Geometric properties of certain linear integral transforms

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Abstract

For $\lambda > 0$ and $0 < \mu < n$, let $\mathcal{U}_n(\lambda, \mu)$ denote the class of all normalized analytic functions f in the unit disc Δ of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ such that

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} - 1 \right| < \lambda, \ z \in \Delta,$$

where $n \in \mathbb{N}$ is fixed. In addition to the discussion of the basic properties of the class $\mathcal{U}_n(\lambda, \mu)$, we find conditions so that $\mathcal{U}_n(\lambda, \mu)$ is included in $\mathcal{S}_\gamma$, the class of all strongly starlike functions of order γ $(0 < \gamma \leq 1)$. We also find necessary conditions so that $f \in \mathcal{U}_n(\lambda, \mu)$ implies that

$$\left| \frac{zf'(z)}{f(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta}, \quad \text{for all } z \in \Delta,$$

or

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta}, \quad \text{for } |z| < r < 1,$$

where $r = r(\lambda, \mu, n)$ will be specified.

*The work of the first author was supported by a Sponsored Research project (Ref No. DST/MS/092/98) from the Department of Science and Technology (India) and the work of the second author was supported by National Board for Higher Mathematics (India) in the form of a Postdoctoral Fellowship.

Received by the editors September 2002.

Communicated by R. Delanghe.

1991 *Mathematics Subject Classification :* 30C45, 30C55.

Key words and phrases : Univalent, starlike and convex functions, subordination, and integral transform.

1 Introduction

Let $\mathcal{A}$ denote the class of all analytic functions defined on the unit disc $\Delta = \{z : |z| < 1\}$ with the normalization condition $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \Delta\}$. A function $f \in \mathcal{A}$ is said to belong to $\mathcal{S}^*$ iff $f(\Delta)$ is a starlike domain with respect to the origin. A function $f \in \mathcal{A}$ is said to be Bazilevič of type $\mu = \alpha + i\beta$ ($\alpha \geq 0$ and $\beta \in \mathbb{R}$), if f satisfies the differential equation

$$f'(z)\left(\frac{z}{f(z)}\right)^{1-\mu} = \left(\frac{g(z)}{z}\right)^{\alpha} h(z),$$

g being a function in $\mathcal{S}^*$ and $\mathrm{Re}\, e^{i\phi}h(z) > 0$ in Δ for some $\phi \in \mathbb{R}$ [1]. It is a well-known result that Bazilevič functions are in $\mathcal{S}$ and the above differential equation necessarily has a solution analytic in Δ for any choice of g and h. We are interested in the case of $g(z) = z$, $\beta = 0$ and formulate the following class for $0 \leq \lambda < 1$:

$$\mathcal{B}_1(\lambda, \mu) = \left\{ f : \mathrm{Re}\left(f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1}\right) > \lambda \right\}.$$

Here $\mu < 0$ and for convenience, we have avoided the rotation factor and assumed that $h(z) = (1 + (1 - 2\lambda)z)/(1 - z)$. The Bazilevič functions are also discussed in [15] and it is clear that $\mathcal{B}_1(\lambda, \mu) \subseteq \mathcal{S}$. We are interested to know whether $\mathcal{B}_1(\lambda, \mu)$ is extendable to cover certain values of μ with $\mu > 0$. To carry out our investigation, we consider a class $\mathcal{U}_h(\mu)$ as follows: For a univalent function h in Δ and $\mu > 0$, we define

$$\mathcal{U}_h(\mu) = \left\{ f \in \mathcal{A} : f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec h(z),\ z \in \Delta \right\}$$

where $\prec$ denotes the subordination. For the basic results on subordination we refer to the book by P. L. Duren [3]. The choice of $h(z) = 1 + \lambda z$ leads to the class $\mathcal{U}(\lambda, \mu)$,

$$\mathcal{U}(\lambda, \mu) = \left\{ f : f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec 1 + \lambda z,\ z \in \Delta \right\}.$$

Since $\mathcal{B}_1(0, \mu) \subset \mathcal{S}$ for $\mu < 0$, it follows that $\mathcal{U}(\lambda, \mu) \subset \mathcal{S}$ for $\mu < 0$ and $0 < \lambda \leq 1$. We note that the Koebe function belongs to $\mathcal{U}(1, 1)$. On the other hand, Nunokawa and Ozaki [8] have shown that $\mathcal{U}(\lambda, 1)$ is included in $\mathcal{S}$ for $0 < \lambda \leq 1$ whereas, among several other interesting results, Ponnusamy [9] has found conditions on λ and $\mu < 0$ so that $\mathcal{U}(\lambda, \mu)$ is included in $\mathcal{S}^*$ or other well known subclasses. In view of these inclusion results, it is natural to seek condition on λ (depending on μ) so that $\mathcal{U}(\lambda, \mu) \subset \mathcal{S}^*$ for $0 < \mu < 1$. Obradović [5] used the idea of Ponnusamy [9] and Ponnusamy and Singh [10] to fill this gap and proved, for example, the following result.

Theorem 1.1. *If* $f \in \mathcal{U}(\lambda, \mu)$ *with* $0 < \mu < 1$ *and* $0 < \lambda \leq \dfrac{1-\mu}{\sqrt{(1-\mu)^2 + \mu^2}}$, *then* $f \in \mathcal{S}^*$.

