An Existence Theorem of Solutions for Degenerate Semilinear Elliptic Equations

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Abstract

In this paper we study existence of solutions to a class of semilinear degenerate elliptic equations in Weighted Sobolev spaces.

1 Introduction

In this paper we prove the existence of a solution in $H_0(\Omega)$ (see definition in section 2) for the semilinear Dirichlet problem

$$(P)\begin{cases} Lu(x) - \mu u(x)g_1(x) + h(u(x))g_2(x) = f(x), & \text{in } \Omega \\ u(x) = 0, & \text{in } \partial\Omega \end{cases}$$

where L is an elliptic operator in divergence form

$$Lu(x) = -\sum_{i,j=1}^{n} D_j\Big(a_{ij}(x)D_i u(x)\Big), \quad \text{with} \quad D_j = \frac{\partial}{\partial x_j} \tag{1.1}$$

where the coefficients a_{ij} are measurable, real-valued functions whose coefficient matrix $\mathcal{A} = (a_{ij})$ is symmetric and satisfies the degenerate ellipticity condition

$$|\xi|^2\omega(x) \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq |\xi|^2 v(x), \tag{1.2}$$

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for all $\xi\in\mathbb{R}^n$ and almost everywhere $x\in\Omega$, $\Omega\subset\mathbb{R}^n$ is bounded and open, ω and v are weight functions (locally integrable, nonnegative functions on $\mathbb{R}^n$) and $\mu\in\mathbb{R}$.

The following will be proved in section 3.

THEOREM 1. Suppose that: (H1) The function $h:\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and bounded ($|h(t)|\leq M$, for all $t\in\mathbb{R}$); (H2) $(v,\omega)\in A_2$; (H3) $g_1/v\in L^\infty(\Omega)$, $g_2/\omega\in L^2(\Omega,\omega)$ and $f/\omega\in L^2(\Omega,\omega)$; (H4) $\mu>0$ is not an eigenvalue of the linearized problem

$$(LP)\begin{cases} Lu(x)-\mu u(x)g_1(x)=0, & \text{in } \Omega\\ u(x)=0, & \text{in } \partial\Omega.\end{cases}$$

Then the problem (P) has a solution $u\in H_0(\Omega)$.

Simple example. Let $\Omega=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}$. By theorem 1, with $h(t)=t\mathrm{e}^{-t^2}$, $f(x,y)=\mathrm{e}^{-(x^2+y^2)}$, $g_1(x,y)=(x^2+y^2)^{-1/3}\cos(xy)$, $g_2(x,y)=(x^2+y^2)^{-1/2}\mathrm{sen}(xy)$, $\omega(x,y)=(x^2+y^2)^{-1/2}$ and $v(x,y)=(x^2+y^2)^{-1/3}$ the problem

$$\begin{cases} Lu(x,y)-\mu u(x,y)g_1(x,y)+h(u(x,y))g_2(x,y)=f(x,y), & \text{in } \Omega\\ u(x,y)=0, & \text{in } \partial\Omega\end{cases}$$

where

$$Lu(x,y)=-\frac{\partial}{\partial x}\left((x^2+y^2)^{-1/2}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left((x^2+y^2)^{-1/3}\frac{\partial u}{\partial y}\right),$$

has solution $u\in H_0(\Omega)$ if $\mu>0$ is not an eigenvalue of the linearized problem (LP).

2 Definitions and basic results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0<\omega<\infty$ almost everywhere. We say that ω belongs to a Muckenhoupt class A_p, $1<p<\infty$, or that ω is an A_p-weight, if there is a constant $C_1=C_{p,\omega}$ such that

$$\left(\frac{1}{|B|}\int_B\omega(x)dx\right)\left(\frac{1}{|B|}\int_B\omega^{1/(1-p)}(x)dx\right)^{p-1}\leq C_1$$

for all balls B in $\mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1<q\leq p$ then $A_q\subset A_p$ (see [HKM] or [GR] for more information about A_p-weights). As an example of A_p-weights, if $x\in\mathbb{R}^n$, the function $\omega(x)=|x|^\alpha$ is A_p if and only if $-n<\alpha<n(p-1)$. Let $\Omega\subset\mathbb{R}^n$ be a bounded open set. We shall denote by $L^p(\Omega,\omega)$ ($1\leq p<\infty$) the Banach space of all measurable functions, f, defined in Ω for which

$$\|f\|_{L^p(\Omega,\omega)}=\left(\int_\Omega|f(x)|^p\omega(x)dx\right)^{1/p}<\infty.$$

