha \mapsto \lambda(a) \circ \alpha$, it follows that $f(e^1)e^2a = f(ae^1)e^2$ for each $f \in \mathrm{End}\, A_B$. Finally then computing in $A \otimes_B A$:

$$ae = \sum_k a_k \otimes f_k(ae^1)e^2 = \sum_k a_kf_k(e^1) \otimes e^2a = ea,$$

for each $a \in A$, whence e is a separability element of A/B. ∎

Example 3.2. *Suppose again that A/B is an H-Galois extension. The multiplication mapping $A \otimes_B A \to A$ corresponds under the Galois isomorphism β to $A \otimes_k H \to A$ given by $a \otimes h \mapsto a\varepsilon(h)$. It follows from the theorem that A/B is a separable extension iff H is semisimple, since H is semisimple iff the left integral $t \in H$ may be chosen so that $\varepsilon(t) = 1$, whence $e = \beta^{-1}(1 \otimes t)$ is a separability element. This recovers a theorem of Doi [8].*

For the next theorem, we recall that any R-coring $\mathcal{C}$ is *cosplit* if there is $e \in \mathcal{C}^{R'}$ such that $\varepsilon(e) = 1$, i.e., the counit $\varepsilon : \mathcal{C} \to R'$ is a split R'-R'-epi. An ring extension A'/B' is *split* if there is a B'-B'-epimorphism $E : A' \to B'$ such that $E(1) = 1$ (cf. [3, 6]).

Example 3.3. *If A/B is a split extension, the Sweedler A-coring $A \otimes_B A$ [32] is coseparable [3]; similarly one shows that if A/B is D2 and split, $\mathcal{T}$ is coseparable.*

If A/B is separable and D2, then $\mathcal{T}$ is a cosplit R-coring, since a separability element $e \in \mathcal{T}$ satisfies $\varepsilon(e) = e^1 e^2 = 1$ and $e \in \mathcal{T}^R$. Define a ring extension A/B to be Procesi if $BR = A$; e.g., centrally projective extensions or extensions of commutative rings are Procesi. Conversely then, $\mathcal{T}$ cosplit implies A/B is separable if A/B is a D2 Procesi ring extension.

Theorem 3.4. *Suppose A/B is a D2 extension with double centralizer condition $C_A(C_A(B)) = B$. Then A/B is a split extension iff $\mathcal{S}$ is a cosplit R-coring.*

Proof. The proof only requires $C_A(R) = B$ in the direction $\Leftarrow$.

($\Rightarrow$) If $E : A \to B$ splits the inclusion map, then $E \in \mathcal{S}^R$ since $rE(a) = rE(a)$ for each $r \in C_A(B), a \in A$. Moreover, $\varepsilon(E) = E(1) = 1$ and we conclude $\mathcal{S}$ is cosplit.

($\Leftarrow$) Suppose $e \in \mathcal{S}^R$ such that $e(1) = 1$. Since $e(a)r = re(a)$ for $a \in A$, $e(a) \in C_A(R) = B$, whence $e : A \to B$ splits the inclusion $B \hookrightarrow A$. ∎

Example 3.5. *As noted in [16], an H-separable extension is D2. If A/B is an H-separable extension and A_B is balanced, then A/B is D2 and $C_A(C_A(B)) = B$: see Lemma 6.3.*

Another example: if A/B is an H-separable extension of simple rings with A_B f.g. projective, then A/B is D2 and $C_A(C_A(B)) = B$. (Cf. Prop. 6.4.)

It is a problem which would generalize and improve results of Noether-Brauer-Artin on simple rings, if A/B a right progenerator H-separable extension implies A/B is split [31].

Recall that an R'-coring $\mathcal{C}$ is *Frobenius* if there is an R'-R'-coring $\gamma : \mathcal{C} \otimes_{R'} \mathcal{C} \to R'$ and $e \in \mathcal{C}^{R'}$ such that $\gamma(c \otimes e) = \varepsilon(c) = \gamma(e \otimes c)$ and $\gamma(c \otimes c'_{(1)})c'_{(2)} = c_{(1)}\gamma(c_{(2)} \otimes c')$ for every $c, c' \in \mathcal{C}$ (cf. [3, 6]).