Our aim in this paper is not only to extend Theorem 1.1 but also to obtain a number of new results extending several other interesting results in this direction, eg. [7]. Let $\mathcal{A}_n$ denote the class of all functions $f \in \mathcal{A}$ such that f has the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \tag{1.2}$$

where $n \in \mathbb{N}$ is fixed. Clearly, $\mathcal{A} := \mathcal{A}_1$. For $\lambda > 0$ and $\mu \geq 0$, we define

$$\mathcal{U}_n(\lambda, \mu) = \left\{ f \in \mathcal{A}_n : \left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} - 1 \right| < \lambda, \quad z \in \Delta \right\} \equiv \mathcal{A}_n \cap \mathcal{U}(\lambda, \mu).$$

For the case $\mu = 1$, this class has been discussed in detail by Ponnusamy and Vasundhara [12]. More recently, the special situations, namely, the classes

$$\mathcal{U}_1(\lambda, \mu) := \mathcal{U}(\lambda, \mu), \;\; \mathcal{U}_1(1,1) := \mathcal{U}, \;\; \text{and} \;\; \mathcal{U}_1(\lambda, 1) := \mathcal{U}(\lambda)$$

under the restriction $\lambda \in (0,1]$ and $\mu \in (0,1)$, have been studied extensively in [5, 6, 7, 11, 13]. In the present paper, we enforce "missing coefficients" and extend the range of μ beyond the unit interval.

A function $f \in \mathcal{A}$ is said to be *strongly starlike of order* γ, $0 < \gamma \leq 1$, if and only if f satisfies the analytic condition

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z} \right)^{\gamma}, \quad z \in \Delta.$$

We denote the class of strongly starlike functions of order γ by $\mathcal{S}_\gamma$. Clearly, $\mathcal{S}^* \equiv \mathcal{S}_1$. If $0 < \gamma < 1$, then $\mathcal{S}_\gamma$ is completely contained in the class of bounded starlike functions [2]. Set

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z}, \; z \in \Delta \right\},$$

so that $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. For our investigation we need a number of preliminary results.

2 Discussion of $\mathcal{U}_n(\lambda, \mu)$

Let $\lambda > 0$ and $\mu \in (0, n)$, where $n \in \mathbb{N}$ is fixed. Then, each $f \in \mathcal{U}_n(\lambda, \mu)$ can be written as

$$\left(\frac{z}{f(z)} \right)^{\mu+1} f'(z) = 1 + \lambda w(z) = 1 + (n-\mu) a_{n+1} z^n + \cdots, \tag{2.1}$$

for some $w \in \mathcal{B}_n$, where

$$\mathcal{B}_n = \{ w \in \mathcal{H} : w(0) = w'(0) = \cdots = w^{(n-1)}(0) = 0, \;\; \text{and } |w(z)| < 1 \text{ for } z \in \Delta \}.$$

Here $\mathcal{H}$ denotes the class of all analytic functions in Δ. If we set

$$p(z) = \left(\frac{z}{f(z)} \right)^{\mu} = 1 + p_n z^n + \cdots,$$

then (2.1) is seen to be equivalent to

$$p(z) - \frac{1}{\mu} z p'(z) = 1 + \lambda w(z).$$

An algebraic computation implies that

$$p(z) = 1 - \lambda \int_1^\infty w(t^{-1/\mu} z)\, dt. \tag{2.2}$$

As $w(z) \in \mathcal{B}_n$, Schwarz' lemma gives that $|w(z)| \leq |z|^n$ for $z \in \Delta$ and therefore

$$|p(z) - 1| \leq \frac{\lambda\mu}{n-\mu} |z|^n, \quad z \in \Delta,$$

which is

$$\left| \left(\frac{z}{f(z)} \right)^\mu - 1 \right| \leq \frac{\lambda\mu}{n-\mu} |z|^n \tag{2.3}$$

so that

$$1 - \frac{\lambda\mu}{n-\mu} |z|^n \leq \mathrm{Re} \left(\frac{z}{f(z)} \right)^\mu \leq 1 + \frac{\lambda\mu}{n-\mu} |z|^n. \tag{2.4}$$

Equality holds in each of the last two inequalities (2.3) and (2.4) for functions of the form

$$f(z) = \frac{z}{(1 \pm (\lambda\mu/(n-\mu))\, z^n)^{1/\mu}}.$$

By (2.3), it follows that

$$\left(\frac{z}{f(z)} \right)^\mu \in \{w : \, |w - 1| < 1\} \quad \text{for} \quad |z| < \left(\frac{n-\mu}{\lambda\mu} \right)^{1/n}.$$

Thus, for $f \in \mathcal{U}_n(\lambda, \mu)$, we have

$$\mathrm{Re} \left(\frac{z}{f(z)} \right)^\mu > 0 \quad \text{for} \quad |z| < \left(\frac{n-\mu}{\lambda\mu} \right)^{1/n}.$$

In particular, for $0 < \lambda \leq (n-\mu)/\mu$, we have

$$\mathrm{Re} \left(\frac{z}{f(z)} \right)^\mu > 0 \quad \text{for} \quad z \in \Delta.$$

