Relative groups in surgery theory*

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Abstract

In this paper we consider various types of relative groups which naturally arise in surgery theory, and describe algebraic properties of them. Then we apply the obtained results to investigate the splitting obstruction groups LS_* and the surgery obstruction groups LP_* for a manifold pair. Finally, we introduce the lower LS_*– and LP_*–groups, and describe connections between them and the corresponding lower L_*–groups and surgery exact sequence.

1 Introduction

Surgery theory is actually the main method for classifying compact manifolds in higher dimensions. The surgery obstruction groups $L_*(\pi, w)$ were introduced by Wall in [44]. Let us consider a normal map $(b, f) : M \to X$, where X is a simple Poincaré complex of formal dimension n, M is an n–dimensional compact manifold, and $b : \nu_M \to \xi$ is a bundle map which covers $f : M \to X$. Then there is an obstruction $\theta(b, f) \in L_n(\pi_1(X), w)$ for the existence of a simple homotopy equivalence in the normal cobordism class $[(b, f)]$. The Wall results described originally the L_n–groups as functors from the category $F(\mathbf{2}^n, \mathcal{G}pd)$ to the category of abelian groups (see [44, pp.34–36]). The category $\mathcal{G}pd$ was defined in [44, p.35] as the category of oriented groupoids of finite type equipped with orientation maps (see for example [35, p.537]),

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while $\mathbf{2}^n$ (see [43, §0]) is the category of subsets of $\{1, 2, ..., n\}$ whose morphisms are inclusions. In particular, any morphism $f : \pi \to \pi'$ of oriented groupoids induces an exact sequence

$$\cdots \longrightarrow L_n(\pi) \xrightarrow{f_*} L_n(\pi') \longrightarrow L_n(\pi \to \pi') \longrightarrow L_{n-1}(\pi) \longrightarrow \cdots \tag{1.1}$$

where $L_n(\pi \to \pi') = L_n(f)$ are called the *relative Wall groups*, or the *relative groups for the map f_* induced by f*.

Let $f : \pi \to \pi'$ and $g : G \to G'$ be morphisms of oriented groupoids, and let $F : f \to g$ be given by the following commutative diagram:

$$F = \left(\begin{array}{ccc} \pi & \xrightarrow{\quad f \quad} & \pi' \\ \downarrow & & \downarrow \\ G & \xrightarrow[\quad g \quad]{} & G' \end{array} \right).$$

Then the L–groups $L_n(F)$ of the triad F fit in the following exact sequence (see [35] and [44])

$$\cdots \longrightarrow L_n(f) \longrightarrow L_n(g) \longrightarrow L_n(F) \longrightarrow L_{n-1}(f) \longrightarrow \cdots \tag{1.2}$$

This approach gives good possibilities to study systematically the relative L–groups. In fact, Theorem 3.1 of [44] describes the general case of this situation which can be considered as an iteration of the previous two steps.

The L_n–groups and their natural maps are realized on the spectra level (see for example [15], [34], and [44]). Let $f : \pi \to \pi'$ be a morphism of oriented groupoids. Then there exist spectra $\mathbb{L}(\pi)$, $\mathbb{L}(\pi')$, and $\mathbb{L}(f)$ with $\pi_n(\mathbb{L}(\pi)) = L_n(\pi)$, and similar relations for the other spectra. These spectra fit in a cofibration of spectra

$$\mathbb{L}(\pi) \longrightarrow \mathbb{L}(\pi') \longrightarrow \mathbb{L}(f). \tag{1.3}$$

The homotopy long exact sequence of this cofibration gives rise to the exact sequence in (1.1). By using the square F, we can construct a spectrum $\mathbb{L}(F)$ which is the natural cofiber of the map $\mathbb{L}(f) \to \mathbb{L}(g)$ induced by the following homotopy commutative diagram of spectra:

$$\begin{array}{ccccc} \mathbb{L}(\pi) & \longrightarrow & \mathbb{L}(\pi') & \longrightarrow & \mathbb{L}(f) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}(G) & \longrightarrow & \mathbb{L}(G') & \longrightarrow & \mathbb{L}(g). \end{array} \tag{1.4}$$

Note that the maps of the horizontal rows in diagram (1.4) are cofibrations. Iterating this process yields a functor $\mathbb{L}$ from the category $F(\mathbf{2}^n, \mathcal{G}pd)$ to the category of spectra (see [44, p.250]). Moreover, we have isomorphisms

$$\pi_n(\mathbb{L}(F(\mathbf{2}^n, \mathcal{G}pd))) \cong L_n(F(\mathbf{2}^n, \mathcal{G}pd)).$$

Let $p : E \to Y$ be a fibration whose fiber is a compact m–manifold M^m. Then the transfer map

$$p^! : \mathbb{L}(\pi_1(Y)) \to \mathbb{L}(\pi_1(E))$$

can be defined on the spectra level (see [15], [21], [22], [35], and [44, p.252]). So we can construct a cofibration of spectra

$$\mathbb{L}(\pi_1(Y)) \longrightarrow \mathbb{L}(\pi_1(E)) \longrightarrow \mathbb{L}(p^!).$$

The homotopy long exact sequence of this cofibration gives the relative exact sequence of the transfer map $p^!$, that is,

$$\cdots \longrightarrow L_n(\pi_1(Y)) \longrightarrow L_n(\pi_1(E)) \longrightarrow L_n(p^!) \longrightarrow L_{n-1}(\pi_1(Y)) \longrightarrow \cdots$$

In particular, for a $(\Delta^q, \partial\Delta^q)$–bundle map $p : (E, \partial E) \to Y$ (where Δ^q denotes the standard q–simplex), we obtain maps of spectra which fit in the following commutative diagram:

$$\begin{tikzcd}
\mathbb{L}(\pi_1(Y)) \arrow[r, "p^!"] \arrow[d, "p_1^!"] & \Omega^q\mathbb{L}(\pi_1(\partial E) \to \pi_1(E)) \arrow[d] \\
\Omega^{q-1}\mathbb{L}(\pi_1(\partial E)) \arrow[r, equal] & \Omega^{q-1}\mathbb{L}(\pi_1(\partial E)).
\end{tikzcd} \tag{1.5}$$

Let X^n be a compact topological n–manifold, and $Y^{n-q} \subset X^n$ a submanifold of codimension q. Then the splitting obstruction groups $LS_{n-q}(F)$ and the surgery obstructions groups $LP_{n-q}(F)$ were defined in [35] and [40]. These groups depend only on $n-q$ (mod 4), and functorially on the square F formed by oriented fundamental groupoids and morphisms between them:

$$F = \begin{pmatrix} \pi_1(\partial U) & \longrightarrow & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(Y) & \longrightarrow & \pi_1(X) \end{pmatrix} = \begin{pmatrix} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{pmatrix}.$$

Here U denotes a tubular neighbourhood of the submanifold Y in X. Of course, the diagram F is a push-out square by Van Kampen's theorem. We shall only consider the cases when the codimension q is either 1 or 2. Indeed, for $q \geq 3$ the situation degenerates, and it is completely described in terms of the L_*–groups of $\pi_1(X)$ and $\pi_1(Y)$, and direct sums of them (see for example [35, p.563]). Let us consider the diagram

$$\begin{tikzcd}
\mathbb{L}(B) \arrow[r] \arrow[d] & \Omega^q\mathbb{L}(A \to B) \arrow[r] \arrow[d] & \Omega^q\mathbb{L}(C \to D) \arrow[d] \\
\Omega^{q-1}\mathbb{L}(A) \arrow[r, equal] & \Omega^{q-1}\mathbb{L}(A) \arrow[r] & \Omega^{q-1}\mathbb{L}(C)
\end{tikzcd} \tag{1.6}$$

which arises from diagrams (1.4) and (1.5). Now we can define the spectra $\mathbb{L}S(F)$ and $\mathbb{L}P(F)$ which fit in the cofibrations

$$\Omega\mathbb{L}(B) \longrightarrow \Omega^{q+1}\mathbb{L}(C \to D) \longrightarrow \mathbb{L}S(F) \tag{1.7}$$

and

$$\Omega\mathbb{L}(B) \longrightarrow \Omega^q\mathbb{L}(C) \longrightarrow \mathbb{L}P(F) \tag{1.8}$$

with $\pi_i(\mathbb{L}S(F)) \cong LS_i(F)$ and $\pi_i(\mathbb{L}P(F)) \cong LP_i(F)$ (see [9], [14], [23], [25], and [29]). In diagram (1.5) the cofibers of the maps $p^!$ and $p_1^!$ coincide with the spectra

$\mathbb{L}N(\pi_1(\partial E) \to \pi_1(E))$ and $\mathbb{L}P(F)$, respectively (see the papers quoted above). Here the spectrum $\mathbb{L}N$ gives splitting obstruction groups while $\mathbb{L}P(F)$ gives a spectrum in the surgery problem for manifold pairs.

The realization of the LS_*– and LP_*–groups on the spectra level gives us the possibility to obtain a functorial description of these groups which is similar to the description of the L_*–groups as functors from the category $F(\mathbf{2}^n, \mathcal{G}pd)$ to the category of abelian groups. However, the existence of the relative LS_*– and LP_*–groups already permits to obtain deep results in surgery theory as shown for example in [9], [12], [14], [16], and [26]. In the present paper we study systematically the relative splitting obstruction groups and the relative surgery obstruction groups for a manifold pair. Our main results in this direction are collected in the statements of Theorems 2.1, 2.2, and 2.3. As corollaries we obtain new relations between the surgery and splitting obstruction groups of a manifold pair $Y \subset X$ and those of the manifold pair $(Y \subset X) \times M = Y \times M \subset X \times M$ (see Section 3). Following a geometrical approach, we can have many different types of obstruction L^q–groups which depend on the decoration q. The most significant cases are given by the simple L^s–groups, the free L^h–groups, and the projective L^p–groups (see [15], [31], [36], [38–40], and [44]). Here we shall mainly consider the groups with decoration s, and point out what happens for other decorations.