For $p\geq1$ and k a nonnegative integer, the Weighted Sobolev spaces $W^{k,p}(\Omega,\omega)$ is defined by

$$W^{k,p}(\Omega,\omega)=\{u\in L^p(\Omega,\omega) : D^\alpha u\in L^p(\Omega,\omega),\ \ 1\leq|\alpha|\leq k\}$$

with norm

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)dx + \sum_{1 \le |\alpha| \le k} \int_\Omega |D^\alpha u(x)|^p \omega(x)dx \right)^{1/p}. \tag{2.1}$$

If $\omega \in A_p$ then $W^{k,p}(\Omega,\omega)$ is a closure of $C^\infty(\overline{\Omega})$ with respect to the norm (2.1) (see proposition 3.5 in [CS]). The space $W_0^{k,p}(\Omega,\omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{k,p}(\Omega,\omega)} = \left(\sum_{1 \le |\alpha| \le k} \int_\Omega |D^\alpha u(x)|^p \omega(x)dx \right)^{1/p}.$$

When $k = 1$ and $p = 2$ the spaces $W^{1,2}(\Omega,\omega)$ and $W_0^{1,2}(\Omega,\omega)$ are Hilbert spaces. The space $H(\Omega)$ is defined to be the completion of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\|u\|_{H(\Omega)} = \left(\int_\Omega u^2 v dx + \int_\Omega < \mathcal{A}\nabla u, \nabla u > dx \right)^{1/2}$$

where $\mathcal{A} = (a_{ij})$ is the coefficient matrix of operator L defined in (1.1) , $< .,. >$ denotes the usual inner product in $\mathbb{R}^n$ and the symbol ∇ indicates the gradient. The space $H_0(\Omega)$ is defined to be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H_0(\Omega)} = \left(\int_\Omega < \mathcal{A}\nabla u, \nabla u > dx \right)^{1/2}.$$

We say that the pair (v,ω) of nonnegative locally integrable functions v and ω satisfies the condition A_p, $1 < p < \infty$, and we write $(v,\omega) \in A_p$, if there is a constant $C_2 = C_{p,v,\omega}$ such that

$$\left(\frac{1}{|B|} \int_B v(x)dx \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)}(x)dx \right)^{p-1} \le C_2,$$

for all balls B in $\mathbb{R}^n$.

Remark 2. If $(v,\omega) \in A_p$ and $\omega \le v$ then $v \in A_p$ and $\omega \in A_p$. In this cases, for $p = 2$ and using condition (1.2) we obtain

$$\int_\Omega |\nabla u|^2 \omega dx \le \int_\Omega < \mathcal{A}\nabla u, \nabla u > dx \le \int_\Omega |\nabla u|^2 v dx.$$

Therefore $W_0^{1,2}(\Omega,v) \subset H_0(\Omega) \subset W_0^{1,2}(\Omega,\omega)$. ∎

We make the following basic assumption on the weights ω and v.

The Weighted Sobolev Inequality (WSI). Let Ω be a bounded open set in $\mathbb{R}^n$. There is an index $q = 2\sigma$, $\sigma > 1$, such that for every ball B and every $f \in \mathrm{Lip}_0(B)$ (i.e., $f \in \mathrm{Lip}(B)$ and whose support is contained in the interior of B),

$$\left(\frac{1}{v(B)} \int_B |f|^q v dx \right)^{1/q} \le C R_B \left(\frac{1}{\omega(B)} \int_B |\nabla f|^2 \omega dx \right)^{1/2}$$

with the constant C independent of f and B, R_B is the radius of B, $v(B) = \int_B v(x)dx$ and $\omega(B) = \int_B \omega(x)dx$. Thus, we can write

$$\|f\|_{L^q(B,v)} \leq C_S \| |\nabla f| \|_{L^2(B,\omega)}$$

where C_S is called the Sobolev constant and

$$C_S = \frac{C[v(B)]^{1/q} R_B}{[\omega(B)]^{1/2}}.$$

For instance, the WSI holds if ω and v are as in Theorem 4.8, chapter X of [T] or if ω and v are as in Theorem 1.5 of [CW]. ■