Also, with the inequality $0 < \lambda \leq (n-\mu)/\mu$, (2.3) is equivalent to

$$\left| \left(\frac{f(z)}{z} \right)^\mu - \frac{1}{1 - (\lambda\mu/(n-\mu))^2 |z|^{2n}} \right| \leq \frac{[\lambda\mu/(n-\mu)]|z|^n}{1 - (\lambda\mu/(n-\mu))^2 |z|^{2n}}$$

which implies that

$$\mathrm{Re} \left(\frac{f(z)}{z} \right)^\mu \geq \frac{1}{1 + (\lambda\mu/(n-\mu))\, |z|^n} \geq \frac{n-\mu}{n-\mu+\lambda\mu}.$$

If $(n-\mu)/\mu < \lambda$, then for $f \in \mathcal{U}_n(\lambda, \mu)$ we have

$$\mathrm{Re} \left(\frac{f(z)}{z} \right)^\mu \geq \frac{1}{1 + (\lambda\mu/(n-\mu))\, |z|^n} > \frac{n-\mu}{n-\mu+\lambda\mu} \quad \text{for} \quad |z| < \left(\frac{n-\mu}{\lambda\mu} \right)^{1/n} < 1.$$

3 Starlikeness and Convexity of $f \in \mathcal{U}_n(\lambda, \mu)$

Although the Koebe function belongs to $\mathcal{U}(1,1)$, the class $\mathcal{U}(1,1)$ is not included in $\mathcal{S}^*$. On the other hand, $f \in \mathcal{U}_2(\lambda, 1)$ is seen to be in $\mathcal{S}^*$ whenever $0 < \lambda \leq 1/\sqrt{2}$. We are now in a position to state our first result.

Theorem 3.1. *Let $\gamma \in (0,1]$, $n \geq 1$, $\mu \in (0,n)$ and*

$$\lambda_*(\gamma, \mu, n) = \frac{(n-\mu)\sin(\gamma\pi/2)}{\sqrt{(n-\mu)^2 + \mu^2 + 2\mu(n-\mu)\cos(\gamma\pi/2)}}.$$

If $f \in \mathcal{U}_n(\lambda, \mu)$, then $f \in \mathcal{S}_\gamma$ for $0 < \lambda \leq \lambda_*(\gamma, \mu, n)$.

Note that $\lambda_*(\gamma, \mu, n)$ is an increasing function of n and

$$\lambda_*(\gamma, \mu, n) \to \sin(\gamma\pi/2) \quad \text{as } n \to \infty.$$

Theorem 3.1 for $\mu = 1$ (under the restriction $n \geq 2$) is due to Ponnusamy and Vasundhara [12]. In the case $\gamma = 1$, Theorem 3.1 yields criteria for starlike functions for missing coefficients.

Corollary 3.2. *If $f \in \mathcal{U}_n(\lambda, \mu)$ and $0 < \lambda \leq \dfrac{n-\mu}{\sqrt{(n-\mu)^2 + \mu^2}}$, then $f \in \mathcal{S}^*$.*

For $n = 1$, this result gives Theorem 1.1 which is due to Obradović [5]. Also for $n = 2$ (i.e. $f \in \mathcal{A}$ with $f''(0) = 0$) and $\mu = 1$, Theorem 3.1 gives a recent result of Obradović et al [7].

Theorem 3.3. *Let $\alpha \in [0,1)$, $n \geq 1$ and $\mu \in (0,n)$. If $f(z) \in \mathcal{U}_n(\lambda, \mu)$, then $f \in \mathcal{S}^*(\alpha)$ for $0 < \lambda \leq \lambda^*(\alpha, \mu, n)$, where*

$$\lambda^*(\alpha, \mu, n) = \begin{cases} \dfrac{(n-\mu)\sqrt{1-2\alpha}}{\sqrt{(n-\mu)^2 + \mu^2(1-2\alpha)}} & \text{for } 0 \leq \alpha \leq \dfrac{\mu}{n+\mu} \\ \dfrac{(n-\mu)(1-\alpha)}{n-\mu+\mu\alpha} & \text{for } \dfrac{\mu}{n+\mu} < \alpha < 1. \end{cases}$$

The case $\alpha = 0$ of Theorem 3.3 also gives Corollary 3.2 and the case $\mu = 1$ has been obtained by Ponnusamy and Vasundhara [12] whereas the case $\mu = 1$, $n = 2$ and $\alpha = 0$ of Theorem 3.3 has been obtained by Obradović et al [7].

The same reasoning indicated in the proof of Theorem 3.1 helps to obtain the following results.

Theorem 3.4. *Let $f \in \mathcal{U}_n(\lambda, \mu)$ and $\lambda_*(\gamma, \mu, n)$ be as in Theorem 3.1. Then, for $\lambda_*(\gamma, \mu, n) \leq \lambda$, f is strongly starlike in $|z| < r = r(\lambda, \gamma, \mu, n)$, where*

$$r = r(\lambda, \gamma, \mu, n) = \left\{ \frac{(n-\mu)\sin(\gamma\pi/2)}{\lambda\sqrt{(n-\mu)^2 + \mu^2 + 2\mu(n-\mu)\cos(\gamma\pi/2)}} \right\}^{1/n}.$$

Corollary 3.5. *If $f\in\mathcal{U}_n(\lambda,\mu)$ and $\lambda_*(\mu,n)=\dfrac{n-\mu}{\sqrt{(n-\mu)^2+\mu^2}}$, then, for $\lambda_*(\mu,n)\le\lambda$, $f\in\mathcal{S}^*$ in $|z|<r=r(\lambda,\mu,n)$, where*

$$r(\lambda,\mu,n)=\left\{\frac{n-\mu}{\lambda\sqrt{(n-\mu)^2+\mu^2}}\right\}^{1/n}.$$

In the following theorem, we consider similar results for a subset of the set of all starlike functions. To do this, we define

$$\mathcal{S}_b^*(\beta)=\left\{f\in\mathcal{A}:\left|\frac{zf'(z)}{f(z)}-\frac{1}{2\beta}\right|<\frac{1}{2\beta},\ z\in\Delta\right\},$$

where $0<\beta<1$. Clearly, $\mathcal{S}_b^*(\beta)\subsetneq\mathcal{S}^*$.