Let us be given a ring R with an involution. Then the involution on the ring $R[z, z^{-1}]$ is generated by the involution on the ring R and $\overline{z} = z^{-1}$. For a ring $R = \mathbb{Z}[\pi_1(X)]$, the relations between the decorated L_*– and L^h_*–groups are given by the Rothenberg exact sequence

$$\cdots \to L_n(R) \to L^h_n(R) \to H^n(\mathbb{Z}_2, Wh(\pi_1(X))) \to \cdots \tag{1.9}$$

and the splitting formula

$$L_n(R[z, z^{-1}]) = L^s_n(R[z, z^{-1}]) = L_n(R) \oplus L^h_{n-1}(R). \tag{1.10}$$

Here $H^n(\mathbb{Z}_2, Wh(\pi_1(X)))$ is the Tate cohomology group of the Whitehead group $Wh(\pi_1(X))$ (see for example [33], [35, pp.648–651], [36], [38], [39], and [40, Appendix C]). The relations between the decorated L^h_*– and L^p_*–groups are given by the splitting formula

$$L^h_n(R[z, z^{-1}]) = L^h_n(R) \oplus L^p_{n-1}(R) \tag{1.11}$$

and by the Rothenberg exact sequence having $\tilde{K}_0(\mathbb{Z}(\pi_1(X)))$ as third member (see [15], [32], [33], and [35, p.649–650]).

Some relations between the decorated $LS^q_*(F)$– and $LP^q_*(F)$–groups for $q = s, h$, and p were obtained in [35, pp.648–650]. In the present paper we obtain new relations between the relative groups $WhS_*(F)$ and $WhP_*(F)$ (compare with [35, Proposition 7.5.2]) which fit in the exact sequences

$$\cdots \to LS_n(F) \to LS^h_n(F) \to WhS_n(F) \to \cdots \tag{1.12}$$

and

$$\cdots \to LP_n(F) \to LP^h_n(F) \to WhP_n(F) \to \cdots \tag{1.13}$$

The main result in this direction is given by Theorem 3.2. Similar properties hold for the relative groups which correspond to the decorations h and p. In some cases concerning with one–sided submanifolds of codimension 1, the corresponding results were obtained in [9], [25], and [29].

The lower $L_*^{<-i>}$–groups were introduced by Ranicki (see [33], [38], and [39]) as a natural analogue of the lower K–groups. The lower L–groups are closely related to the surgery exact sequence and certain maps in L–theory which can be defined geometrically in a very natural way (see [36], and [40, Appendix C]). In Section 4 we introduce the lower $LS_*^{<-i>}$– and $LP_*^{<-i>}$–groups for any $i \geq 1$, and describe some properties of them. These groups preserve all relations to the lower $L_*^{<-i>}$–groups which are similar to those corresponding to the cases when the decoration q is either s, $p =< 0 >$, or $h =< 1 >$. We also describe some relations between the introduced groups and the surgery exact sequence. The main results on this topic are given by Theorems 4.1 and 4.2.

For general references on algebraic K– and L–theory see for example [2], [15], [17], [33], [37], and [45]. Connections between algebraic K–theory and surgery on compact manifolds can be found in [10], [12], [27], [29], [32], [35], [40], and [44] (recent results on surgery theory for compact four–manifolds with special fundamental groups were obtained in [4–7]). For algebraic properties of many decorated obstruction groups arising from the splitting and surgery problems, and their computations we refer for example to [9], [14], [16], [23], [25–30], and [41].

2 Relative LS– and LP–groups

In this section we study some general functorial properties of the LS– and LP–groups. First, we recall the definitions of these groups (for more details see [35], [40], and [44]). We shall only consider the cases $q = 1$ and $q = 2$. In fact, for any $q \geq 3$, the splitting obstruction groups coincide with the surgery obstruction groups, and these groups for a manifold pair split into direct sums of the ordinary obstruction groups (see [35, p.563]). Let us consider a push–out square of groupoids with orientations

$$F = \begin{pmatrix} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{pmatrix}$$

and a $(q-1)$–spherical fibration $p : X \to Y$ which induces the morphism $A = \pi_1(X) \to B = \pi_1(Y)$ of fundamental groupoids with orientations (see [35, §7.2], and [44, §11]). Let Z denote the Eilenberg-MacLane space (with a cellular structure) $K(C, 1)$ which intersects the cylinder M_Y of the map p in X. By Van Kampen's theorem, the fundamental groupoid of the space $M_Y \cup Z$ is isomorphic to D. Then the splitting obstruction groups $LS_n(F)$ and the surgery obstruction groups $LP_n(F)$ for the manifold pair are defined for any n (mod 4). These groups depend functorially on the square F. Following [44, §0], let $\mathbf{2}^n$ denote the category of all lattices of subsets of $\{1, 2, \ldots, n\}$ together with inclusions as morphisms.

Now we introduce the following categories:

i) $S_q\mathcal{G}r_2$ is the category of push-out squares of groupoids which arise from the splitting problem in codimension $q = 2$ (see [23], [35, §7.8], and [44, §12B]). The morphisms of this category are homomorphisms between the above squares which preserve the orientations of all groupoids;

ii) $S_q\mathcal{G}r_1$ is the category of push-out squares of groupoids which arise from the splitting problem for one-sided submanifolds of codimension $q = 1$ (that is, non–trivial bundles) (see [9], [14], [25], [27], [35, §7.6, C], [44, §12C]). The morphisms are defined similarly to case i);

iii) $S_q\mathcal{G}r_{10}$ is the category of push-out squares of groupoids which arise from the splitting problem in codimension $q = 1$ concerning with the case of two–sided separating submanifolds (see [35, §7.6 A], and [44, §12A]). The morphisms are defined similarly to case i);

iv) $S_q\mathcal{G}r_{11}$ is the category of push-out squares which arise from the splitting problem in codimension $q = 1$ along two–sided non–separating submanifolds (see [35, §7.6 B], and [44, §12B]). The morphisms are defined similarly to case i).

Now we can define objects of type $\mathbf{2}^n$ in the categories $S_q\mathcal{G}r_i$ for $i \in \{1, 2, 10, 11\}$. According to [44, §0] an object of type $\mathbf{2}^n$ in a category $\mathcal{C}at$ is a functor from $\mathbf{2}^n$ to $\mathcal{C}at$. The category of such functors is denoted by $F(\mathbf{2}^n, \mathcal{C}at)$ (see [44, p.35]). We shall briefly refer to these functors for $\mathcal{C}at = S_q\mathcal{G}r_i,\ (i \in \{1, 2, 10, 11\})$ as "squares" of type $\mathbf{2}^n$. We also recall the standard functors between categories of type $\mathbf{2}^n$ (see for example [44, §0]). The functor

$$\partial_i : \mathbf{2}^{n-1} \to \mathbf{2}^n \qquad (1 \le i \le n)$$

is induced by the map

$$\partial_i(j) = \begin{cases} j & \text{if} \quad j < i \\ j+1 & \text{if} \quad j \ge i. \end{cases}$$

The functor

$$\delta_i : \mathbf{2}^{n-1} \to \mathbf{2}^n \qquad (1 \le i \le n)$$

is generated by

$$\delta_i(j) = \partial_i(j) \cup \{i\}.$$

Let $F(\mathbf{2}^n, S_q\mathcal{G}r)$ denote one of the categories $F(\mathbf{2}^n, S_q\mathcal{G}r_i)$ for any $i \in \{1, 2, 10, 11\}$, and $\mathbb{S}p$ the category of spectra and homotopy classes of maps (see [43]). Then we have the following result.

Theorem 2.1. *There are functors from $F(\mathbf{2}^n, S_q\mathcal{G}r)$ into the homotopy category of spectra*

$$\mathbb{L}S : F(\mathbf{2}^n, S_q\mathcal{G}r) \to \mathbb{S}p$$

and

$$\mathbb{L}P : F(\mathbf{2}^n, S_q\mathcal{G}r) \to \mathbb{S}p$$

which induce the following homotopy cofibrations of spectra

$$\mathbb{L}S(\partial_i\Phi) \to \mathbb{L}S(\delta_i\Phi) \to \mathbb{L}S(\Phi)$$

and

$$\mathbb{L}P(\partial_i\Phi) \to \mathbb{L}P(\delta_i\Phi) \to \mathbb{L}P(\Phi)$$

for any $\Phi \in F(\mathbf{2}^n, S_q\mathcal{G}r)$, and $i = 1, \ldots, n$, respectively.

Proof. First we recall that an object $\partial_i\Phi$ is by definition the object of type $\mathbf{2}^{n-1}$ which is defined by the composite functor $\Phi\circ\partial_i$ into the same category $S_q\mathcal{G}r$ where the object Φ takes values (see [44, §0]). In a similar way, an object $\delta_i\Phi$ is the object of type $\mathbf{2}^{n-1}$ which is defined by the composite functor $\Phi\circ\delta_i$. By using forgetful functors, every square Φ of type $\mathbf{2}^n$ gives rise to four objects (A,B,C,D) of type $\mathbf{2}^n$ in the category of groupoids. These objects fit in the following commutative diagram

$$\begin{pmatrix} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{pmatrix}$$

where the maps are given by Φ. Then we can consider the relative objects for the category of objects of type $\mathbf{2}^n$ in the category of groupoids, that is, $A\to B$ and $C\to D$.