Lemma 3. If $\omega \in A_2$ then $W_0^{1,2}(\Omega,\omega) \hookrightarrow L_2(\Omega,\omega)$ is compact and

$$\|u\|_{L_2(\Omega,\omega)} \leq C_2 \|u\|_{W_0^{1,2}(\Omega,\omega)}.$$

Proof. The proof of this lemma follows the lines of theorem 4.6 in [FS]. ■

Remark 4. Let $q = 2\sigma$, $\sigma > 1$ be as in (WSI). We have that: (i) If $u \in L^q(\Omega, v)$ then $u \in L^2(\Omega, v)$ and $\|u\|_{L^2(\Omega,v)} \leq [v(\Omega)]^{1/2\sigma'} \|u\|_{L^q(\Omega,v)}$. (ii) If $u \in H_0(\Omega)$ then

$$\int_\Omega |\nabla u|^2 \omega dx \leq \int_\Omega < \mathcal{A}\nabla u, \nabla u > dx < \infty.$$

Using (WSI) we obtain

$$\|u\|_{L^q(\Omega,v)} \leq C_S \left(\int_B |\nabla u|^2 \omega dx \right)^{1/2},$$

that is, $u \in L^q(\Omega, v)$. Hence, using (i),we get $u \in L^2(\Omega, v)$. Therefore $H_0(\Omega) \subset L^2(\Omega, v)$ and

$$\|u\|_{L^2(\Omega,v)} \leq C_S [v(\Omega)]^{1/2\sigma'} \|u\|_{H_0(\Omega)}.$$

Definition 5. We say that an element $u \in H_0(\Omega)$ is a (weak) solution of problem (P) if

$$\int_\Omega \Big(a_{ij}(x) D_i u(x) D_j \varphi(x) - \mu u(x) g_1(x) \varphi(x) \Big) dx + \int_\Omega h(u(x)) g_2(x) \varphi(x) dx = \int_\Omega f(x) \varphi(x) dx$$

for every $\varphi \in H_0(\Omega)$.

3 Proof of theorem 1

The basic idea is to reduce (P) to an operator equation $Bu + Nu = T$ and apply the following theorem.
Theorem A. Let $B, N : X \longrightarrow X^*$ be forms on the real separable reflexive Banach space X. Assume:
(a) The operator $B : X \longrightarrow X^*$ is linear and continuous;
(b) The operator $N : X \longrightarrow X^*$ is demicontinuous and bounded;
(c) $B + N$ is asymptotically linear;
(d) For each $T \in X^*$ and each $t \in [0, 1]$ the operator $A_t(u) = Bu + t(Nu - T)$ satisfies condition (S) in X.
If $Bu = 0$ implies $u = 0$, then for each $T \in X^*$ the operator equation $Bu + Nu = T$ has a solution in X.
Proof. See [H] or theorem 29.C in [EZ]. ■

Remark 6. Let X be a real separable reflexive Banach space.
(i) The operator $N : X \longrightarrow X^*$ is said to be demicontinuous if

$$u_n \longrightarrow u \text{ implies } Nu_n \rightharpoonup Nu, \text{ as } n \longrightarrow \infty.$$

(ii) The operator N is strongly continuous if

$$u_n \rightharpoonup u \text{ implies } Nu_n \longrightarrow Nu, \text{ as } n \rightarrow \infty.$$

(iii) $B + N : X \longrightarrow X^*$ is asymptotically linear if B is linear and

$$\frac{\|Nu\|}{\|u\|} \longrightarrow 0 \text{ as } \|u\| \longrightarrow \infty.$$