Proposition 3.6. *Let A/B be a D2 right progenerator Procesi extension. Then A/B is a Frobenius extension iff $\mathcal{T}$ is a Frobenius coring.*

Proof. ($\Rightarrow$) Suppose $(E : A \to B, e \in (A \otimes_B A)^A)$ is a Frobenius system for A/B; i.e., for each $a \in A$, we have $a = e^1 E(e^2 a) = E(ae^1)e^2$. Define $\gamma : \mathcal{T} \otimes_R \mathcal{T} \cong (A \otimes_B A \otimes_B A)^B \to R$ by $\gamma(a \otimes a' \otimes a'') = aE(a')a'' \in R$. It follows that: $(d \in \mathcal{T})$

$$\gamma(d \otimes e) = \gamma(d^1 \otimes d^2 e^1 \otimes e^2) = d^1 E(d^2 e^1)e^2 = d^1 d^2 = \varepsilon(d),$$

and similarly $\gamma(e \otimes d) = \varepsilon(d)$. Recalling the E-multiplication $\cdot_E$ on $A \otimes_B A$ induced

by $A \otimes_B A \cong \text{End} A_B$, we note:

$$
\begin{aligned}
d_{(1)}\gamma(d_{(2)} \otimes d') &= \sum_i b_i\beta_i(d^1)E(d^2 d'^1)d'^2 \\
&= d^1 \otimes_B E(d^2 d'^1)d'^2 = d \cdot_E d' \\
&= \sum_j \gamma(d \otimes_R d'^1 \otimes_B \gamma_j(d'^2))c_j \\
&= \gamma(d \otimes d'_{(1)})d'_{(2)}
\end{aligned}
$$

($\Leftarrow$) We now assume that $BR = A$ and that A_B is a progenerator. We see from [6, 3.3.10] that $R \to \mathcal{T}^*$ given by $r \mapsto r\varepsilon$ is a Frobenius extension. But the R-dual $\mathcal{T}^* \cong \mathcal{S}$ via $\phi \mapsto \sum_i \phi(b_i)\beta_i$ with inverse

$$
\alpha \mapsto (t \mapsto \alpha(t^1)t^2).
$$

The composite $R \to \mathcal{S}$ is the left regular map $\lambda : R \to \mathcal{S}$, which is therefore Frobenius. Let $\rho_k, \eta_k \in \mathcal{S}$, $E' : \mathcal{S} \to R$ be a Frobenius system satisfying ($\alpha \in \mathcal{S}, r, r' \in R$)

$$
\sum_k \lambda(E'(\alpha\rho_k))\eta_k = \alpha = \sum_k \rho_k\lambda(E'(\eta_k\alpha)) \tag{23}
$$

$$
E'(\lambda(r)\alpha\lambda(r')) = rE'(\alpha)r' \tag{24}
$$

The last equation is equivalent to $E'(\lambda(r\alpha_{(1)}(r'))\alpha_{(2)}) = r\alpha_{(1)}(r')E'(\alpha_{(2)}) = rE'(\alpha)r'$. Since $\alpha(r)b = b\alpha(r)$ for all $b \in B, r \in R, \alpha \in \mathcal{S}$, it follows that for every $a, a' = \sum_i b_i r_i \in A$

$$
a\alpha_{(1)}(a')E'(\alpha_{(2)}) = \sum_i a\alpha_{(1)}(r_i)E'(\alpha_{(2)})b_i = aE'(\alpha)a' \tag{25}
$$

where of course $b_i \in B, r_i \in R$.

We now claim $\lambda : A \hookrightarrow \text{End} A_B$ is a Frobenius extension, from which it follows that A/B is Frobenius by a converse of the endomorphism ring theorem, since A_B is a progenerator [14]. This follows from $A \rtimes \mathcal{S} \cong \text{End} A_B$ via $a \rtimes \alpha \mapsto \lambda(a)\alpha$ and the assumption $A = BR$. Define $E : \text{End} A_B \to A$ by $E(\lambda(a)\alpha) = aE'(\alpha)$. Now $E \in \text{Hom}_{A-A}(\text{End} A_B, A)$ by eq. (25), since $\lambda(a)\alpha\lambda(a') = \lambda(a\alpha_{(1)}(a'))\alpha_{(2)}$. It follows that

$$
\sum_k E(\lambda(a)\alpha\rho_k)\eta_k = \lambda(a)\alpha
$$

by eqs. (23), which also imply (assuming $a = \sum_j b_j r_j \in BR$)

$$
\sum_k \rho_k E(\eta_k\lambda(a)\alpha) = \sum_{k,j} \lambda(r_j)\rho_k\lambda(b_j)E'(\eta_k\alpha) = \lambda(a)\alpha. \tag{26}
$$