We apply an inductive argument on n. For $n=1$, the object Φ is given by a morphism of squares of commutative groupoids

$$F_\varnothing = \begin{pmatrix} A_\varnothing & \longrightarrow & C_\varnothing \\ \downarrow & & \downarrow \\ B_\varnothing & \longrightarrow & D_\varnothing \end{pmatrix} \xrightarrow{\varphi} \begin{pmatrix} A_1 & \longrightarrow & C_1 \\ \downarrow & & \downarrow \\ B_1 & \longrightarrow & D_1 \end{pmatrix} = F_1$$

in the corresponding category $S_q\mathcal{G}r_i$ for $i\in\{1,2,10,11\}$. In this case the object A is $A_\varnothing\to A_1$, and similarly for the other objects. We can write the homotopy commutative diagram of spectra:

$$\begin{tikzcd} \Omega\mathbb{L}(B_\varnothing) \arrow[r] \arrow[d] & \Omega^{q+1}\mathbb{L}(C_\varnothing\to D_\varnothing) \arrow[r] \arrow[d] & \mathbb{L}S(F_\varnothing) \arrow[d, "\varphi_*"] \\ \Omega\mathbb{L}(B_1) \arrow[r] & \Omega^{q+1}\mathbb{L}(C_1\to D_1) \arrow[r] & \mathbb{L}S(F_1). \end{tikzcd}$$

Here the horizontal maps arise from diagram (1.7), and the vertical maps are induced by φ. By [43], the map φ_* is well–defined. We shall denote the cofiber of φ_* by $\mathbb{L}S(\Phi)$. In this case, we have only $i=1$. By construction, the spectrum $\mathbb{L}S(\Phi)$ fits in the cofibration of spectra

$$\mathbb{L}S(\partial_i\Phi) \longrightarrow \mathbb{L}S(\delta_i\Phi) \longrightarrow \mathbb{L}S(\Phi)$$

as $\partial_i\Phi=\Phi\circ\partial_1=F_\varnothing$ and $\delta_i\Phi=\Phi\circ\delta_1=F_1$. These arguments are preserved by the inductive step. Now we must consider a spectrum $\Omega\mathbb{L}(B)$ for the object B of type $\mathbf{2}^n$ in the category of groupoids. Hence we have spectra $\Omega\mathbb{L}(\partial_iB)$ and $\Omega\mathbb{L}(\delta_iB)$ and a cofibration [44, Theorem 3.1]

$$\Omega\mathbb{L}(\partial_iB)\to\Omega\mathbb{L}(\delta_iB)\to\Omega\mathbb{L}(B).$$

We have also an Ω–spectrum $\Omega^{q+1}\mathbb{L}(C\to D)$ for the relative object $C\to D$. Similarly to the absolute case, we obtain a cofibration of spectra

$$\Omega^{q+1}\mathbb{L}(\partial_iC\to\partial_iD)\to\Omega^{q+1}\mathbb{L}(\delta_iC\to\delta_iD)\to\Omega^{q+1}\mathbb{L}(C\to D).$$

By the inductive assumption we can write down a commutative diagram

$$\begin{array}{ccccc}
\Omega\mathbb{L}(\partial_i B) & \longrightarrow & \Omega^{q+1}\mathbb{L}(\partial_i C \to \partial_i D) & \longrightarrow & \mathbb{L}S(\partial_i \Phi) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega\mathbb{L}(\delta_i B) & \longrightarrow & \Omega^{q+1}\mathbb{L}(\delta_i C \to \delta_i D) & \longrightarrow & \mathbb{L}S(\delta_i \Phi) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega\mathbb{L}(B) & \longrightarrow & \Omega^{q+1}\mathbb{L}(C \to D) & \longrightarrow & \mathbb{L}S(\Phi)
\end{array}$$

where the right upper vertical map exists by [43]. The right vertical column gives the necessary cofibration. The case of the $\mathbb{L}P$–spectrum can be treated in a similar way. It is necessary only to consider the cofibration in (1.8) instead of that in (1.7). So the proof is complete. ■

Let $\mathcal{A}b$ be the category of abelian groups and morphisms between them. Theorem 2.1 implies immediately the following result.

Corollary 2.1. *There are functors*

$$LS_m : F(\mathbf{2}^n, S_q\mathcal{G}r) \to \mathcal{A}b$$

and

$$LP_m : F(\mathbf{2}^n, S_q\mathcal{G}r) \to \mathcal{A}b$$

which induce the long exact sequences

$$\cdots \longrightarrow LS_m(\partial_i\Phi) \longrightarrow LS_m(\delta_i\Phi) \longrightarrow LS_m(\Phi) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow LP_m(\partial_i\Phi) \longrightarrow LP_m(\delta_i\Phi) \longrightarrow LS_m(\Phi) \longrightarrow \cdots$$

for any object Φ of type $\mathbf{2}^n$ in the category $S_q\mathcal{G}r$, and for any $i = 1, \ldots, n$, where the subscripts m are taken mod 4.

Proof. We use notation $LS_m(\Phi) = \pi_m(\mathbb{L}S(\Phi))$, and similarly for the other groups. Now the result follows from Theorem 2.1 by considering the homotopy long exact sequences and the standard properties of the LS– and LP–groups. ■

Any push-out square of groupoids

$$F = \begin{pmatrix} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{pmatrix}$$

for a splitting problem in codimension $q \in \{1, 2\}$ defines a push–out square

$$\Psi = \begin{pmatrix} A & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \end{pmatrix}$$

in the same category of squares. Furthermore, there is a natural map $\Psi \to F$ (see [44], p.146). The obstruction groups $LS_*(\Psi)$ are denoted by $LN_*(A \to B)$ (see again [44, p.147]). In a similar way, we can define a full subcategory $SR\mathcal{G}r_i$ of the category $S_q\mathcal{G}r_i$ for $i \in \{1, 2, 10, 11\}$. Thus for any object Φ of type $\mathbf{2}^n$ in the category $S_q\mathcal{G}r_i$, we obtain an object Ψ of type $\mathbf{2}^n$ in the subcategory $SR\mathcal{G}r_i$ and a natural morphism $\Psi \to \Phi$ in the category $S_q\mathcal{G}r_i$. Then we shall use notation $\mathbb{L}S(\Psi) := \mathbb{L}N(A \to B)$. In the statement of the next theorem we use notations from Theorem 2.1.

Theorem 2.2. *Let Φ be an object of type $\mathbf{2}^n$ in the category $S_q\mathcal{G}r_i$ for any $i \in \{1, 2, 10, 11\}$. Then there exist the following homotopy commutative push–out diagrams of spectra*

$$\begin{tikzcd} \Omega^{q+1}\mathbb{L}(C \to D) \arrow[r] \arrow[d] & \mathbb{L}S(\Phi) \arrow[d] \\ \Omega^q\mathbb{L}(C) \arrow[r] & \mathbb{L}P(\Phi) \end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd} \Omega^{q+1}\mathbb{L}(A \to B) \arrow[r] \arrow[d] & \Omega^{q+1}\mathbb{L}(C \to D) \arrow[d] \\ \mathbb{L}N(A \to B) \arrow[r] & \mathbb{L}S(\Phi) \end{tikzcd} \tag{2.2}$$

$$\begin{tikzcd} \Omega^q\mathbb{L}(A) \arrow[r] \arrow[d] & \mathbb{L}P(\Psi) \arrow[d] \\ \Omega^q\mathbb{L}(C) \arrow[r] & \mathbb{L}P(\Phi) \end{tikzcd} \tag{2.3}$$

and

$$\begin{tikzcd} \Omega^{q+1}\mathbb{L}(A \to B) \arrow[r] \arrow[d] & \mathbb{L}N(A \to B) \arrow[d] \\ \Omega^q\mathbb{L}(A) \arrow[r] & \mathbb{L}P(\Psi). \end{tikzcd} \tag{2.4}$$

Proof. It suffices to prove the absolute case, i.e., $n = 0$. Then the statement follows by induction since the cofibers of the maps (which are morphisms between push–out squares) give rise to a push–out square. In the absolute case, square (2.1) is obtained from a realization on the spectra level commutative diagram as shown in [44, p.264] (see also [13]). The natural map $\Psi \to \Phi$ induces the map of corresponding cofibrations (1.7). So we obtain a commutative diagram from which push–out square (2.2) follows. Remark here that this square is a realization on the spectra level commutative diagram as in [44, Proposition 12.3]. For some particular cases we refer to [23] and [25]. Square (2.3) can be obtained in a similar way by using cofibration (1.8) instead of (1.7). Square (2.4) is a particular case of (2.1) written for the object Ψ instead of Φ. ∎

Each square in the statement of Theorem 2.2 gives rise to a braid of exact sequences. This can be obtained by considering the homotopy long exact sequences of all maps involved in the square, and then identifying the homotopy groups of the cofibers of the parallel maps. Thus we have the following consequence.

Corollary 2.2. *Under the assumptions of Theorem 2.2, there exist the following braids of exact sequences*

$$
\begin{array}{ccccccccccccc}
\to & & L_{n+q+1}(D) & & \to & & LS_n(\Phi) & & \to & & L_n(B) & & \to \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & & L_{n+q+1}(C \to D) & & & & LP_n(\Phi) & & & & \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & L_{n+1}(B) & & \to & & L_{n+q}(C) & & \to & & L_{n+q}(D) & & \to
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
\to & & L_{n+1}(B) & & \to & & L_{n+q+1}(C \to D) & & \to & & L_{n+q+1}(\Phi) & & \to \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & & L_{n+q+1}(A \to B) & & & & LS_n(\Phi) & & & & \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & L_{n+q+1}(\Phi) & & \to & & LN_n(A \to B) & & \to & & L_n(B) & & \to
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
\to & & L_{n+q+1}(A \to C) & & \to & & LP_n(\Psi) & & \to & & L_n(B) & & \to \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & & L_{n+q}(A) & & & & LP_n(\Phi) & & & & \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & L_{n+1}(B) & & \to & & L_{n+q}(C) & & \to & & L_{n+q}(A \to C) & & \to
\end{array}
$$

and

$$
\begin{array}{ccccccccccccc}
\to & & L_{n+q+1}(B) & & \to & & LN_n(A \to B) & & \to & & L_n(B^-) & & \to \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & & L_{n+q+1}(A \to B) & & & & LP_n(\Psi) & & & & \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & L_{n+1}(B^-) & & \to & & L_{n+q}(A) & & \to & & L_{n+q}(B) & & \to
\end{array}
$$

where the object B^- in the last diagram represents the object B endowed with the orientation corresponding to the left column of the object Ψ.