Theorem 3.6. *Let $n\in\mathbb{N}$, $\mu\in(0,n)$, $\lambda\in(0,1]$ and $r^*_{\lambda,\mu,n}(\beta)=\left(\lambda^*_\beta/\lambda\right)^{1/n}$, where*

$$\lambda^*_\beta=\frac{\beta^*(n-\mu)}{\beta(n-\mu)+\mu}\quad\text{and}\quad\beta^*=\begin{cases}\beta & \text{if } 0<\beta\le 1/2\\ 1-\beta & \text{if } 1/2\le\beta<1\end{cases}.$$

Then

(i) *for $0<\lambda\le\lambda^*_\beta$, we have $\mathcal{U}_n(\lambda,\mu)\subset\mathcal{S}_b^*(\beta)$.*

(ii) *for $\lambda^*_\beta<\lambda\le 1$, $f\in\mathcal{U}_n(\lambda,\mu)$, we have $f\in\mathcal{S}_b^*(\beta)$ for $|z|<r^*_{\lambda,\mu,n}(\beta)$.*

Taking $n=2$ and $\mu=1$ in Theorem 3.6, we obtain the following

Corollary 3.7. *Let $\lambda\in(0,1]$ and $r^*_{\beta,\lambda}(\beta)=\left(\lambda^*_\beta/\lambda\right)^{1/2}$, where*

$$\lambda^*_\beta=\frac{\beta^*}{\beta+1}\quad\text{and}\quad\beta^*=\begin{cases}\beta & \text{if } 0<\beta\le 1/2\\ 1-\beta & \text{if } 1/2\le\beta<1\end{cases}.$$

Then

(i) *for $0<\lambda\le\lambda^*_\beta$, we have $\mathcal{U}_2(\lambda)\subset\mathcal{S}_b^*(\beta)$.*

(ii) *for $\lambda^*_\beta<\lambda\le 1$, $f\in\mathcal{U}_2(\lambda)$, we have $f\in\mathcal{S}_b^*(\beta)$ for $|z|<r^*_\lambda(\beta)$.*

This corollary is a special case of Theorem 1.9 of [7].

Theorem 3.8. *Let $n\in\mathbb{N}$, $\mu\in(0,n)$ and $\lambda\in(0,1]$. If $f\in\mathcal{U}_n(\lambda,\mu)$, then for $0<\beta\le 1$ we have*

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{1}{2\beta}\right|<\frac{1}{2\beta}\quad\text{for}\ \ |z|<r_{\lambda,\mu,n}(\beta),$$

where $r=r_{\lambda,\mu,n}(\beta)$ is the smallest positive root of the equation

$$2\beta\lambda^2\mu r^{2n+1}+2\lambda[\beta(\mu+1)n+\mu(1-\beta)]r^{n+2}-2\beta\lambda(n-\lambda\mu-\mu)r^{n+1}- \tag{3.9}$$

$$2\lambda[\beta(\mu+1)n+(1-\beta)\mu)]r^n-[(1-|2\beta-1|)r^2+2\beta\lambda r+|2\beta-1|-1](n-\mu)=0.$$

In particular, $f(rz)\in\mathcal{K}$.

Here $\mathcal{K}$ denotes the class of all convex functions g, i.e. $zg' \in S^*$.

If we choose $\beta = 1/2$, we obtain

Corollary 3.10. *Let* $f \in \mathcal{U}_n(\lambda, \mu)$. *Then*

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 \quad \text{for} \quad |z| < r_{\lambda,\mu}(1/2),$$

where $r_{\lambda,\mu,n}(1/2)$ *is the smallest positive root of the equation*

$$\lambda^2\mu r^{2n+1} + \lambda[(\mu+1)n+\mu]r^{n+2} \\ -\lambda(n-\lambda\mu-\mu)r^{n+1} - \lambda[(\mu+1)n+\mu]r^n - (r^2+\lambda r-1)(n-\mu) = 0.$$

The case $\mu = 1$ and $n = 2$ of Theorem 3.8 is due to [7]. The proof of these theorems will be given in Section 4.

4 Proofs of the Main Theorems

4.1. *Proof of Theorem 3.1.* Suppose that $f \in \mathcal{U}_n(\lambda, \mu)$ for some $\lambda \in (0, 1]$ and $\mu \in (0, n)$. Then, by the definition of $\mathcal{U}_n(\lambda, \mu)$, we have

$$\left|\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) - 1\right| < \lambda$$

and, by (2.3), we get

$$\left|\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < \frac{\lambda\mu}{n-\mu}|z|^n < \frac{\lambda\mu}{n-\mu}.$$

Therefore, it follows that

$$\left|\arg\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)\right| < \arcsin(\lambda) \tag{4.2}$$

and

$$\left|\arg\left(\frac{z}{f(z)}\right)^{\mu}\right| < \arcsin\left(\frac{\lambda\mu}{n-\mu}\right). \tag{4.3}$$