∎

4   Hopf algebroids

For the convenience of the reader and the sake of convention, let's recall some facts
about Lu's Hopf algebroid, which consists of a left bialgebroid $(H, R, \tilde{s}, \tilde{t}, \Delta, \varepsilon)$, and
and *antipode* τ for H. H and R are k-algebras and all maps are k-linear. First,
recall from [23] (and compare [4, 18]) that the *source* and *target* maps $\tilde{s}$ and $\tilde{t}$ are
algebra homomorphism and anti-homomorphism, respectively, of R into H such that
$\tilde{s}(r)\tilde{t}(r') = \tilde{t}(r')\tilde{s}(r)$ for all $r, r' \in R$. This induces an R-R-bimodule structure on
H (from the left in this case) by $r \cdot h \cdot r' = \tilde{s}(r)\tilde{t}(r')h$ ($h \in H$). With respect to
this bimodule structure, (H, Δ, ε) is an R-coring (cf. [32]), i.e. with coassociative
coproduct and R-R-bimodule map $\Delta : H \to H \otimes_R H$ and counit $\varepsilon : H \to R$ (also an
R-bimodule mapping). The image of Δ, written in Sweedler notation, is required
to satisfy

$$a_{(1)}\tilde{t}(r) \otimes a_{(2)} = a_{(1)} \otimes a_{(2)}\tilde{s}(r) \qquad (27)$$

for all $a \in H, r \in R$. It then makes sense to require that Δ be homomorphic:

$$\Delta(ab) = \Delta(a)\Delta(b), \quad \Delta(1) = 1 \otimes 1 \qquad (28)$$

for all $a, b \in H$. The counit must satisfy the following modified augmentation law:

$$\varepsilon(ab) = \varepsilon(as(\varepsilon(b))) = \varepsilon(at(\varepsilon(b))), \quad \varepsilon(1_H) = 1_R. \qquad (29)$$

The axioms of a right bialgebroid H' are opposite those of a left bialgebroid in the
sense that H' obtains its R-bimodule structure from the right via its source and tar-
get maps and, from the left bialgebroid H above, we have that $(H^{\mathrm{op}}, R, \tilde{t}^{\mathrm{op}}, \tilde{s}^{\mathrm{op}}, \Delta, \varepsilon)$
in this precise order is a right bialgebroid: for the explicit axioms, see [18, Section 2].

The left R-bialgebroid H is a *Hopf algebroid* (H, R, τ) if $\tau : H \to H$ is an algebra
anti-automorphism (called an *antipode*) such that

1. $\tau\tilde{t} = \tilde{s}$;

2. $\tau(a_{(1)})a_{(2)} = \tilde{t}(\varepsilon(\tau(a)))$ for every $a \in A$;

3. there is a linear section $\eta : H \otimes_R H \to H \otimes_K H$ to the natural projection
   $H \otimes_k H \to H \otimes_R H$ such that:

$$\mu(H \otimes \tau)\eta\Delta = \tilde{s}\varepsilon.$$

The following lemma covers some examples in the literature (e.g. [20, 3.2]).

Lemma 4.1. *If $(H, R, s, t, \Delta, \varepsilon, \tau)$ and $(H', R', s', t', \Delta', \varepsilon', \tau')$ are Hopf algebroids,
then*
$$(H \otimes H', \ R \otimes R', \ s \otimes s', \ t \otimes t', \ (1 \otimes \sigma \otimes 1)\Delta \otimes \Delta', \ \varepsilon \otimes \varepsilon', \ \tau \otimes \tau')$$
is (the tensor) Hopf algebroid.

Proof. The proof is straightforward and left to the reader, σ denoting the twist and
the linear section being given up to two twists by $\eta \otimes \eta'$ if η, η' are the sections for
H and H' as in axiom (3) above.   ∎

Lu's examples of bialgebroids and Hopf algebroids are the following. Given an algebra C over commutative ground ring K such that C is finitely generated projective as K-module, the following two are left bialgebroids over C (with $\otimes = \otimes_K$):

Example 4.2. *The endomorphism algebra $E := \mathrm{End}_K\, C$ with $\tilde{s}(c) = \lambda(c)$, $\tilde{t}(c') = \rho(c')$, coproduct $\Delta(f)(c \otimes c') = f(cc')$ for $f \in \mathrm{End}_K C$ after noting that $E \otimes_C E \cong \mathrm{Hom}_K (C \otimes C, C)$ via $f \otimes g \mapsto (c \otimes c' \mapsto f(c)g(c'))$. The counit is given by $\varepsilon(f) = f(1)$. We see that this is the left bialgebroid S above when $B = K$, a subring in the center of $A = C$.*

Example 4.3. *The ordinary tensor algebra $C \otimes C^{\mathrm{op}}$ with $\tilde{s}(c) = c \otimes 1$, $\tilde{t}(c') = 1 \otimes c'$ with bimodule structure $c \cdot c' \otimes c'' \cdot c''' = cc' \otimes c''c'''$. Coproduct $\Delta(c \otimes c') = c \otimes 1 \otimes c'$ after a simple identification, with counit $\varepsilon(c \otimes c') = cc'$ for $c, c' \in C$. $C \otimes C^{\mathrm{op}}$ is a left C-bialgebroid by arguing as in [23], or [18, $N = K$] since $C|K$ is D2. In addition, $\tau : C \otimes C^{\mathrm{op}} \to C \otimes C^{\mathrm{op}}$ defined as the twist $\tau(c \otimes c') = c' \otimes c$ is an antipode satisfying the axioms of a Hopf algebroid (in addition, $\tau^2 = \mathrm{id}$, an involutive antipode).*