A further relation can be obtained as follows. Let us consider a subcategory of such squares for which the right column gives the object

$$F_r = \begin{pmatrix} C & \longrightarrow & C \\ \downarrow & & \downarrow \\ D & \longrightarrow & D \end{pmatrix}$$

for the splitting problem, too. In this case, the square F is said to be a *geometric diagram* as in [1], [9], [14], [23], [25], [27], and [29]. An object Φ is called *geometric* if it is an object in the category of geometric diagrams. For geometric objects, there is a natural map $\Phi \to \Phi_r$, where Φ is an object of the category $S_q\mathcal{G}r_i$ and Φ_r is the corresponding object of the subcategory $SR\mathcal{G}r_i$. In this case, we have $\mathbb{L}S(\Phi_r) = \mathbb{L}N(C \to D)$ by definition. Composing this map with $\Psi \to \Phi$ (obtained above) yields a map $\Psi \to \Phi_r$ in the subcategory $SR\mathcal{G}r_i$.

Theorem 2.3. *Let Φ be a geometric object of type $\mathbf{2}^n$ in the category $S_q\mathcal{G}r_i$ for $i \in \{1, 2, 10, 11\}$. Then there are the following homotopy commutative push-out squares of spectra*

$$\begin{tikzcd}
\mathbb{L}S(\Phi) \arrow[r] \arrow[d] & \mathbb{L}(B) \arrow[d] \\
\mathbb{L}N(C \to D) \arrow[r] & \mathbb{L}(D)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{L}P(\Phi) \arrow[r] \arrow[d] & \mathbb{L}P(\Phi_r) \arrow[d] \\
\mathbb{L}(B) \arrow[r] & \mathbb{L}(D)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{L}S(\Phi) \arrow[r] \arrow[d] & \mathbb{L}P(\Phi) \arrow[d] \\
\mathbb{L}N(C \to D) \arrow[r] & \mathbb{L}P(\Phi_r)
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega\mathbb{L}N(\Phi) \arrow[r] \arrow[d] & \Omega\mathbb{L}(B \to D) \arrow[d] \\
\mathbb{L}N(A \to B) \arrow[r] & \mathbb{L}S(F)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Omega\mathbb{L}P(\Psi \to \Phi_r) \arrow[r] \arrow[d] & \Omega\mathbb{L}(B \to D) \arrow[d] \\
\mathbb{L}P(\Psi) \arrow[r] & \mathbb{L}P(\Phi).
\end{tikzcd}$$

Proof. The results follow by using the methods discussed in the proof of Theorem 2.2. ∎

Corollary 2.3. *Under the assumptions of Theorem 2.3, there exist the following braids of exact sequences:*

$$\begin{tikzcd}
{} \arrow[r] & L_{n+1}(B \to D) \arrow[rr] \arrow[dr] & & L_n(B) \arrow[rr] \arrow[dr] & & L_{n+q}(C \to D) \arrow[r] \arrow[dr] & {} \\
{} \arrow[ur] \arrow[dr] & & LS_n(\Phi) \arrow[ur] \arrow[dr] & & L_n(D) \arrow[ur] \arrow[dr] & & {} \\
{} \arrow[r] & L_{n+q+1}(C \to D) \arrow[rr] \arrow[ur] & & LN_n(C \to D) \arrow[rr] \arrow[ur] & & L_n(B \to D) \arrow[r] \arrow[ur] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & L_{n+q}(C) \arrow[rr] \arrow[dr] & & LP_n(\Phi_r) \arrow[rr] \arrow[dr] & & L_n(B \to D) \arrow[r] \arrow[dr] & {} \\
{} \arrow[ur] \arrow[dr] & & LP_n(\Phi) \arrow[ur] \arrow[dr] & & L_n(D) \arrow[ur] \arrow[dr] & & {} \\
{} \arrow[r] & L_n(B \to D) \arrow[rr] \arrow[ur] & & L_n(B) \arrow[rr] \arrow[ur] & & L_{n+q-1}(C) \arrow[r] \arrow[ur] & {}
\end{tikzcd}$$

$$
\begin{array}{ccccccccccc}
\to & L_{n+1}(B\to D) & & \to & & LP_n(\Phi) & & \to & & L_{n+q}(D) & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & & LS_n(\Phi) & & & & LP_n(\Phi_r) & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & L_{n+q+1}(D) & & \to & & LN_n(C\to D) & & \to & & L_n(B\to D) & \to
\end{array}
$$

$$
\begin{array}{ccccccccccc}
\to & LN_n(C\to D) & & \to & & L_n(B\to D) & & \to & & L_{n+q}(\Phi) & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & & LN_n^{rel}(\Psi\to\Phi_r) & & & & LS_{n-1}(\Phi) & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & L_{n+q+1}(\Phi) & & \to & & LN_{n-1}(A\to B) & & \to & & LN_{n-1}(C\to D) & \to
\end{array}
$$

and

$$
\begin{array}{ccccccccccc}
\to & LP_n(\Phi_r) & & \to & & L_n(B\to D) & & \to & & L_{n+q-1}(A\to C) & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & & LP_n(\Psi\to\Phi_r) & & & & LP_{n-1}(\Phi) & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & L_{n+q}(A\to C) & & \to & & LP_{n-1}(\Psi) & & \to & & LP_{n-1}(\Phi_r) & \to
\end{array}
$$

where $LN_n^{rel}(\Psi \to \Phi_r)$ are the relative LN–groups of the map $\Psi \to \Phi_r$ and $LP_n(\Psi \to \Phi_r)$ are the corresponding relative LP–groups.

Proof. The results can be obtained by considering the homotopy long exact sequences of the push–out squares listed in the statement of Theorem 2.3. ∎

3 Surgery and splitting obstruction groups for a manifold pair

Here we use the results of Section 2 to investigate the behaviour of the splitting and surgery obstruction groups for a manifold pair under some natural changes of the squares of fundamental groupoids. Then we study how the obstruction groups change for different decorations, and find new relations among these decorated groups. Let

$$
F = \begin{pmatrix} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{pmatrix} \tag{3.1}
$$

denote a push–out square of groupoids corresponding to the splitting problem for a manifold pair (X, Y) in codimension $q \in \{1, 2\}$. Taking the topological product of (X, Y) with a further compact manifold N with $\pi_1(N) = H$ yields a splitting problem with the square

$$
F \times H = \begin{pmatrix} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{pmatrix} \times H = \begin{pmatrix} A\times H & \longrightarrow & C\times H \\ \downarrow & & \downarrow \\ B\times H & \longrightarrow & D\times H \end{pmatrix}. \tag{3.2}
$$

Let us denote by

$$
\Lambda = \begin{pmatrix} A\times H & \longrightarrow & C \\ \downarrow & & \downarrow \\ B\times H & \longrightarrow & D \end{pmatrix} \tag{3.3}
$$

the square of fundamental groupoids which corresponds to a splitting problem in the same codimension considered for squares (3.1) and (3.2). We have a natural inclusion $F \to F \times H$ and a natural projection $F \times H \to F$ by assuming the trivial orientation on the groupoid H. In a similar way, we have a natural map $F \times H \to \Lambda$ in the case of trivial orientation on the group H. Now we describe connections between the splitting obstruction groups and the surgery obstruction groups for manifold pairs with associated squares (3.1), (3.2), and (3.3).

Theorem 3.1. *Let H be a group equipped with the trivial orientation. Then there are isomorphisms*

$$LS_n(\Lambda) \cong LS_n(F) \oplus L_{n+1}(B \times H \to B)$$

and

$$LP_n(\Lambda) \cong LP_n(F) \oplus L_{n+1}(B \times H \to B)$$

for any $n \equiv 0, 1, 2, 3$ (mod 4). Furthermore, the projections $LS_n(\Lambda) \to LS_n(F)$ and $LP_n(\Lambda) \to LP_n(F)$ are induced by the natural map $p : \Lambda \to F$ of push–out squares.

Proof. We have natural maps $F \to \Lambda \to F$ of push–out squares such that the composition is the identity. Hence the relative exact sequences of Corollary 2.1 (written for p) give the following exact sequences

$$0 \to LS_{n+1}(p) \to LS_n(\Lambda) \to LS_n(F) \to 0$$

and

$$0 \to LP_{n+1}(p) \to LP_n(\Lambda) \to LP_n(F) \to 0.$$

By Corollary 2.2, we obtain the following braid of exact sequences (use the objects given by the map p):

$$\begin{tikzcd}[column sep=small]
{} \arrow[dr] & & & & & & {} \\
& L_{n+q}(D\to D) \arrow[rr] \arrow[dr] & & LS^{\mathrm{rel}}_{n-1}(p) \arrow[rr] \arrow[dr] & & L_{n-1}(B\times H\to B) \arrow[r] \arrow[dr] & {} \\
& & L_{n+q}\begin{pmatrix} C & \longrightarrow & D \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{pmatrix} \arrow[ur] \arrow[dr] & & LP^{\mathrm{rel}}_{n-1}(p) \arrow[ur] \arrow[dr] & & {} \\
& L_n(B\times H\to B) \arrow[rr] \arrow[ur] & & L_{n+q-1}(C\to C) \arrow[rr] \arrow[ur] & & L_{n+q-1}(D\to D) \arrow[r] \arrow[ur] & {} \\
{} \arrow[ur] & & & & & & {}
\end{tikzcd}$$

From this we get isomorphisms

$$LS^{\mathrm{rel}}_{n-1}(p) \cong LP^{\mathrm{rel}}_{n-1}(p) \cong L_{n-1}(B \times H \to B)$$

as

$$L_*(D \to D) \cong L_*(C \to C) \cong L_* \begin{pmatrix} C & \longrightarrow & D \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{pmatrix} \cong 0.$$

Thus the isomorphisms of the statement follow. ∎

Now, let us consider the square

$$\Phi = \begin{pmatrix} \pi \times H & \longrightarrow & \pi \\ \downarrow & & \downarrow \\ G \times H & \longrightarrow & G \end{pmatrix}$$

for a splitting problem, where the horizontal maps are given by the natural projections, and the orientation on the group H is trivial.