Using the formulae (4.2) and (4.3) and the addition formula for the inverse of sine function, namely,

$$\arcsin(x) + \arcsin(y) = \arcsin[x\sqrt{1-y^2} + y\sqrt{1-x^2}],$$

we find that

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f(z)}\right| &\le \left|\arg\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)\right| + \left|\arg\left(\frac{z}{f(z)}\right)^{\mu}\right| \\ &< \arcsin(\lambda) + \arcsin\left(\frac{\lambda\mu}{n-\mu}\right) \\ &= \arcsin\left[\lambda\sqrt{1-\left(\frac{\lambda\mu}{n-\mu}\right)^2} + \frac{\lambda\mu}{n-\mu}\sqrt{1-\lambda^2}\right].\end{aligned}$$

Thus, $f \in \mathcal{S}_\gamma$ whenever $\lambda \in (0, \lambda_*(\gamma, \mu, n)]$. Here $\lambda_*(\gamma, \mu, n)$ is the solution of the equation

$$\phi(\lambda) = \lambda\sqrt{1 - \left(\frac{\lambda\mu}{n-\mu}\right)^2} + \frac{\lambda\mu}{n-\mu}\sqrt{1-\lambda^2} - \sin\left(\frac{\pi\gamma}{2}\right) = 0$$

which proves the theorem. ■

4.4. *Proof of Theorem 3.3***.** Suppose that $f(z) \in \mathcal{U}_n(\lambda, \mu)$. Then, by the representation (2.1) and (2.2), it follows that

$$\frac{zf'(z)}{f(z)} = \frac{1 + \lambda w(z)}{1 - \lambda \int_1^\infty w(t^{-1/\mu}z)\,dt}$$

and therefore,

$$\frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)} - \alpha\right) = \frac{1 + \dfrac{\lambda w(z)}{1-\alpha} + \dfrac{\alpha\lambda}{1-\alpha}\displaystyle\int_1^\infty w(t^{-1/\mu}z)\,dt}{1 - \lambda\displaystyle\int_1^\infty w(t^{-1/\mu}z)\,dt}.$$

We want f to be in $\mathcal{S}^*(\alpha)$. To do this, according to a well known result [14] and the last equation, it suffices to show that

$$\frac{1 + \dfrac{\lambda w(z)}{1-\alpha} + \dfrac{\alpha\lambda}{1-\alpha}\displaystyle\int_1^\infty w(t^{-1/\mu}z)\,dt}{1 - \lambda\displaystyle\int_1^\infty w(t^{-1/\mu}z)\,dt} \neq -iT, \quad T \in \mathbb{R},$$

which is equivalent to

$$\lambda\left[\frac{w(z) + (\alpha - i(1-\alpha)T)\displaystyle\int_1^\infty w(t^{-1/\mu}z)\,dt}{(1-\alpha)(1+iT)}\right] \neq -1, \quad T \in \mathbb{R}.$$

If we let

$$M = \sup_{z\in\Delta,\, w\in\mathcal{B}_n,\, T\in\mathbb{R}}\left|\frac{w(z) + (\alpha - i(1-\alpha)T)\displaystyle\int_1^\infty w(t^{-1/\mu}z)\,dt}{(1-\alpha)(1+iT)}\right|$$

then, in view of the rotation invariance property of the space $\mathcal{B}_n$, we obtain that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{if } M\lambda \leq 1.$$

This observation shows that it suffices to find M. First we notice that

$$M \leq \sup_{T\in\mathbb{R}}\left\{\frac{(n-\mu) + \mu\sqrt{\alpha^2 + (1-\alpha)^2T^2}}{(n-\mu)(1-\alpha)\sqrt{1+T^2}}\right\}.$$

Define $\phi:\ [0,\infty) \longrightarrow \mathbb{R}$ by

$$\phi(x) = \frac{(n-\mu) + \mu\sqrt{\alpha^2 + (1-\alpha)^2 x}}{(n-\mu)(1-\alpha)\sqrt{1+x}}.$$

Then, by differentiating ϕ with respect to x, we get

$$\phi'(x) = \frac{\mu(1-2\alpha) - (n-\mu)\sqrt{\alpha^2 + (1-\alpha)^2 x}}{2(n-\mu)(1-\alpha)(1+x)^{3/2}\sqrt{\alpha^2 + (1-\alpha)^2 x}}.$$

Case(i) : Let $0 < \alpha < \mu/(n+\mu)$. Then we see that ϕ has its only critical point in the positive real line at

$$x_0 = \frac{1}{(1-\alpha)^2}\left[\left(\frac{\mu(1-2\alpha)}{n-\mu}\right)^2 - \alpha^2\right].$$

Further, we easily observe that $\phi'(x) > 0$ for $0 \le x < x_0$ and $\phi'(x) < 0$ for $x > x_0$. Therefore, $\phi(x)$ attains maximum value at x_0 and hence,

$$\phi(x) \le \phi(x_0) = \frac{\sqrt{(n-\mu)^2 + \mu^2(1-2\alpha)}}{\sqrt{1-2\alpha}(n-\mu)} \quad \text{for } \ x \ge 0. \tag{4.5}$$

Case(ii) : Let $\alpha > \mu/(n+\mu)$. We can easily observe that

$$\phi'(x) \le 0 \iff \mu(1-2\alpha) < (n-\mu)\sqrt{\alpha^2 + (1-\alpha)^2 x}\,, \qquad \text{for } \ x \ge 0.$$