A bialgebroid homomorphism from $(H_1, R_1, s_1, t_1, \Delta_1, \varepsilon_1)$ into $(H_2, R_2, s_2, t_2, \Delta_2, \varepsilon_2)$ consists of a pair of algebra homomorphisms, $F : H_1 \to H_2$ and $f : R_1 \to R_2$, such that four squares commute: $F s_1 = s_2 f$, $F t_1 = t_2 f$, $\Delta_2 F = p(F \otimes F)\Delta_1$ and $\varepsilon_2 F = f \varepsilon_1$, where f induces an R_1-R_1-bimodule structure on H_2 via "restriction of scalars," $p : H_2 \otimes_{R_1} H_2 \to H_2 \otimes_{R_2} H_2$ is the canonical mapping and $F : {}_{R_1} H_{1 R_1} \to {}_{R_1} H_{2 R_1}$ is a bimodule homomorphism since

$$F(r \cdot h \cdot r') = F(s_1(r)t_1(r')h) = s_2(f(r))t_2(f(r'))F(h) = r \cdot_f F(h) \cdot_f r'.$$

If F and f are both inclusions, we say H_1 is a *sub-bialgebroid* of H_2; if moreover H_1 and H_2 are both Hopf algebroids with antipodes τ_1 and τ_2 such that $F\tau_1 = \tau_2 F$, we call H_1 a *Hopf subalgebroid* of H_2. We say that a bialgebroid H is minimal over its base ring R if it has no proper R-subbialgebroid.

5 T^{op} is a Hopf algebroid

In this section, we find an antipode for the bialgebroid T^{op} we associated to the H-Galois extension $A|B$ in Section 2. We apply [23, Theorem 5.1], repeated below without proof for the convenience of the reader, after noting that the centralizer $R = C_A(B)$ is a commutative algebra in the Yetter-Drinfel'd category $\mathcal{YD}_H^H$ of modules-comodules over H [28, 3.1].

Theorem 5.1 (Lu Theorem 5.1 [23]). *Let H' be a Hopf algebra with antipode $\tilde{S}$ and $D(H')$ its Drinfel'd double. Let V be a left $D(H')$-module algebra. Assume that the R-matrix $\sum_i (1 \otimes h_i) \otimes (p_i \otimes 1)$ satisfies the following pre-braided commutativity condition:*

$$\sum_i (p_i \cdot u)(h_i \cdot v) = vu \tag{30}$$

for every $u, v \in V$. Then the obvious smash product algebra $V \rtimes H'$ is a Hopf algebroid over V with R-coring structure and antipode τ given by ($v \in V, h \in H'$)

$$\tilde{s}(v) = v \rtimes 1 \tag{31}$$

$$\tilde{t}(v) = \sum_i (p_i \cdot v) \otimes h_i \tag{32}$$

$$\Delta(v \rtimes h) = v \rtimes h_{(1)} \otimes h_{(2)} \tag{33}$$

$$\varepsilon(v \rtimes h) = \varepsilon(h)v \tag{34}$$

$$\tau(v \rtimes h) = \sum_i (1 \rtimes \tilde{S}(h))\tilde{t}(\tilde{S}^2(h_i) \cdot p_i \cdot v) \tag{35}$$

Theorem 5.2. *The left bialgebroid T^{op} associated to an H-Galois extension $A|B$ is a Hopf algebroid of the type covered in [23, Theorem 5.1].*

Proof. We have seen in example 2.4 that

$$T^{\mathrm{op}} \cong R \rtimes H^{\mathrm{op}}$$

as algebras. Schauenburg [28, 3.1] computes that the centralizer $R \in \mathcal{YD}_H^H$ where $\lhd$ denotes the Miyashita-Ulbrich action of H on R, the coaction $A \to A \otimes H$ restricts to $R \to R \otimes H$, and the two intertwine in the following Yetter-Drinfel'd condition: ($r \in R, h \in H$)

$$(r \lhd h_{(2)})_{(0)} \otimes h_{(1)}(r \lhd h_{(2)})_{(1)} = r_{(0)} \lhd h_{(1)} \otimes r_{(1)} h_{(2)} \tag{36}$$