Corollary 3.1. *Under the above notations, there are isomorphisms*

$$LS_n(\Phi) \cong LN_n(\pi \to G) \oplus L_{n+1}(G \times H \to G)$$

and

$$LP_n(\Phi) \cong L_{n+1}(i^!) \oplus L_{n+1}(G \times H \to G)$$

where $i^!$ is the transfer of the map $i : \pi \to G$, and the orientations of the groups in $L_*(G \times H \to H)$ correspond to the orientations of the left–corner groups in the squares arising from a splitting problem.

Corollary 3.2. *If $H \cong \mathbb{Z}$, then there are isomorphisms*

$$LS_n(\Phi) \cong LN_n(\pi \to G) \oplus L_n^h(G^-)$$

and

$$LP_n(\Phi) \cong L_{n+1}(i^!) \oplus L_n^h(G^-).$$

Changing the decorations yields similarly isomorphisms

$$LS_n^h(\Phi) \cong LN_n^h(\pi \to G) \oplus L_n^p(G^-)$$

and

$$LP_n^h(\Phi) \cong L_{n+1}^h(i^!) \oplus L_n^p(G^-).$$

The natural map $F \times H \to \Lambda$ induces maps of LS_*– and LP_*–groups which fit in the corresponding relative exact sequences. The next result gives a description of the third members involved in these exact sequences.

Proposition 3.1. *There exist exact sequences*

$$\cdots \to LS_n(F \times H) \to LS_n(\Lambda) \to LS_n(F \times H \to \Lambda) \to \cdots$$

and

$$\cdots \to LP_n(F \times H) \to LP_n(\Lambda) \to LP_n(F \times H \to \Lambda) \to \cdots$$

with

$$LS_n(F \times H \to \Lambda) \cong L_{n+q+1}\begin{pmatrix} C\times H & \longrightarrow & C \\ \downarrow & & \downarrow \\ D\times H & \longrightarrow & D \end{pmatrix}$$

and

$$LP_n(F \times H \to \Lambda) \cong L_{n+q}(C \times H \to C).$$

Proof. The results follow by using the second and the third diagram in the statement of Corollary 2.2. We have to write these diagrams for the relative groups of the map $F \times H \to \Lambda$. ■

The relations between the splitting and surgery obstruction groups for the manifold pairs related to squares (3.1) and (3.2) (assuming the trivial orientation on the group H) are given by the direct sum decompositions

$$LS_n(F \times H) \cong LS_n(F) \oplus LS_{n+1}^{\mathrm{rel}}(p)$$

and

$$LP_n(F \times H) \cong LP_n(F) \oplus LP_{n+1}^{\mathrm{rel}}(p)$$

where $p : F \times H \to F$ is the natural projection. The groups $LS^{\mathrm{rel}}_*(p)$ and $LP^{\mathrm{rel}}_*(p)$ fit in the braids of exact sequences described in the statement of Corollary 2.2 for the objects given by the map p. In the particular case $H = \mathbb{Z}$, there exist the natural decompositions (see [35, §7.5], and [36])

$$\begin{aligned} LS_n(\Phi \times \mathbb{Z}) &\cong LS_n(\Phi) \oplus LS^h_{n-1}(\Phi) \\ LP_n(\Phi \times \mathbb{Z}) &\cong LP_n(\Phi) \oplus LP^h_{n-1}(\Phi). \end{aligned} \tag{3.4}$$

Note that this case arises naturally by taking the product of a splitting problem with S^1 (see [35, §7.5], and [36]). Similar decompositions can be obtained for the decorations h and p. From now on in this section we shall use notations from [35, §7.5]. Recall that (X, Y) is a CW–pair of codimension $q \in \{1, 2\}$ whose associated square is

$$\Phi = \begin{pmatrix} \pi_1(S(\xi)) & \to & \pi_1(Z) \\ \downarrow & & \downarrow \\ \pi_1(Y) & \to & \pi_1(X) \end{pmatrix} = \begin{pmatrix} A & \to & C \\ \downarrow & & \downarrow \\ B & \to & D \end{pmatrix}.$$

Here the topological block–bundle $E(\xi)$ with boundary $S(\xi)$ gives a structure of tubular neighborhood of Y in X (see [35, §7.5]). The pair $(E(\xi), Y)$ has an associated square

$$\Psi = \begin{pmatrix} A & \to & A \\ \downarrow & & \downarrow \\ B & \to & B \end{pmatrix}.$$

A natural map $\Psi \to \Phi$ of squares is defined.

Recall now that the relative groups $WhS_*(\Phi)$ and $WhP_*(\Phi)$ from exact sequences (1.12) and (1.13) are closely related to the Tate cohomology groups of the Whitehead groups of the groups from the square Φ (see [37, Proposition 7.5.2]). These relations are given by the following exact sequences

$$\longrightarrow H^{n+1}(\mathbb{Z}_2, Wh(C) \to Wh(D)) \longrightarrow WhS_{n-q}(\Phi) \longrightarrow H^{n-q}(\mathbb{Z}_2, Wh(B)) \longrightarrow$$

and

$$\longrightarrow H^{n}(\mathbb{Z}_2, Wh(C)) \longrightarrow WhP_{n-q}(\Phi) \longrightarrow H^{n-q}(\mathbb{Z}_2, Wh(B)) \longrightarrow$$

By [37, §3] and [15], there are relative Whitehead groups $H^n(\mathbb{Z}_2, Wh(\Phi))$ of the triad Φ which fit in the following exact sequences

$$\to H^n(\mathbb{Z}_2, Wh(A) \to Wh(B)) \to H^n(\mathbb{Z}_2, Wh(C) \to Wh(D)) \to H^n(\mathbb{Z}_2, Wh(\Phi)) \to$$

and

$$\to H^n(\mathbb{Z}_2, Wh(A) \to Wh(C)) \to H^n(\mathbb{Z}_2, Wh(B)) \to Wh(D)) \to H^n(\mathbb{Z}_2, Wh(\Phi)) \to$$

Now we obtain new relations between the relative groups $WhS_*(\Phi)$ and $WhP_*(\Phi)$ and the Tate cohomology groups of different Whitehead groups related to the square Φ.

Theorem 3.2. *Under the above hypotheses, the relative groups $WhS_*(\Phi)$ and $WhP_*(\Phi)$ fit into the following braids of exact sequences:*

$$
\begin{array}{ccccccccc}
\to H^{n+1}(\mathbb{Z}_2,Wh(C)) & & \longrightarrow & & H^{n+1}(\mathbb{Z}_2,Wh(D)) & & \longrightarrow & & WhS_{n-q}(\Phi) \to \\
\nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow \\
& & WhP_{n-q+1}(\Phi) & & & & H^{n+1}(\mathbb{Z}_2,Wh(C)\to Wh(D)) & & \\
\searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow \\
\to WhS_{n-q+1}(\Phi) & & \longrightarrow & & H^{n-q+1}(\mathbb{Z}_2,Wh(B)) & & \longrightarrow & & H^{n}(\mathbb{Z}_2,Wh(C)) \to
\end{array}
$$

$$
\begin{array}{ccccccccc}
\to H^{n+1}(\mathbb{Z}_2,Wh(B)) & & \longrightarrow & & H^{n+q+1}(\mathbb{Z}_2,Wh(C)\to Wh(D)) & & \longrightarrow & & H^{n+q+1}(\mathbb{Z}_2,Wh(\Phi)) \to \\
\nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow \\
& & H^{n+q}(\mathbb{Z}_2,Wh(A)\to Wh(B)) & & & & WhS_{n}(\Phi) & & \\
\searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow \\
\to H^{n+q+1}(\mathbb{Z}_2,Wh(\Phi)) & & \longrightarrow & & WhS_{n}(\Psi) & & \longrightarrow & & H^{n}(\mathbb{Z}_2,Wh(B)) \to
\end{array}
$$

$$
\begin{array}{ccccccccc}
\to H^{n+q+1}(\mathbb{Z}_2,Wh(A)\to Wh(C)) & & \longrightarrow & & WhP_{n}(\Psi) & & \longrightarrow & & H^{n}(\mathbb{Z}_2,Wh(B)) \to \\
\nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow \\
& & H^{n+q}(\mathbb{Z}_2,Wh(A)) & & & & WhP_{n}(\Phi) & & \\
\searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow \\
\to H^{n+1}(\mathbb{Z}_2,Wh(B)) & & \longrightarrow & & H^{n+q}(\mathbb{Z}_2,Wh(C)) & & \longrightarrow & & H^{n+q}(\mathbb{Z}_2,Wh(A)\to Wh(C)) \to
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
\to H^{n+1}(\mathbb{Z}_2,Wh(A)) & & \longrightarrow & & H^{n+1}(\mathbb{Z}_2,Wh(B)) & & \longrightarrow & & WhS_{n-q}(\Psi) \to \\
\nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow \\
& & WhP_{n-q+1}(\Psi) & & & & H^{n+1}(\mathbb{Z}_2,Wh(A)\to Wh(B)) & & \\
\searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow \\
\to WhS_{n-q+1}(\Psi) & & \longrightarrow & & H^{n-q+1}(\mathbb{Z}_2,Wh(B^-)) & & \longrightarrow & & H^{n}(\mathbb{Z}_2,Wh(A)) \to
\end{array}
$$

Proof. Let us consider the squares of spectra listed in the statement of Theorem 2.2 for the absolute case $n = 0$ with decorations "s" and "h". Each square with decoration "s" maps in a natural way to a similar square equipped with decoration "h". The cofibres of these maps give rise to push–out squares of spectra. The homotopy long exact sequences of the obtained squares give the braids of exact sequences in the statement of the theorem. ∎

Now it is easy to state and to prove similar results for the case when the right column of the square Φ gives a splitting problem in codimension q. These results will be obtained by using Theorem 2.3 instead of Theorem 2.2 in the considered case. Remark that the comparison braid of exact sequences for the decorations "h" and "p" can be constructed in a similar way (compare also with [35, p.649]).