This implies that $\phi'(x) \le 0$ for all $x \ge 0$ whenever $\mu(1-2\alpha) < (n-\mu)\alpha$. Therefore, if $\alpha \ge \mu/(n+\mu)$, ϕ is decreasing on $[0,\infty)$ and hence,

$$\phi(x) \le \phi(0) = \frac{n-\mu+\mu\alpha}{(n-\mu)(1-\alpha)} \quad \text{for all } \ x \ge 0. \tag{4.6}$$

The required conclusion follows from (4.5) and (4.6). ■

4.7. *Proof of Theorem 3.4***.** Let $f \in \mathcal{U}_n(\lambda,\mu)$ for some $\mu \in (0,n)$. Following the proof of Theorem 3.1, we obtain that

$$\left|\arg\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)\right| < \arcsin(\lambda r^n)$$

and

$$\left|\arg\left(\frac{z}{f(z)}\right)^{\mu}\right| < \arcsin\left(\frac{\lambda\mu}{n-\mu}r^n\right).$$

Combining the last two inequalities we get

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \le \arcsin\left[\lambda r^n\sqrt{1-\left(\frac{\lambda\mu}{n-\mu}\right)^2 r^{2n}} + \frac{\lambda\mu}{n-\mu}r^n\sqrt{1-\lambda^2 r^2}\right].$$

By a calculation, we see that the right hand side of the last inequality is less than or equal to $\pi\gamma/2$ provided

$$r^n \le \frac{(n-\mu)\sin(\gamma\pi/2)}{\lambda\sqrt{(n-\mu)^2 + \mu^2 + 2\mu(n-\mu)\cos(\gamma\pi/2)}}$$

which completes the proof. ■

4.8. *Proof of Theorem 3.6*. Let $f \in \mathcal{U}_n(\lambda,\mu)$. Then, by the representations (2.1) and (2.2), it follows that

$$\frac{zf'(z)}{f(z)} = \frac{1+\lambda w(z)}{1-\lambda\displaystyle\int_1^\infty w(t^{-1/\mu}z)\,dt}, \tag{4.9}$$

where $w \in \mathcal{B}_n$. Then, $|w(z)| \le |z|^n$. We proceed with the method of the proof of Theorem 1.9 in [7]. According to this

$$\begin{aligned}
\left|\frac{zf'(z)}{f(z)} - \frac{1}{2\beta}\right| &= \frac{1}{2\beta}\frac{\left|2\beta - 1 + 2\beta\lambda w(z) + \lambda\displaystyle\int_1^\infty w(t^{-1/\mu}z)\,dt\right|}{\left|1-\lambda\displaystyle\int_1^\infty w(t^{-1/\mu}z)\,dt\right|} \\
&\le \frac{1}{2\beta}\frac{|2\beta-1| + 2\beta\lambda|z|^n + \dfrac{\lambda\mu}{n-\mu}|z|^n}{1-\dfrac{\lambda\mu}{n-\mu}|z|^n} \\
&= \frac{1}{2\beta}\left[\frac{|2\beta-1|(n-\mu) + 2\beta\lambda(n-\mu)|z|^n + \lambda\mu|z|^n}{1-\lambda\mu|z|^n}\right].
\end{aligned}$$

It is a simple exercise to see that the square bracketed term in the last step is less than 1 provided

$$|z| < \left[\frac{\beta^*(n-\mu)}{\beta\lambda(n-\mu)+\lambda\mu}\right]^{1/n} =: r^{\beta}_{\lambda,\mu,n}$$

where $2\beta^* = 1 - |2\beta - 1|$. We remark that $r^{\beta}_{\lambda,\mu,n} \ge 1$ if and only if

$$\lambda \le \frac{\beta^*(n-\mu)}{\beta(n-\mu)+\mu}.$$

The desired result follows. ∎

4.10. *Proof of Theorem 3.8*. Let $f \in \mathcal{U}_n(\lambda,\mu)$. Then taking logarithmic derivative of the representation given by (2.1), we have

$$1 + \frac{zf''(z)}{f'(z)} = (\mu+1)\frac{zf'(z)}{f(z)} - \mu + \frac{\lambda z w'(z)}{1+\lambda w(z)}.$$

In view of this equation and (4.9) we see that

$$1 + \frac{zf''(z)}{f'(z)} - \frac{1}{2\beta} = (\mu+1)\frac{1+\lambda w(z)}{1-\lambda\displaystyle\int_1^\infty w(t^{-1/\mu}z)\,dt} - \mu + \frac{\lambda z w'(z)}{1+\lambda w(z)} - \frac{1}{2\beta}.$$

Since $w \in \mathcal{B}_n$, by the definition of $\mathcal{B}_n$, we have $|w(z)| \le |z|^n$. By the well-known Schwarz-Pick lemma, we find that

$$|w'(z)| \le \frac{1-|w(z)|^2}{1-|z|^2}.$$

It follows that (as $\lambda \leq 1$)

$$\left|\frac{zw'(z)}{1+\lambda w(z)}\right| \leq \frac{|z|}{1-\lambda|w(z)|}\left(\frac{1-|w(z)|^2}{1-|z|^2}\right) \leq \frac{|z|(1+|z|^n)}{1-|z|^2}.$$

With the help of this inequality and the fact that $|w(z)| \leq |z|^n$, after computation, we obtain