Moreover, the following pre-braided commutativity is satisfied: ($r, r' \in R$)

$$r'r = r_{(0)}(r' \lhd r_{(1)}). \tag{37}$$

Comparing eq. (36) with the left-right Yetter-Drinfel'd condition [25, 10.6.12], one easily computes that

$$\mathcal{YD}_H^H = {}_{H^{\mathrm{op}}}\mathcal{YD}^{H^{\mathrm{op}}}$$

when we note that right modules over an algebra correspond exactly to left modules over its opposite algebra, and that H^{op} has the same coalgebra structure as H (but with antipode $\overline{S} := \tilde{S}^{-1}$). In other words, there are natural actions of H^{op} and its dual on R from the left; the dual acting via the dual of the coaction (i.e., $p \cdot r = r_{(0)}p(r_{(1)})$) and H^{op} acting via the Miyashita-Ulbrich action. But Majid [24] computes that the left-right Yetter-Drinfel'd condition [25, 10.6.12] is equivalent to the anti-commutation relation in the Drinfel'd double $D(H) = H^{*\,\mathrm{cop}} \bowtie H$ (cf. [25, 19]) given by

$$(1 \bowtie h)(p \bowtie 1) = (h_{(1)} \rightharpoonup p_{(2)}) \bowtie (h_{(2)} \leftharpoonup p_{(1)})$$

where $\rightharpoonup$ and $\leftharpoonup$ denote the right and left coadjoint actions of H on H^* and H^* on H [25, 10.3.1]; whence the left $D(H)$-modules correspond exactly to left-right Yetter-Drinfel'd modules, or equivalently,

$$_{D(H^{\mathrm{op}})}\mathbf{Mod} = {}_{H^{\mathrm{op}}}\mathcal{YD}^{H^{\mathrm{op}}}.$$

Then R is a left $D(H^{\mathrm{op}})$-module; since the coalgebra structure of $D(H^{\mathrm{op}})$ is just $H^* \otimes H$, we see this action is measuring as well. It follows that $R \rtimes H^{\mathrm{op}}$ in example 2.4

is a smash product $V \rtimes H$ of the type satisfying the conditions in Theorem 5.1 with $V = R$ and $H' = H^{\mathrm{op}}$, for $D(H^{\mathrm{op}}$ has the R-matrix $\sum_i (1 \bowtie h_i) \otimes (p_i \bowtie 1)$ (where $\sum_i p_i(x) h_i = x$ for each $x \in H$ and $p_i(h_j) = \delta_{ij}$), so we compute using Eq. (37):

$$\sum_i (p_i \cdot y)(h_i \cdot x) = y_{(0)} p_i(y_{(1)})(x \triangleleft h_i)$$

$$= y_{(0)}(x \triangleleft y_{(1)}) = xy$$

Finally we compute that the bialgebra structure on $R \rtimes H^{\mathrm{op}}$ coming from T^{op} in example 2.4 is identical with that of eqs. (31)-(34).

$$\tilde{t}(r) = \sum_i (p_i \cdot r) \otimes h_i = \sum_i r_{(0)} \otimes h_i p_i(r_{(1)}) = r_{(0)} \otimes r_{(1)}.$$

This and the other R-coring structures are then clearly the same.

We conclude that T^{op} is a Hopf algebroid with antipode τ on $R \rtimes H^{\mathrm{op}}$ given by

$$\tau(r \rtimes h) = (1 \rtimes \overline{S}(h))(r_{(0)} \triangleleft \overline{S}^2(r_{(1)}))_{(0)} \rtimes (r_{(0)} \triangleleft \overline{S}^2(r_{(1)}))_{(1)}. \tag{38}$$

■

6   A Galois correspondence for H-separable extensions of simple rings

Although Hopf-Galois extensions in general lack a main theorem of Galois theory [27], we expose results of Sugano in light of obtaining a Galois correspondence for a depth two cousin of Hopf-Galois extensions, namely H-separable extensions. Their definition and part of the proposition below are due to [12, 13, Hirata]. We will require the Hopf algebroids introduced for H-separable extensions in [16]. We must eventually narrow our focus to certain H-separable extensions of *simple rings*, which in this section will denote rings with no proper two-sided ideals; such a ring is not necessarily artinian or finite dimensional over a field. Again let B be a subring of A with centralizer subring R, endomorphism ring $S = \mathrm{End}\ _B A_B$ and ring $T = (A \otimes_B A)^B$.

Lemma & Definition 6.1. *$A|B$ is H-separable if $A \otimes_B A \oplus * \cong \oplus^n A$ as A-A-bimodules. Equivalently, $A|B$ is H-separable if there are elements $e_i \in (A \otimes_B A)^A$ and $r_i \in R$ (a so-called H-separability system) such that*

$$1 \otimes 1 = \sum_i r_i e_i. \tag{39}$$

We note that $e_i \in T$, and for $a, a' \in A$

$$a \otimes a' = \sum_i e_i \rho_{r_i}(a) a' = \sum_i a \lambda_{r_i}(a') e_i,$$

whence e_i, λ_{r_i} is a right D2 quasibasis and e_i, ρ_{r_i} is a left D2 quasibasis for $A|B$.