To complete the section we give some examples of explicit computations of the LS– and LP–groups for certain manifold pairs (here we use the direct sum notation for the product of commutative groups). Let us consider a pair (X, Y) of compact manifolds with boundary, where Y is the Möbius band (i.e., the twisted I–bundle over $\mathbb{S}^1$) and X is a non–trivial D^1–bundle over $\mathbb{R}P^2$. The inclusion $Y \subset X$ is generated by the natural inclusion $\mathbb{S}^1 = \mathbb{R}P^1 \subset \mathbb{R}P^2$, where $\mathbb{S}^1$ is the central line of the Möbius band. In this case, the square for the splitting problem is

$$
F = \begin{pmatrix} \mathbb{Z}^+ & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ \mathbb{Z}^- & \longrightarrow & \mathbb{Z}_2^+ \end{pmatrix}.
$$

The LS_*– and LP_*–groups of F were computed in [41], Section 3. Taking the topological product of (X, Y) with a real projective space of dimension $2k+1$ ($k \geq 1$) gives the manifold pair $(X \times \mathbb{R}P^{2k+1}, Y \times \mathbb{R}P^{2k+1})$ which has the square $F \oplus \mathbb{Z}_2^+$

for the corresponding splitting problem. By using the relative groups we compute explicitly the obstruction groups $LS_n(F\oplus\mathbb{Z}_2^+)$ and $LP_n(F\oplus\mathbb{Z}_2^+)$ for all n. We have a natural inclusion $j: F\to F\oplus\mathbb{Z}_2^+$ which induces the direct sum decompositions

$$LS_n(F\oplus\mathbb{Z}_2^+)=LS_n(F)\oplus LS_n^{rel}(j),\ LP_n(F\oplus\mathbb{Z}_2^+)=LP_n(F)\oplus LP_n^{rel}(j), \quad (3.5)$$

where $LS_n^{rel}(j)$ and $LP_n^{rel}(j)$ denote the relative groups. The map j defines maps of the groups involved in the squares F and $F\oplus\mathbb{Z}_2^+$, and hence the corresponding relative L-groups are defined. We shall use the following notations for these groups:

$$\begin{array}{ll} L_n^{rel}(D)=L_n(\mathbb{Z}_2\to\mathbb{Z}_2\oplus\mathbb{Z}_2^+), & L_n^{rel}(C)=L_n(1\to\mathbb{Z}_2^+), \\ L_n^{rel}(B^-)=L_n(\mathbb{Z}^-\to\mathbb{Z}^-\oplus\mathbb{Z}_2^+), & L_n^{rel}(C\to D)=L_n\begin{pmatrix} 1 & \to & \mathbb{Z}_2^+ \\ \downarrow & & \downarrow \\ \mathbb{Z}_2 & \to & \mathbb{Z}_2\oplus\mathbb{Z}_2^+\end{pmatrix}. \end{array} \quad (3.6)$$

To compute the $LS_n^{rel}(j)$– and $LP_n^{rel}(j)$–groups we use a diagram chasing in the first diagram of relative groups shown in the statement of Corollary 2.2. According to our notations this diagram takes the following form:

$$\begin{array}{ccccccccccc} \cdots\to & L_n^{rel}(C) & & \longrightarrow & & L_n^{rel}(D) & & \longrightarrow & & LS_{n-2}^{rel}(j) & \longrightarrow\cdots \\ & \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\ & & & LP_{n-1}^{rel}(j) & & & & L_n^{rel}(C\to D) & & & \\ & \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\ \cdots\to & LS_{n-1}^{rel}(j) & & \longrightarrow & & L_{n-1}^{rel}(B^-) & & \longrightarrow & & L_{n-1}^{rel}(C) & \longrightarrow\cdots \end{array}. \quad (3.7)$$

Now we recall standard results about surgery obstruction groups (see [44, §13 A] and [46, §3.3]). There are isomorphisms

$$\begin{array}{ccccc} L_n(*) & n=0 & n=1 & n=2 & n=3 \\ L_n(1) & \mathbb{Z} & 0 & \mathbb{Z}_2 & 0 \\ L_n(\mathbb{Z}_2^+) & \mathbb{Z}\oplus\mathbb{Z} & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ L_n(\mathbb{Z}_2^+\oplus\mathbb{Z}_2^+) & \mathbb{Z}^4 & 0 & \mathbb{Z}_2 & \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2 \\ L_n(\mathbb{Z}^-) & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ L_n(\mathbb{Z}^-\oplus\mathbb{Z}_2^+) & \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z}_2\oplus\mathbb{Z}_2 \end{array} \quad (3.8)$$

The map j defines natural inclusions of the oriented groups of F into the corresponding groups of $F\oplus\mathbb{Z}_2^+$. The functoriality of the L–groups yields the following decompositions:

$$\begin{array}{c} L_n(\mathbb{Z}_2\oplus\mathbb{Z}_2^+)=L_n^{rel}(D)\oplus L_n(\mathbb{Z}_2), L_n(\mathbb{Z}_2^+)=L_n^{rel}(C)\oplus L_n(1), \\ L_n(\mathbb{Z}^-\oplus\mathbb{Z}_2^+)=L_n^{rel}(B^-)\oplus L_n(\mathbb{Z}^-). \end{array} \quad (3.9)$$

Now (3.8) and (3.9) provide isomorphisms

$$\begin{array}{ccccc} L_n^{rel}(*) & n=0 & n=1 & n=2 & n=3 \\ L_n^{rel}(D) & \mathbb{Z}\oplus\mathbb{Z} & 0 & 0 & \mathbb{Z}_2\oplus\mathbb{Z}_2 \\ L_n^{rel}(C) & \mathbb{Z} & 0 & 0 & \mathbb{Z}_2 \\ L_n^{rel}(B^-) & \mathbb{Z}_2\oplus\mathbb{Z}_2 & 0 & 0 & \mathbb{Z}_2 \end{array} \quad (3.10)$$

The vertical maps of the square

$$\begin{pmatrix} 1 & \to & \mathbb{Z}_2^+ \\ \downarrow & & \downarrow \\ \mathbb{Z}_2 & \to & \mathbb{Z}_2\oplus\mathbb{Z}_2^+\end{pmatrix}$$

define an inclusion of the upper row on the direct summand of the lower row. Hence we have a decomposition

$$L_n^{rel}(D) = L_n^{rel}(C) \oplus L_n^{rel}(C \to D).$$

From this decomposition, (3.6), and (3.10), we obtain isomorphisms

$$L_n^{rel}(C \to D) = \mathbb{Z},\ 0,\ 0,\ \mathbb{Z}_2$$

for $n = 0, 1, 2, 3 \pmod 4$, respectively. A diagram chasing in (3.6) provides the following isomorphisms:

$$\begin{array}{rcccc} & n=0 & n=1 & n=2 & n=3 \\ LP_n^{rel}(j) = & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z} \oplus \mathbb{Z}_2 \\ LS_n^{rel}(j) = & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & LS_1^r & \mathbb{Z} & \mathbb{Z}_2 \end{array} \tag{3.11}$$

Here the group LS_1^r is defined by an extension

$$1 \to \mathbb{Z}_2 \to LS_1^r \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 1. \tag{3.12}$$

The isomorphisms

$$\begin{array}{rcccc} & n=0 & n=1 & n=2 & n=3 \\ LS_n(F) = & \mathbb{Z}_2 & 0 & \mathbb{Z} & \mathbb{Z}_2 \\ LP_n(F) = & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z} \end{array} \tag{3.13}$$

were obtained in [41] (Theorem 3.1).

Proposition 3.2. *Let $(X \times \mathbb{R}P^{2k+1}, Y \times \mathbb{R}P^{2k+1})$, $k \geq 1$, be the manifold pair described above which has $F \oplus \mathbb{Z}_2^+$ for the corresponding splitting problem. Then there are isomorphisms*

$$\begin{array}{rcccc} & n=0 & n=1 & n=2 & n=3 \\ LP_n(F \oplus \mathbb{Z}_2^+) = & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ LS_n(F \oplus \mathbb{Z}_2^+) = & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & LS_1^r & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{array}$$

where the group LS_1^r fits in the short exact sequence (3.12).

Proof. The result follows from decomposition (3.5) and isomorphisms (3.11) and (3.13). ∎

Now we take the topological product of the pair (X, Y) with a real projective space of dimension $2k$, $k \geq 1$. We obtain the manifold pair $(X \times \mathbb{R}P^{2k}, Y \times \mathbb{R}P^{2k})$ which admits the square

$$F \oplus \mathbb{Z}_2^- = \begin{pmatrix} \mathbb{Z}^+ \oplus \mathbb{Z}_2^- & \longrightarrow & \mathbb{Z}_2^- \\ \downarrow & & \downarrow \\ \mathbb{Z}^- \oplus \mathbb{Z}_2^- & \longrightarrow & \mathbb{Z}_2^+ \oplus \mathbb{Z}_2^- \end{pmatrix} \tag{3.14}$$

for the corresponding splitting problem. We need standard results about surgery obstruction groups obtained in [44, §13 A] and [46, §3.5]:

$$\begin{array}{ccccc} L_n(*) & n=0 & n=1 & n=2 & n=3 \\ L_n(\mathbb{Z}_2^-) & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 \\ L_n(\mathbb{Z}_2^+ \oplus \mathbb{Z}_2^-) & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 \\ L_n(\mathbb{Z}^- \oplus \mathbb{Z}_2^-) & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \end{array} \tag{3.15}$$

The right vertical map in (3.14) is an inclusion on the second summand. It follows from (3.15) that the induced maps

$$L_n(\mathbb{Z}_2^-) \to L_n(\mathbb{Z}_2^+ \oplus \mathbb{Z}_2^-)$$

are isomorphisms for all n. Hence the relative groups $L_n(\mathbb{Z}_2^- \to \mathbb{Z}_2^+ \oplus \mathbb{Z}_2^-)$ vanish for all n. Let us consider the first diagram of Corollary 2.2 written for $\Phi = F \oplus \mathbb{Z}_2^-$. Then we obtain $L_n(D) = L_n(\mathbb{Z}_2^+ \oplus \mathbb{Z}_2^-)$, $L_n(C) = L_n(\mathbb{Z}_2^-)$, $L_n(B) = L_n(\mathbb{Z}^- \oplus \mathbb{Z}_2^-)$, and $L_n(C \to D) = 0$. A diagram chasing in this diagram provides the following result.