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{1}{2\beta}\right| < \frac{1}{2\beta}R_\mu(\lambda,\beta,|z|),$$

where $R_\mu(\lambda,\beta,|z|) := R_\mu$ with

$$R_\mu = \frac{(|2\beta-1|+2\beta\lambda(\mu+1)|z|^n)(n-\mu)+(2\beta\mu+1)\lambda\mu|z|^n}{n-\mu-\lambda\mu|z|^n} + \frac{2\beta\lambda(|z|+|z|^{n+1})}{1-|z|^2}.$$

It can be easily seen that the equality $R_\mu(\lambda,\beta,|z|) < 1$ is equivalent to (3.9). The desired conclusion follows. ■

5 Integral Transforms

In this section we consider the following integral transform $I(f)$ of $f \in \mathcal{A}$ defined by

$$[I(f)](z) = F(z) = z\left[\frac{c+1-\mu}{z^{c+1-\mu}}\int_0^z \left(\frac{t}{f(t)}\right)^\mu t^{c-\mu}\,dt\right]^{1/\mu}, \quad c+1-\mu>0. \quad (5.1)$$

This transform is similar to the Alexander transform when $c=\mu=1$ and is similar to Bernardi transformation when $\mu=1$ and $c>0$.

Theorem 5.2. *Let $f \in \mathcal{U}_n(\lambda,\mu)$ for some $\lambda>0$, $n\geq 2$ and $\mu\in(0,n)$. For $c+1-\mu>0$ and $\alpha<1$, let $F(z)$ be defined by* (5.1)*. Then $F\in\mathcal{S}^*_\alpha$ whenever c, λ are related by*

$$0<\lambda\leq\frac{(1-\alpha)(n-\mu)(c+1-\mu+n)}{(c+1-\mu)(n+(1-\alpha)\mu)}. \quad (5.3)$$

Proof. Assume that $f(z) = z+\sum_{k=n+1}^\infty a_kz^k \in \mathcal{U}_n(\lambda,\mu)$. By (5.1), we see that

$$(c+1-\mu)\left(\frac{F(z)}{z}\right)^\mu + z\frac{d}{dz}\left(\frac{F(z)}{z}\right)^\mu = (c+1-\mu)\left(\frac{z}{f(z)}\right)^\mu.$$

It is a simple exercise to show that

$$\frac{1}{\mu(c+1-\mu)}\left[(c-\mu)(\mu+1)\left(\frac{F(z)}{z}\right)^\mu - (c-2\mu)\frac{d}{dz}\left(z\left(\frac{F(z)}{z}\right)^\mu\right)\right.$$
$$\left.-z\frac{d^2}{dz^2}\left(z\left(\frac{F(z)}{z}\right)^\mu\right)\right] = \left(\frac{z}{f(z)}\right)^{\mu+1}f'(z).$$

If we set

$$P(z) = z\left(\frac{F(z)}{z}\right)^\mu, \quad (5.4)$$

then, from the last equation and the assumption $f \in \mathcal{U}_n(\lambda,\mu)$, it follows that $P(z)$ satisfies the second order differential equation

$$\frac{(c-\mu)(\mu+1)}{\mu(c+1-\mu)}\frac{P(z)}{z} - \frac{c-2\mu}{\mu(c+1-\mu)}P'(z) - \frac{1}{\mu(c+1-\mu)}zP''(z) = 1+\lambda w(z) \quad (5.5)$$

where $w \in \mathcal{B}_n$. If we take $P(z) = z + \sum_{k=n+1}^{\infty} c_k z^k$ and $w(z) = \sum_{k=n}^{\infty} w_k z^k$ in (5.5), then by equating the coefficients of z^n we get the representations

$$\frac{P(z)}{z} = 1 - \frac{\lambda(c+1-\mu)}{c+1}\int_1^{\infty} w(t^{-1/\mu}z)(1-t^{-(c+1)/\mu})\,dt \quad (5.6)$$

and

$$P'(z) = 1 - \frac{\lambda(c+1-\mu)}{c+1}\int_1^{\infty} w(t^{-1/\mu}z)(\mu+1+(c-\mu)t^{-(c+1)/\mu})\,dt\,. \quad (5.7)$$

In view of the equality

$$\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) = \left(\frac{z}{f(z)}\right)^{\mu} - \frac{z}{\mu}\left\{\left(\frac{z}{f(z)}\right)^{\mu}\right\}' = 1+\lambda w(z),$$

where $w \in \mathcal{B}_n$, it follows that (see Section 2)

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1 - \lambda\int_1^{\infty} w(t^{-1/\mu}z)dt\,.$$

From (5.4) we have

$$\frac{zF'(z)}{F(z)} - 1 = \frac{1}{\mu}\left(\frac{zP'(z)}{P(z)} - 1\right). \quad (5.8)$$

