Euclidean Geometric Objects in the Clifford Geometric Algebra of {Origin, 3-Space, Infinity}

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Abstract

This paper concentrates on the homogeneous (conformal) model of Euclidean space (*Horosphere*) with subspaces that intuitively correspond to Euclidean geometric objects in three dimensions. Mathematical details of the construction and (useful) parametrizations of the 3D Euclidean object models are explicitly demonstrated in order to show how 3D Euclidean information on positions, orientations and radii can be extracted.

1 Introduction

The Clifford geometric algebra of three dimensional (3D) Euclidean space with vectors

$$\mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3. \tag{1}$$

given in terms of an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, nicely encodes the algebra of 3D subspaces with algebraic basis

$$\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{i}_1 = \mathbf{e}_2\mathbf{e}_3, \mathbf{i}_2 = \mathbf{e}_3\mathbf{e}_1, \mathbf{i}_3 = \mathbf{e}_1\mathbf{e}_2, i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\}, \tag{2}$$

providing geometric multivector product expressions of rotations and set theoretic operations[1]. But in this framework line and plane subspaces always contain the origin.

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The homogeneous (conformal) model of 3D Euclidean space in the Clifford geometric algebra $\mathbb{R}_{4,1}$ provides a way out.[5, 7] Here positions of points, lines and planes, etc. off the origin can be naturally encoded. Other advantages are the unified treatment of rotations and translations and ways to encode point pairs, circles and spheres. The creation of such elementary geometric objects simply occurs by algebraically joining a minimal number of points in the object subspace. The resulting multivector expressions completely encode in their components positions, orientations and radii.

The geometric algebra $\mathbb{R}_{4,1}$ can be intuitively pictured as the algebra of origin $\bar{\mathbf{n}}$, Euclidean 3D space and infinity $\mathbf{n}$, where origin and infinity are represented by additional linearly independent null-vectors.

$$\{\bar{\mathbf{n}}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{n}\}, \;\; \mathbf{n}^2 = \bar{\mathbf{n}}^2 = 0. \tag{3}$$

This algebra seems most suitable for applications in computer graphics, robotics and other fields.[2, 3]

This paper concentrates on giving explicit details for the construction of fundamental geometric objects in this model, detailing how the 3D geometric information can be extracted. How the simple multivector representations of these objects can be manipulated in order to move them in three dimensions and to express set theoretic operations of union (join), intersection (meet), projections and rejections is described in [8, 7, 2].

This algebraic encoding of geometric objects and their manipulations strongly suggests an object oriented software implementation. This does allow computers to calculate with this algebra and provides programmers with the means to most suitably represent fundamental geometric objects, their 3D properties and ways (methods) to manipulate these objects. This happens on a higher algebraic level, so that the programmer actually is freed of the need to first investigate suitable less intuitive matrix representations.[8, 9, 10, 11, 12]

The mathematical notation used in this paper for multivectors, geometric products and products derived from the geometric product is fairly standard: Italic capital letters are used for conformal vectors and multivectors, bold lower case vectors represent Euclidean vectors (except the origin and infinity null vectors), the wedge $\wedge$ signifies the outer product, the asterisk $*$ the scalar product, the angles $\rfloor$ and $\lfloor$ left and right contractions and mere juxtaposition the full geometric product of multivectors. Other notations are explicitly defined where they are used.[6, 13]

2 3D Information in Homogeneous Objects

We will see how homogenous multivectors completely encode positions, directions, moments and radii of the corresponding three dimensional (3D) objects in Euclidean space. An overview of this is give in Table 1. In the rest of this section we will look at the details of extracting the encoded 3D information from each homogeneous multivector object. Where suitable, we will also give useful alternative parametrizations of homogeneous multivector objects.[1]

[1]Dorst and Fontijne[14] give similar parametrizations, but with the rather strong simplification, that objects are centered at the origin $\bar{\mathbf{n}}$ or contain the origin $\bar{\mathbf{n}}$, e.g. $C = \bar{\mathbf{n}}$ in eq. (15), etc.

homogeneous object	3D information
point X	position $\mathbf{x}$
point pair $P_1 \wedge P_2$	positions $\mathbf{p}_1, \mathbf{p}_2$
line	direction vector, moment bivector
circle	plane bivector, center, radius
plane	plane bivector, location vector
sphere	center, radius

Table 1: 3D geometric information in homogeneous objects. The left column lists the homogeneous multivectors, that represent the geometric objects.

2.1 Point and Pair of Points

The Euclidean position $\mathbf{p}$ of a conformal point

$$P = \mathbf{p} + \frac{1}{2}p^2\mathbf{n} + \bar{\mathbf{n}} \tag{4}$$

is obtained with the help of the (additive[7]) *conformal split*, which is an example of a rejection[4]: The conformal point vector P is rejected off the Minkowski plane represented by the bivector $N = \mathbf{n} \wedge \bar{\mathbf{n}}$

$$\mathbf{p} = (P \wedge N)N. \tag{5}$$

Equation (4) shows how to achieve the opposite, i.e. how to get back to the conformal point P from just knowing the Euclidean position $\mathbf{p}$.