Proposition 3.3. *Let $(X \times \mathbb{R}P^{2k}, Y \times \mathbb{R}P^{2k})$, $k \geq 1$, be the manifold pair described above which has the square $F \oplus \mathbb{Z}_2^-$ for the corresponding splitting problem. Then there are isomorphisms*

$$\begin{array}{rcccc} & n=0 & n=1 & n=2 & n=3 \\ LP_n(F \oplus \mathbb{Z}_2^-) = & \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ LS_n(F \oplus \mathbb{Z}_2^-) = & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \end{array}$$

The natural forgetful maps

$$LS_n(F \oplus \mathbb{Z}_2^-) \to L_n(\pi_1(Y \times \mathbb{R}P^{2k}))$$

are isomorphisms for all $n = 0, 1, 2, 3(\mathrm{mod}4)$.
The natural forgetful maps

$$LP_n(F \oplus \mathbb{Z}_2^-) \to L_n(\pi_1(Y \times \mathbb{R}P^{2k}))$$

and

$$LP_n(F \oplus \mathbb{Z}_2^-) \to L_n(\pi_1(X \times \mathbb{R}P^{2k}))$$

are epimorphisms for all $n = 0, 1, 2, 3(\mathrm{mod}4)$.

Let us consider the square

$$\Psi = \begin{pmatrix} \mathbb{Z}^+ & \longrightarrow & \mathbb{Z}^+ \\ \downarrow & & \downarrow \\ \mathbb{Z}_2^+ & \longrightarrow & \mathbb{Z}_2^+ \end{pmatrix} \tag{3.16}$$

for a splitting problem in codimension 2. The question about the computation of such splitting obstruction groups arises from [44, §14 E, p.183 and p.265]. It is related to the classification of fake lens spaces. For square (3.16) the splitting obstruction groups and the surgery obstruction groups were computed in [23]. Now we consider the product of this splitting problem with a real projective space $\mathbb{R}P^{2k}$ of dimension $2k$, $k \geq 1$. We obtain a splitting problem whose square is

$$\Psi \oplus \mathbb{Z}_2^- = \begin{pmatrix} \mathbb{Z} \oplus \mathbb{Z}_2^- & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}_2^- \\ \downarrow p & & \downarrow p \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2^- & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2^- \end{pmatrix}. \tag{3.17}$$

In this case we shall denote

$$\begin{gathered} L_n(C) = L_n(\mathbb{Z}^+ \oplus \mathbb{Z}_2^-) = L_n(\mathbb{Z}^- \oplus \mathbb{Z}_2^-), \\ L_n(D) = L_n(\mathbb{Z}_2^+ \oplus \mathbb{Z}_2^-), L_n(B) = L_n(\mathbb{Z}_2^+ \oplus \mathbb{Z}_2^-). \end{gathered} \tag{3.18}$$

All groups from (3.18) are given in (3.15). Let us consider the composition

$$\mathbb{Z}_2^- \to \mathbb{Z}^+ \oplus \mathbb{Z}_2^- \to \mathbb{Z}_2^+ \oplus \mathbb{Z}_2^- \to \mathbb{Z}_2^- \tag{3.19}$$

where the left map is an inclusion on the first summand, the middle map is the vertical map p from (3.17), and the right map is a projection on the second summand. Since the middle map is the identity on the second summand, composition (3.19) is the identity. From this and (3.15), it follows that the induced map $p_* : L_{2n}(C) = L_{2n}(\mathbb{Z}^+ \oplus \mathbb{Z}_2^-) \to L_{2n}(D) = L_{2n}(\mathbb{Z}_2^+ \oplus \mathbb{Z}_2^-)$ is an isomorphism for all n. So we obtain $L_{2n}(C \to D) = \mathbb{Z}_2$ and $L_{2n+1}(C \to D) = 0$ for all n. The natural map of square (3.16) into square (3.17) induces a commutative diagram

$$\begin{array}{ccc} L_2(\mathbb{Z}_2) & \longrightarrow & L_3(\mathbb{Z}) \\ \downarrow & & \downarrow \\ L_2(\mathbb{Z}_2 \oplus \mathbb{Z}_2^-) & \longrightarrow & L_3(\mathbb{Z} \oplus \mathbb{Z}_2^-) \end{array} \tag{3.20}$$

where the upper horizontal map is a transfer map for square (3.16), and the lower horizontal map is a transfer map for square (3.17) (see [44, Theorem 11.6]). The map $L_3(\mathbb{Z}) \to L_3(\mathbb{Z}_2)$ is an isomorphism [44, Lemma 13A.9]. Then the upper horizontal map in (3.20) is trivial. In fact it fits in the chain complex

$$L_2(\mathbb{Z}_2) \to L_3(\mathbb{Z}) \to L_3(\mathbb{Z}_2)$$

which is a part of the last diagram of Corollary 2.2 (see [44, p. 264]) for the square Ψ. The left vertical map in (3.20) is an isomorphism since it preserves the Arf–invariant. The right vertical map is an isomorphism, too. To see this, we consider the following part of the upper and lower rows in the last diagram of Corollary 2.2

$$\begin{array}{ccccc} L_3(\mathbb{Z}) & \rightarrow & Ł_3(\mathbb{Z} \oplus \mathbb{Z}_2^-) & \rightarrow & LN_1 = L_1(\mathbb{Z}) = \mathbb{Z} \\ | & & | & & | \\ LN_2(\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2^-) & \rightarrow & L_2(\mathbb{Z} \oplus \mathbb{Z}_2^+) & \rightarrow & L_2(\mathbb{Z}) \end{array}$$

for the index 2 inclusion $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2^-$. These rows are chain complexes with isomorphic homology groups. By [44, Corollary 12.9.2] it follows that the left lower map is an isomorphism. Hence the homology group in the second member is trivial, and the map

$$\mathbb{Z}_2 = L_3(\mathbb{Z}) \to L_3(\mathbb{Z} \oplus \mathbb{Z}_2^-) = \mathbb{Z}_2$$

is an isomorphism. Thus the two vertical maps in (3.20) are isomorphisms, and the upper horizontal map is trivial. This implies that the lower transfer map

$$L_2(\mathbb{Z}_2 \oplus \mathbb{Z}_2^-) \longrightarrow L_3(\mathbb{Z} \oplus \mathbb{Z}_2^-)$$

in (3.20) is trivial. Using the natural isomorphism $\text{Ł}_n(\pi \times \mathbb{Z}_2^-) = \text{Ł}_{n+2}(\pi \times \mathbb{Z}_2^-)$ (see [44, Proposition 13A.7]), we can conclude that the transfer map

$$\mathbb{Z}_2 = L_0(\mathbb{Z}_2 \oplus \mathbb{Z}_2^-) \longrightarrow L_1(\mathbb{Z} \oplus \mathbb{Z}_2^-) = \mathbb{Z}_2$$

is trivial, too. A diagram chasing in the last diagram of Corollary 2.2 written for the square $\Psi \oplus \mathbb{Z}_2^-$ (3.17) gives the following result.

Proposition 3.4. *Let $(X \times \mathbb{R}P^{2k}, Y \times \mathbb{R}P^{2k})$, $k \geq 1$, be a manifold pair of codimension 2 with square (3.17) for the corresponding splitting problem. Then there are isomorphisms*

$$\begin{array}{rcccc} & n=0 & n=1 & n=2 & n=3 \\ LP_n(\Psi \oplus \mathbb{Z}_2^-) = & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \\ LS_n(\Psi \oplus \mathbb{Z}_2^-) = & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \end{array}$$

The natural forgetful maps

$$LS_n(\Psi \oplus \mathbb{Z}_2^-) \to L_n(\pi_1(Y \times \mathbb{R}P^{2k}))$$

are epimorphisms for all $n = 0, 1, 2, 3(\mathrm{mod} 4)$.
The natural forgetful maps

$$LP_n(\Psi \oplus \mathbb{Z}_2^-) \to L_n(\pi_1(Y \times \mathbb{R}P^{2k}))$$

and

$$LP_n(\Psi \oplus \mathbb{Z}_2^-) \to L_n(\pi_1(X \times \mathbb{R}P^{2k}))$$

are epimorphisms for all $n = 0, 1, 2, 3(\mathrm{mod} 4)$.
The natural forgetful maps

$$LS_n(\Psi \oplus \mathbb{Z}_2^-) \to LP_n(\Psi \oplus \mathbb{Z}_2^-)$$

are isomorphisms for n odd and monomorphisms for n even.