For example, given an Azumaya algebra $D|K$ and an arbitrary K-algebra B then $A := D \otimes_K B$ is an H-separable extension of B [12]. If B is a type II_1 factor and $D = M_n(\mathbb{C})$, this example covers all H-separable finite Jones index subfactors $B \subseteq A$ by Proposition 6.2(2) and Proposition 6.4 below.

We next let Z denote the center of A.

Proposition 6.2. *If $A|B$ is an H-separable extension, then*

1. *R is f.g. projective Z-module;*

2. *$A \otimes_Z R^{\mathrm{op}} \cong \mathrm{End}\, A_B$ via $a \otimes r \mapsto \lambda_a \rho_r$;*

3. *$R \otimes_Z R^{\mathrm{op}} \cong S$ via $r \otimes r' \mapsto \lambda_r \rho_{r'}$ is an isomorphism of bialgebroids;*

4. *$A \otimes_B A \cong \mathrm{Hom}_Z(R, A)$ via $a \otimes a' \mapsto \lambda_a \rho_{a'}$.*

5. *$T^{\mathrm{op}} \cong \mathrm{End}\,_Z R$ via $t \mapsto \lambda(t^1)\rho(t^2)$ is an isomorphism of bialgebroids.*

Conversely, if A_B is f.g. projective, the first two conditions imply that A is an H-separable extension of B.

Proof. We offer some short alternative proofs to these facts. R_Z is f.g. projective since for each $r \in R$, we note that $r = r_i e_i^1 r e_i^2$ where summation over i is understood and for each i, $r \mapsto e_i^1 r e_i^2$ defines a map in $\mathrm{Hom}\,_Z(R, Z)$.

The inverse $\mathrm{End}\, A_B \to A \otimes_Z R^{\mathrm{op}}$ to the ring homomorphism above is given by $f \mapsto f(e_i^1)e_i^2 \otimes r_i$, since $f(e_i^1)e_i^2 a r_i = f(a e_i^1)e_i^2 r_i = f(a)$ $(a \in A)$, while $a e_i^1 r e_i^2 \otimes_Z r_i = a \otimes e_i^1 r e_i^2 r_i = a \otimes r$ for $r \in R$ and $a \in A$.

The ring isomorphism $R^e \cong S$ follows from noting the previous isomorphism is an A-A-bimodule morphism. That this ring isomorphism preserves the R-bialgebroid structures on S (cf. Section 2) with respect to Lu's R-bialgebroid structure on $R \otimes_Z R^{\mathrm{op}}$ follows from [16, 5.1].

The inverse $\mathrm{Hom}_Z(R, A) \to A \otimes_B A$ to the ring homomorphism above is given by $g \mapsto g(r_i)e_i$, since for each $a \otimes a' \in A \otimes_B A$, $a r_i a' e_i = a r_i e_i a' = a \otimes a'$, while for each $r \in R$, $g \in \mathrm{Hom}\,(R, A)$, $g(r_i)e_i^1 r e_i^2 = g(r_i e_i^1 r e_i^2) = g(r)$.

The ring isomorphism $\mathrm{End}\,_Z R \cong T^{\mathrm{op}}$ follows from noting that the previous mapping is an A-A-bimodule homomorphism; this preserves the Lu and depth two R-bialgebroid structures by [16, 5.2].

The converse follows from the general fact that

$$H := \mathrm{Hom}\,(A \otimes_B A_A, A_A) \cong \mathrm{End}\, A_B.$$

Since R is f.g. projective over Z, $\mathcal{E} := \mathrm{End}\, A_B \cong A \otimes_Z R$ implies that $\mathcal{E}$ is centrally projective over A:

$$\mathcal{E} \oplus * \cong A \oplus \cdots \oplus A,$$

whence the same is true of $A \otimes_B A \cong \mathrm{Hom}\,(_A H, {}_A A)$, which follows from $A \otimes_B A_A$ being finite projective. ∎

The R-bialgebroid S is in fact an Hopf algebroid since the obvious antipode on R^e (cf. [23, Lu]) is transferred via part (3) of the proposition [16].

We prove a lemma relevant to section 2 but independent of the rest of this section.

Lemma 6.3. *If A/B is H-separable and A_B is balanced, then $C_A(C_A(B)) = B$.*

Proof. Since $R^e \cong \mathcal{S}$, we have for each $\alpha \in S$ an $r^1 \otimes r^2 \in R^e$ such that $\alpha = \lambda(r^1)\rho(r^2)$. Then for $t \in C_A(R)$:

$$\alpha \triangleright t = \lambda(r)\rho(r)t = rr't = \alpha(1)t.$$

So $t \in A^S$, an invariant under the action, whence $t \in B$ since A_B is balanced [18, Theorem 4.1]. ∎

The proposition and theorem below are due to Sugano, recapitulated below in a hopefully useful expository manner.