Using (5.6), (5.7) and (5.8), we compute that

$$\begin{aligned}
&\frac{zF'(z)}{F(z)} - 1 \\
&= \frac{1}{\mu}\left[-1 + \frac{1 - \dfrac{\lambda(c+1-\mu)}{c+1}\displaystyle\int_1^{\infty} w(t^{-1/\mu}z)\left(\mu+1+(c-\mu)t^{-(c+1)/\mu}\right)\,dt}{1 - \dfrac{\lambda(c+1-\mu)}{c+1}\displaystyle\int_1^{\infty} w(t^{-1/\mu}z)\left(1-t^{-(c+1)/\mu}\right)\,dt}\right] \\
&= -\frac{\dfrac{\lambda(c+1-\mu)}{\mu(c+1)}\displaystyle\int_1^{\infty} w(t^{-1/\mu}z)\left(\mu+(c+1-\mu)t^{-(c+1)/\mu}\right)\,dt}{1 - \dfrac{\lambda(c+1-\mu)}{c+1}\displaystyle\int_1^{\infty} w(t^{-1/\mu}z)\left(1-t^{-(c+1)/\mu}\right)\,dt}
\end{aligned}$$

so that

$$\begin{aligned}
\left|\frac{zF'(z)}{F(z)} - 1\right| &< \frac{\dfrac{\lambda(c+1-\mu)}{\mu(c+1)}\displaystyle\int_1^{\infty} t^{-n/\mu}\left(\mu+(c+1-\mu)t^{-(c+1)/\mu}\right)\,dt}{1 - \dfrac{\lambda(c+1-\mu)}{c+1}\displaystyle\int_1^{\infty} t^{-n/\mu}\left(1-t^{-(c+1)/\mu}\right)\,dt} \\
&< \frac{\dfrac{\lambda(c+1-\mu)}{c+1}\left[\dfrac{\mu}{n-\mu} + \dfrac{c+1-\mu}{c+1+n-\mu}\right]}{1 - \dfrac{\lambda\mu(c+1-\mu)}{c+1}\left[\dfrac{1}{n-\mu} - \dfrac{1}{c+1+n-\mu}\right]} \le 1-\alpha, \quad \text{by } (5.3).
\end{aligned}$$

This completes the proof. ■

The case $\mu = 1$ of Theorem 5.2 has been obtained in [11] (see also [12] for further discussion on this operator for $\mu = 1$). Taking $\alpha = 0$ in Theorem 5.2 we have

Corollary 5.9. *Let* $n \geq 1$*,* $\mu \in (0, n)$*,* $c + 1 - \mu > 0$ *and* $f \in \mathcal{U}_n(\lambda, \mu)$*, for some* λ *such that* $0 < \lambda \leq \dfrac{(n-\mu)(c+1-\mu+n)}{(c+1-\mu)(n+\mu)}$*. Then* F *defined in* (5.1) *satisfies the condition*

$$\left|\frac{zF'(z)}{F(z)} - 1\right| < 1, \quad z \in \Delta,$$

and, in particular, F *is starlike in* Δ*.*

In particular, if $f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{U}(\lambda)$ for some $0 < \lambda \leq n - 1$ and $n > 1$, then

$$\int_0^z \frac{t}{f(t)}\, dt$$

is starlike in Δ.

We end the paper with the following remark: It would be interesting to know whether the bounds/estimates in Theorems 3.1, 3.3, 3.4 and 5.2 are all sharp.

References

[1] I.E. Bazilevič: On a case of integrability in quadratures of the Löewner-Kufarev equation, *Mat. Sb.* **37**(79)(1955), 471-476.

[2] D. Brannan and W. Kirwan: On some classes of bounded univalent functions, *J. London Math. Soc.* (2) **1**(1969), 431-443.

[3] P. L. Duren: Univalent functions (Grundlehren der mathematischen Wissenschaften 259, New York, Berlin, Heidelberg, Tokyo), Springer-Verlag, 1983.

[4] A.W. Goodman: Univalent functions, Vol. I & II, Mariner Publ. Co. Tampa, Florida 1983.

[5] M. Obradović: A class of univalent functions, *Hokkaido Mathematical Journal* **27**(1998), 329-335.

[6] M. Obradović and S. Ponnusamy: New criteria and distortion theorems for univalent functions, *Complex Variables: Theory and Appl.* **44**(2001), 173-191. (Also Reports of the Department of Mathematics, Preprint 190, June 1998, University of Helsinki, Finland).

[7] M. Obradović, S. Ponnusamy, V. Singh and P. Vasundhra: Univalency, starlikeness and convexity applied to certain classes of rational functions, *Analysis (Munich)* **22**(3)(2002), 225-242.

[8] S. Ozaki and M. Nunokawa: The Schwarzian derivative and univalent functions, *Proc. Amer. Math. Soc.* **33**(2)(1972), 392-394.

[9] S. Ponnusamy: Pólya-Schoenberg conjecture by Carathéodory functions, *J. London Math. Soc.* (2)**51**(1995), 93-104.

[10] S. Ponnusamy and V. Singh: Convolution properties of some classes analytic functions, *Zapiski Nauchnych Seminarov POMI* **226**(1996), 138-154.

[11] S. Ponnusamy, V. Singh and P. Vasundhra: Starlikeness and convexity of an integral transform, *Integral Transforms and Special Functions*, **15**(3)(2004), 267–280.

[12] S. Ponnusamy and P. Vasundhra: Univalent functions with missing Taylor coefficients, *Hokkaido Mathematical Journal*, **33**(2004), 341–355.

[13] S. Ponnusamy and P. Vasundhra: Criteria for univalence, starlikeness and convexity, Annales Polonici Mathematici, To appear.

[14] St. Ruscheweyh: Convolutions in geometric function theory, Les Presses de l'Université de Montréal, Montréal, 1982.

[15] T. Sheil-Small: Some remarks on Bazilevič functions, *J. Analyse Math.* **43** (1983/84), 1-11.

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