4 Lower LS_*– and LP_*–groups

First we recall the definition of the lower $L_*^{<-i>}$–groups according to Ranicki (see [33, Appendix C], [38], and [39]). For any ring R with an involution, let $R[z, z^{-1}]$ be the Laurent extension of R with the involution $z \to z^{-1}$. Then we define the lower $L_*^{<-i>}(R)$–groups, for any $i \geq 1$, by using splittings (1.10) and (1.11). For $i = 1, 0$, we set $< 1 >= h$ and $< 0 >= p$. Now we can define inductively the $L_*^{<-i>}(R)$-groups, for any $i \geq 1$, which fit in the natural algebraic splitting

$$L_n^{<1-i>}(R[z, z^{-1}]) = L_n^{<1-i>}(R) \oplus L_{n-1}^{<-i>}(R). \tag{4.1}$$

These groups fit in the following generalization of the Rothenberg exact sequence (see [15], [36], [38], [39], and [40])

$$\cdots \to L_n^{<1-i>}(R) \to L_n^{<-i>}(R) \to H^n(\mathbb{Z}_2, \tilde{K}_{-i}(R)) \to \cdots \tag{4.2}$$

where $\tilde{K}_{-i}(R) = K_{-i}(R)$, for any $i \geq 1$, are the lower K–groups of Bass (see [2, XII]). Let $f : R \to R'$ be a morphism of rings with involutions. We can define the lower relative $L_*^{<-i>}(f)$–groups by using the functoriality of the definition of the lower $L_*^{<-i>}$–groups. So we have

$$L_n^{<1-i>}(R[z, z^{-1}] \to R'[z, z^{-1}]) = L_n^{<1-i>}(f) \oplus L_{n-1}^{<-i>}(f). \tag{4.3}$$

Proposition 4.1. *The lower relative groups $L_*^{<-i>}(f)$ fit in the following relative exact sequence*

$$\cdots \to L_*^{<-i>}(R) \to L_*^{<-i>}(R') \to L_*^{<-i>}(f) \to \cdots \tag{4.4}$$

and in the exact sequence of Rothenberg type

$$\cdots \to L_n^{<1-i>}(f) \to L_n^{<-i>}(f) \to H^n(\mathbb{Z}_2, \tilde{K}_{-i}(R) \to \tilde{K}_{-i}(R')) \to \cdots \tag{4.5}$$

Proof. We use the fact that the definition of the lower $L_*^{<i>}$–groups is functorial. Then we consider the following exact sequence of Tate cohomology groups

$$\longrightarrow H^n(\mathbb{Z}_2, \tilde{K}_{-i}(R)) \longrightarrow H^n(\mathbb{Z}_2, \tilde{K}_{-i}(R')) \longrightarrow H^n(\mathbb{Z}_2, \tilde{K}_{-i}(R) \to \tilde{K}_{-i}(R')) \longrightarrow$$

So the results follow. ∎

Recall that for $q = s, h$, and p the algebraic surgery exact sequence

$$\cdots \to H_n(X, \mathbb{L}_\bullet) \to L_n^q(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n^q(X) \to H_{n-1}(X, \mathbb{L}_\bullet) \to \cdots \tag{4.6}$$

is defined (see [34], [35], [36], and [40]), where $\mathbb{L}_\bullet$ is an 1-connective Ω-spectrum with $\pi_n(\mathbb{L}_\bullet) = L_n(1)$. For any space X with $\pi = \pi_1(X)$ the relation between the lower $L_*^{<-i>}$–groups and the surgery exact sequence is given by the following braid of exact sequences (see [36], and [40, Appendix C])

$$\begin{tikzcd}[column sep=small]
{} \arrow[r] & H_n(X,\mathbb{L}_\bullet) \arrow[rr] \arrow[dr] & & L_n^{<-i>}(\mathbb{Z}[\pi]) \arrow[rr] \arrow[dr] & & H^n(\mathbb{Z}_2,\tilde{K}_{-i}(\mathbb{Z}[\pi])) \arrow[r] \arrow[dr] & {} \\
{} \arrow[ur] \arrow[dr] & & L_n^{<1-i>}(\mathbb{Z}[\pi]) \arrow[ur] \arrow[dr] & & \mathbb{S}_n^{<-i>}(X) \arrow[ur] \arrow[dr] & & {} \\
{} \arrow[r] & H^{n+1}(\mathbb{Z}_2,\tilde{K}_{-i}(\mathbb{Z}[\pi])) \arrow[rr] \arrow[ur] & & \mathbb{S}_n^{<1-i>}(X) \arrow[rr] \arrow[ur] & & H_{n-1}(X,\mathbb{L}_\bullet) \arrow[r] \arrow[ur] & {}
\end{tikzcd} \tag{4.7}$$

where

$$\mathbb{S}_n^{<1>}(X) = \mathbb{S}_n^h(X), \quad \mathbb{S}_n^{<0>}(X) = \mathbb{S}_n^p(X)$$

and

$$\mathbb{S}_n^{<1-i>}(X \times S^1) = \mathbb{S}_n^{<1-i>}(X) \oplus \mathbb{S}_{n-1}^{<-i>}(X) \qquad (i \geq 1).$$

Let (X, Y) be a CW–pair of codimension $q \in \{1, 2\}$ (see [35, §7.5]) with associated square

$$\Phi = \begin{pmatrix} \pi_1(S(\xi)) & \to & \pi_1(Z) \\ \downarrow & & \downarrow \\ \pi_1(Y) & \to & \pi_1(X) \end{pmatrix} = \begin{pmatrix} A & \to & C \\ \downarrow & & \downarrow \\ B & \to & D \end{pmatrix}. \tag{4.8}$$

Taking the topological product of (X, Y) with the 1–sphere S^1 gives the CW–pair $(X \times S^1, Y \times S^1)$, where the submanifold has the same codimension q. The square associated to $(X \times S^1, Y \times S^1)$ is $\Phi \times \mathbb{Z}$. We have a natural injection $\overline{\epsilon} : \Phi \to \Phi \times \mathbb{Z}$ and a natural projection $\epsilon : \Phi \times \mathbb{Z} \to \Phi$. We set

$$LS_*^{<1>}(\Phi) = LS_*^h(\Phi), \quad LP_*^{<1>}(\Phi) = LP_*^h(\Phi)$$

and

$$LS_*^{<0>}(\Phi) = LS_*^p(\Phi), \quad LP_*^{<0>}(\Phi) = LP_*^p(\Phi).$$

Since $\overline{\epsilon} \circ \epsilon = Id$, we can define inductively the $LS_*^{<-i>}(\Phi)$– and $LP_*^{<-i>}(\Phi)$–groups for any $i \geq 1$ as follows:

$$\begin{aligned} LS_n^{<1-i>}(\Phi \times \mathbb{Z}) &= LS_n^{<1-i>}(\Phi) \oplus LS_{n-1}^{<-i>}(\Phi) \\ LP_n^{<1-i>}(\Phi \times \mathbb{Z}) &= LP_n^{<1-i>}(\Phi) \oplus LP_{n-1}^{<-i>}(\Phi). \end{aligned} \tag{4.9}$$

Remark that the maps $\overline{\epsilon}$ and ϵ commute with the map $\Psi \to \Phi$ of push–out squares.

Theorem 4.1. *The lower $LS_*^{<-i>}(\Phi)$– and $LP_*^{<-i>}(\Phi)$–groups relate to the lower $L_*^{<-i>}$–groups by means of braids of exact sequences which are similar to the diagrams in Corollary 2.2. It suffices to consider all the groups of those diagrams equipped with the decoration $< -i >$ for any $i \geq 1$.*

Proof. The result follows from the functoriality of the diagrams in Corollary 2.2 and from the definitions of the lower L_*–, LS_*–, and LP_*–groups. ∎

The map $L_n^{<1-i>}(R) \to L_n^{<-i>}(R)$ induces a map of the diagram equipped with the decoration $< 1-i >$ to the corresponding diagram equipped with the decoration $< -i >$ for any $i \geq 0$. In particular, we obtain the following relative exact sequences

$$\cdots \to LS_n^{<1-i>}(\Phi) \to LS_n^{<-i>}(\Phi) \to KS_n^{<-i>}(\Phi) \to \cdots$$

and

$$\cdots \to LP_n^{<1-i>}(\Phi) \to LP_n^{<-i>}(\Phi) \to KP_n^{<-i>}(\Phi) \to \cdots .$$

The relative groups $KS_n^{<-i>}(\Phi)$ and $KP_n^{<-i>}(\Phi)$ are similar to the relative groups $WhS_n(\Phi)$ and $WhP_n(\Phi)$, respectively. They fit in braids of exact sequences similar to those listed in Theorem 3.2. These braids contain the groups $H^n(\mathbb{Z}_2, \tilde{K}_{-i}(\mathbb{Z}(-)))$ instead of the groups $H^n(\mathbb{Z}_2, Wh(-))$, and analogously for the other groups. In case of the decoration "s", the LP_*– and LS_*–groups fit in the following commutative diagram (which contains the surgery exact sequence):

$$\begin{array}{ccccccc} \cdots \to & \mathbb{S}_{n+1}^s(X) & \to & H_n(X; \mathbb{L}_\bullet) & \to & L_n(\mathbb{Z}\pi_1(X)) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow = & \\ \cdots \to & LS_{n-q}(\Phi) & \to & LP_{n-q}(\Phi) & \to & L_n(\mathbb{Z}\pi_1(X)) & \to \cdots \end{array} \tag{4.10}$$

The rows of this diagram are exact sequences (see [35, §7.2], and [44, p.136]).

Theorem 4.2. *For any $i \geq 1$, there exists a commutative diagram of exact sequences*

$$\begin{array}{ccccccc} \cdots \to & \mathbb{S}_{n+1}^{<-i>}(X) & \to & H_n(X;\mathbb{L}_\bullet) & \to & L_n^{<-i>}(\mathbb{Z}\pi_1(X)) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow = & \\ \cdots \to & LS_{n-q}^{<-i>}(\Phi) & \to & LP_{n-q}^{<-i>}(\Phi) & \to & L_n^{<-i>}(\mathbb{Z}\pi_1(X)) & \to \cdots \end{array}$$

in which the upper row is the exact sequence from diagram (4.7) and the lower row is the exact sequence arising from the first diagram in Corollary 2.2 written for lower L_*–groups.

Proof. The natural inclusion $(X,Y) = (X,Y)\times * \to (X,Y)\times S^1$ induces a map from commutative diagram (1.10) to the corresponding commutative diagram for the pair $(X \times S^1, Y \times S^1)$. The induced maps of L_*–, LS_*–, and LP_*–groups coincide with the maps induced by the map $\overline{\epsilon}$. Hence the lower row maps to a direct summand and the upper row maps to a direct summand as follows from [36]. As a direct summand of commutative diagram (1.10) for the pair $(X \times S^1, Y \times S^1)$, we obtain the same diagram equipped with the decoration $h =< 1 >$. Iterating this process yields the requested result. ■

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