Proposition 6.4 (Sugano [29]). *Suppose B is a simple ring and subring of A. Then A is a right f.g. projective H-separable extension of B if and only if*

1. *A is a simple ring,*

2. *$C_A(C_A(B)) = B$, and*

3. *$C_A(B)$ is a simple finite dimensional Z-algebra.*

Proof. ($\Rightarrow$) Since $\mathrm{Hom}\,(A_B, B_B) \neq 0$ and B has no non-trivial ideals, the trace ideal for A_B is B, so A_B is a generator. Let $f_i \in \mathrm{Hom}\,(A_B, B_B)$, $a_i \in A$ such that $\sum_i f_i(a_i) = 1_A$. Then the inclusion $\iota : B \to A$ is split as right B-module mapping by $a \mapsto \sum_i f_i(a_i a)$. Let $e : A_B \to B_B$ be a projection. Given a two-sided ideal $I \subset A$, we have

$$I = (I \cap B)A$$

since $x = \sum_i e(r_i x e_i^1) e_i^2$ for $x \in I$ and H-separability system $\{e_i, r_i\}$; but $e(r_i x e_i^1) \in I \cap B$ by Proposition 6.2(2). Then B simple implies $I = 0$, whence A is simple.

Clearly, $B \subseteq C_A(R)$ where $R = C_A(B)$. Let $v \in C_A(R)$ and ϕ denote the isomorphism in Proposition 6.2(4). Then $\phi(v \otimes 1)(r) = \phi(1 \otimes v)(r)$ for every $r \in R$, whence $1 \otimes v = v \otimes 1$ in $A \otimes_B A$. Applying the projection e, we arrive at $v = e(v) \in B$, whence $B = C_A(R)$.

Since A_B is a progenerator, $\mathrm{End}\, A_B$ is also a simple ring by Morita theorems. Then $A \otimes_Z R$ is simple. Since Z is a field by Schur's lemma, it follows that R is a simple (finite dimensional) Z-algebra.

($\Leftarrow$) The map in Proposition 6.2(2), call it ψ, always exists although it may not be an isomorphism. By conditions (1) and (3) however, ψ is a monomorphism from $\Lambda := A \otimes_Z R$ into $\mathrm{End}\, A_B$. It suffices to show that ψ is an isomorphism and A_B is f.g. projective by the converse in Proposition 6.2. If C is the center of R, it follows from $C \otimes_Z C$ being a Kasch ring and R being a C-separable algebra that $\mathrm{Hom}\,(_\Lambda A, {}_\Lambda \Lambda) \neq 0$ [29], whence $_\Lambda A$ is a left generator, therefore right $\mathrm{End}\, {}_\Lambda A$-f.g. projective. But $\mathrm{End}\, {}_\Lambda A = \mathrm{Hom}_{A-R}\,(A, A) \cong C_A(R) = B$, so A_B is f.g. projective. Again, B is simple and $\mathrm{Hom}\,(A_B, B_B) \neq 0$ implies that A_B is also a generator. It follows from Morita theorems applied to $_\Lambda A_B$ that $\Lambda \cong \mathrm{End}\, A_B$ via ψ. ∎

In [15] a (right) HS-separable extension $A|B$ is defined to be H-separable such that the natural module A_B is a progenerator. What we have then seen above is that a right f.g. projective H-separable extension $A|B$, where B is simple, is HS-separable.

Theorem 6.5 (Sugano [29, 30]). *Suppose A is an HS-separable extension of a simple ring B. Then the class of simple Z-subalgebras V of the centralizer R is in one-to-one correspondence with the class of intermediate simple subrings $B \subseteq D \subseteq A$ where A_D is f.g. projective, via the centralizer in A: $D \mapsto C_A(D)$ with inverse $V \mapsto C_A(V)$. Moreover, A over each such intermediate simple ring D is an HS-separable extension.*

Proof. Of course, A is a simple ring by proposition. Given D as in the theorem, we show D is a right relatively separable extension of B in A, i.e., the multiplication mapping $\mu : A \otimes_B D \to A$ is split as an A-D-bimodule epi. For then

$$A \otimes_D A \oplus * \cong A \otimes_B D \otimes_D A = A \otimes_B A,$$

as A-A-bimodules, the latter being isomorphic itself to a direct summand of $A \oplus \cdots \oplus A$; whence $A|D$ is H-separable, in fact HS-separable since D is simple. It follows from the proposition then that $C_A(D)$ is a simple Z-algebra with $C_A(C_A(D)) = D$, which yields half of the theorem.