(iv) The operator $B : X \longrightarrow X^*$ satisfies condition (S) if

$$u_n \rightharpoonup u \text{ and } \lim_{n \to \infty} (Bu_n - Bu | u_n - u) = 0 \text{ implies } u_n \longrightarrow u,$$

where $(f|x)$ denotes the value of linear functional f at the point x. ■

Step 1. We define the operators $B_1, B_2 : H_0(\Omega) \times H_0(\Omega) \longrightarrow \mathbb{R}$ through

$$B_1(u, \varphi) = \int_\Omega a_{ij}(x) D_i u(x) D_j \varphi(x) dx - \mu \int_\Omega u(x) \varphi(x) g_1(x) dx,$$
$$B_2(u, \varphi) = \int_\Omega h(u(x)) g_2(x) \varphi(x) dx,$$

and $T : H_0(\Omega) \longrightarrow \mathbb{R}$ through

$$T(\varphi) = \int_\Omega f(x) \varphi(x) dx.$$

We have that $u \in H_0(\Omega)$ solves problem (P) if

$$B_1(u, \varphi) + B_2(u, \varphi) = T(\varphi), \text{ for all } \varphi \in H_0(\Omega).$$

Using Hölder inequality, condition (H3) and remark 4(ii) we get

$$\begin{aligned}
|B_1(u,\varphi)| &\leq \int_\Omega | < \mathcal{A}\nabla u, \nabla\varphi > | dx + |\mu| \int_\Omega |u||\varphi||g_1| dx \\
&\leq \int_\Omega < \mathcal{A}\nabla u, \nabla u >^{1/2} < \mathcal{A}\nabla\varphi, \nabla\varphi >^{1/2} dx + |\mu| \int_\Omega |u||\varphi| \left|\frac{g_1}{v}\right| v dx \\
&\leq \left(\int_\Omega < \mathcal{A}\nabla u, \nabla u > dx \right)^{1/2} \left(\int_\Omega < \mathcal{A}\nabla\varphi, \nabla\varphi > dx \right)^{1/2} + \\
&\quad + |\mu| \|g_1/v\|_{L^\infty(\Omega)} \int_\Omega |u||\varphi| v dx \\
&\leq \|u\|_{H_0(\Omega)} \|\varphi\|_{H_0(\Omega)} + |\mu| \|g_1/v\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega,v)} \|\varphi\|_{L^2(\Omega,v)} \\
&\leq \left(1 + C|\mu| \|g_1/v\|_{L^\infty(\Omega)} \right) \|u\|_{H_0(\Omega)} \|\varphi\|_{H_0(\Omega)} \\
&= \mathbf{C} \|u\|_{H_0(\Omega)} \|\varphi\|_{H_0(\Omega)}.
\end{aligned}$$

By conditions (H1) and (H3), Lemma 3 and remark 2, we obtain

$$\begin{aligned}
|B_2(u,\varphi)| &\leq \int_\Omega |h(u)||\varphi||g_2| dx \\
&\leq M \int_\Omega \left|\frac{g_2}{\omega}\right| |\varphi| \omega dx \\
&\leq M \|g_2/\omega\|_{L^2(\Omega,\omega)} \|\varphi\|_{L^2(\Omega,\omega)} \\
&\leq M \|g_2/\omega\|_{L^2(\Omega,\omega)} C_2 \|\varphi\|_{W_0^{1,2}(\Omega,\omega)} \\
&\leq C_2 M \|g_2/\omega\|_{L^2(\Omega,\omega)} \|\varphi\|_{H_0(\Omega)}. \qquad (3.1)
\end{aligned}$$