To show that D is right relatively separable extension of B in A, [30] shows by other means from the hypotheses that $A|D$ is H-separable, hence $C_A(D)$ is simple: not surprisingly then, R is a Frobenius extension of $C_A(D)$, so D is a Frobenius extension of B via an isomorphism, say η, given in Proposition 6.2(4) [30, Theorem 3]. Let $\{E : D \to B, x_i, y_i\}$ be a Frobenius system for $D|B$. Consider the two-sided ideal $I := \sum_i x_i R y_i$ in $C_A(D)$. If $I = 0$, then by Proposition 6.2(4) $\sum_i x_i \otimes y_i = 0$ in $A \otimes_B A$, whence in $D \otimes_B D$ since A_B and $_B D$ are flat modules. It follows that $\sum_i E(x_i) y_i = 0$, which contradicts $\sum_i E(dx_i) y_i = d$ for all $d \in D$. Then $= \sum_i x_i R y_i = C_A(D)$. It follows that there is $r \in R$ such that $\sum_i x_i r \otimes y_i \in A \otimes_B D$ is a right relative separability element which yields a splitting for μ.

The other half of the theorem depends on showing that $C_A(V) = D'$ is a simple ring, which clearly is intermediate to subring B and over-ring A, and furthermore $C_A(D') = V$ as well as $A_{D'}$ being f.g. projective. Since A is a right B-generator, A is left-f.g. projective over $\operatorname{End} A_B \cong A \otimes_Z R$. But R is a f.g. projective V-module, whence A is f.g. projective left $\Omega := A \otimes_Z V$-module. Since Ω is a simple ring, $_\Omega A$ is also a generator, so $A_{D'}$ is a progenerator module and D' is a simple ring by Morita theorems, since $\operatorname{End}_\Omega A \cong C_A(V)$ via $f \mapsto f(1)$.

Now let $V' := C_A(D') \supseteq V$. Clearly there is a mapping $A \otimes_Z V' \to \operatorname{End} A_{D'}$ as in Proposition 6.2(2), which forms commutative squares with two other such mappings $A \otimes_Z V \xrightarrow{\cong} \operatorname{End} A_{D'}$ and $A \otimes_Z R \xrightarrow{\cong} \operatorname{End} A_B$. These squares are joined by inclusions, which forces ψ to be an isomorphism and $A \otimes V = A \otimes V'$ over the field Z, whence $V = V'$. Then $A|D'$ is HS-separable and the correspondence in the theorem is one-to-one.   ∎

We are now in a position to establish a Galois correspondence between intermediate simple rings of $A|B$ and Hopf subalgebroids of S over simple subalgebras of R. The one-to-one correspondence below bears a resemblance to the Jacobson-Bourbaki correspondence for division rings.

Theorem 6.6. *Given an HS-separable extension of simple rings $A|B$, there is a one-to-one correspondence between intermediate simple rings D such that A_D is f.g. projective and Hopf subalgebroids H of S minimal over simple subalgebras $V \subseteq R$. The Galois correspondence is given by $D \mapsto \operatorname{End}_D A_D$, a Hopf algebroid over $C_A(D)$, with inverse given by $H \mapsto A^H$, the fixed points under the canonical action of H.*

Proof. Given a simple intermediate ring D such that A_D is finite projective, we have seen that $A|D$ is an HS-separable extension, hence a depth two right balanced extension of the type considered in [18, Section 4]. It follows that $H := \operatorname{End}_D A_D$ is

a left bialgebroid over $C_A(D)$ such that $A^H = D$ under the action given in Eq. 12; with antipode and Lu Hopf algebroid structure [16] from Proposition 6.2(3). There are clearly inclusions $H \subseteq S$ and another $C_A(D) \subseteq R$ which together show H to be a Hopf subalgebroid of S minimal over $C_A(D)$. Of course, $C_A(D)$ is a simple Z-algebra and $C_A(C_A(D)) = D$ by Proposition 6.4.

Conversely, given a Hopf subalgebroid H of S minimal over a simple subalgebra $V \subseteq R$, we let $D' = C_A(V)$, an intermediate ring between B and A which is simple with $A|D'$ an HS-separable extension by the previous theorem. Now under identification of S with $R \otimes_Z R^{\mathrm{op}}$, we note that $V \otimes_Z V^{\mathrm{op}} \subseteq H$ since $s(v) = v \otimes 1$ and $t(v') = 1 \otimes v'$ for $v, v' \in V$, while $s(v)t(v') = v \otimes v' \in H$ as well. Since $V \otimes V^{\mathrm{op}}$ is a V-bialgebroid and obviously a subbialgebroid of H, the minimality condition forces $H = V \otimes V^{\mathrm{op}}$. Since $\mathrm{End}\,_{D'}A_{D'} = V \otimes V^{\mathrm{op}} = H$ and $A|D'$ is depth two right balanced, it follows from [18] that $A^H = D'$. Therefore the correspondence in the theorem is one-to-one. ∎

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