Moreover, we also have

$$\begin{aligned}
|T(\varphi)| &\leq \int_\Omega |f||\varphi| dx \\
&= \int_\Omega \left(\frac{|f|}{\omega} \right) |\varphi| \omega dx \\
&\leq \|f/\omega\|_{L^2(\Omega,\omega)} \|\varphi\|_{L^2(\Omega,\omega)} \\
&\leq C_2 \|f/\omega\|_{L^2(\Omega,\omega)} \|\varphi\|_{W_0^{1,2}(\Omega,\omega)} \\
&\leq C_2 \|f/\omega\|_{L^2(\Omega,\omega)} \|\varphi\|_{H_0(\Omega)}.
\end{aligned}$$

Step 2. Since $H_0(\Omega)$ is a real Hilbert space with inner product

$$a_0(u,\varphi) = \int_\Omega < \mathcal{A}\nabla u, \nabla\varphi > dx$$

using the Identification Principle (theorem 21.18 in [EZ]) we set $H_0(\Omega) = [H_0(\Omega)]^*$ and $a_0(u,\varphi) = (u|\varphi)$ (if $f \in X^*$ and $u \in X$, then $(f|u) = f(u)$).

We define the operators $B, N : H_0(\Omega) \longrightarrow H_0(\Omega)$ through

$$\begin{aligned}
(Bu|\varphi) &= B_1(u,\varphi); \\
(Nu|\varphi) &= B_2(u,\varphi), \forall u,\varphi \in H_0(\Omega).
\end{aligned}$$

Since $T\in[H_0(\Omega)]^*$, the problem (P) is equivalent to the operator equation

$$Bu + Nu = T, \quad u\in H_0(\Omega).$$

Step 3: Using that $H_0(\Omega)\hookrightarrow L_2(\Omega, v)$ is compact (see Lemma 3 and remark 4(ii)), we have that $B_1(.,.)$ is a regular Gårding form. In fact: since $\mu > 0$ and by condition (1.2) we obtain

$$\begin{aligned} B_1(u,u) &= \int_\Omega a_{ij} D_i u D_j u dx - \mu \int_\Omega u^2 g_1 dx \\ &= \int_\Omega < \mathcal{A}\nabla u, \nabla u > dx - \mu \int_\Omega u^2 \left(\frac{g_1}{v}\right) v dx \\ &\geq \int_\Omega < \mathcal{A}\nabla u, \nabla u > dx - \mu \|g_1/v\|_{L^\infty(\Omega)} \int_\Omega u^2 v dx \\ &= \|u\|^2_{H_0(\Omega)} - \mu \|g_1/v\|_{L^\infty(\Omega)} \|u\|^2_{L_2(\Omega,v)}. \end{aligned}$$

Hence, there exist a decomposition of the form $B = T_1 + T_2$, where T_1 and T_2 are bilinear and bounded, $T_1(.,.)$ is strongly positive and $T_2(.,.)$ is compact (see lemma 22.38 in [EZ]). Thus, B is Fredholm of index zero (see definition 8.13 and theorem 21.F in [EZ]) and B satisfies condition (S) (see proposition 27.12, [EZ]).

Step 4: By (3.1) we get

$$\begin{aligned} |(Nu, \varphi)| &= |B_2(u,\varphi)| \\ &\leq C_2 M \|g_2/\omega\|_{L^\infty(\Omega)} \|\varphi\|_{H_0(\Omega)}. \end{aligned}$$

Hence, $\|Nu\|\leq C$, for all $u\in H_0(\Omega)$. Therefore,

$$\frac{\|Nu\|}{\|u\|} \longrightarrow 0, \quad \text{as} \quad \|u\|_{H_0(\Omega)} \longrightarrow \infty,$$

that is, $B + N$ is asymptotically linear and the operator N is strongly continuous (see corollary 26.14 in [EZ]).

Step 5. For each $t\in[0,1]$, the operator $A_t(u) = Bu + t(Nu - T)$ is a strongly continuous perturbation of the operator B. Thus, the operator A_t also satisfies condition (S) (see proposition 27.12, [EZ]).

If μ is not an eigenvalue of the linearized problem (LP), $Bu = 0$ implies $u = 0$. Therefore, by theorem A, the operator equation $Bu + Nu = T$ has a solution $u\in H_0(\Omega)$ and u is solution for the problem (P). ∎

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