The Euclidean positions $\mathbf{p}_1, \mathbf{p}_2$ of a pair of points represented by the conformal bivector

$$\begin{aligned} V_2 &= P_1 \wedge P_2 \\ &= \mathbf{p}_1 \wedge \mathbf{p}_2 + \frac{1}{2}(p_2^2\mathbf{p}_1 - p_1^2\mathbf{p}_2)\mathbf{n} - (\mathbf{p}_2 - \mathbf{p}_1)\bar{\mathbf{n}} + \frac{1}{2}(p_1^2 - p_2^2)N \\ &= \mathbf{b} + \frac{1}{2}\mathbf{v}\mathbf{n} - \mathbf{u}\bar{\mathbf{n}} - \frac{1}{2}\gamma N \end{aligned} \tag{6}$$

can be fully reconstructed from the components of V_2. We assume without restricting the generality, that $p_1 = \sqrt{\mathbf{p}_1^2} \geq p_2 = \sqrt{\mathbf{p}_2^2}$. Given any conformal bivector V_2 with components $\mathbf{b}$ (a Euclidean bivector), $\mathbf{u}$ and $\mathbf{v}$ (Euclidean vectors of length $u = \sqrt{\mathbf{u}^2}$ and $v = \sqrt{\mathbf{v}^2}$), and γ (a real scalar), the calculation works as follows

$$\sigma = \frac{1}{2}\gamma^2 - \mathbf{u} * \mathbf{v}, \qquad \rho = \sqrt{\sigma^2 - u^2v^2}, \tag{7}$$

$$p_1 = \frac{\sqrt{\sigma + \rho}}{u}, \qquad p_2 = \frac{\sqrt{\sigma - \rho}}{u}, \tag{8}$$

$$\mathbf{p}_1 = p_1\frac{p_1^2\mathbf{u} + \mathbf{v}}{|p_1^2\mathbf{u} + \mathbf{v}|}, \qquad \mathbf{p}_2 = p_2\frac{p_2^2\mathbf{u} + \mathbf{v}}{|p_2^2\mathbf{u} + \mathbf{v}|}. \tag{9}$$

The general and explicit formulas presented in the following, seem to appear nowhere else in the published literature so far.

This calculation is the full solution (of two conformal points $X = P_1, P_2$) to the equation

$$V_2 \wedge X = 0, X^2 = 0. \tag{10}$$

We can further view conformal point pairs as one-dimensional circles and arrive thereby at another highly useful characterization[2]:

$$P_1 \wedge P_2 = 2r\{\hat{\mathbf{p}} \wedge \mathbf{c} + \frac{1}{2}[(c^2 + r^2)\hat{\mathbf{p}} - 2\mathbf{c} * \hat{\mathbf{p}}\ \mathbf{c}]\mathbf{n} + \hat{\mathbf{p}}\bar{\mathbf{n}} + \mathbf{c} * \hat{\mathbf{p}}N\}, \tag{11}$$

with the "radius" r defined as half the Euclidean point pair distance, $\hat{\mathbf{p}}$ a unit vector pointing from $\mathbf{p}_2$ to $\mathbf{p}_1$, and $\mathbf{c}$ the Euclidean midpoint (center) of the point pair:

$$2r = |\ \mathbf{p}_1 - \mathbf{p}_2\ |, \ \ \hat{\mathbf{p}} = \frac{\mathbf{p}_1 - \mathbf{p}_2}{2r}, \ \ \mathbf{c} = \frac{\mathbf{p}_1 + \mathbf{p}_2}{2}. \tag{12}$$

In case that the straight line defined by the point pair contains the origin, i.e. for $\hat{\mathbf{p}} \wedge \mathbf{c} = 0$ ($\hat{\mathbf{p}} \parallel \mathbf{c}$) we get the simplified form

$$P_1 \wedge P_2 = 2r\{C - \frac{1}{2}r^2\mathbf{n}\}\hat{\mathbf{p}}N. \tag{13}$$

In case that the Euclidean midpoint vector $\mathbf{c}$ is perpendicular to $\hat{\mathbf{p}}$ ($\hat{\mathbf{p}} \perp \mathbf{c}$), i.e. if $\hat{\mathbf{p}} * \mathbf{c} = 0$ we get

$$P_1 \wedge P_2 = -2r\{C + \frac{1}{2}r^2\mathbf{n}\}\hat{\mathbf{p}}. \tag{14}$$

In both cases we used the conformal representation of the midpoint as

$$C = \mathbf{c} + \frac{1}{2}c^2\mathbf{n} + \bar{\mathbf{n}}. \tag{15}$$

2.2 Lines

Given two conformal points P_1 and P_2 the conformal trivector

$$V_{line} = P_1 \wedge P_2 \wedge \mathbf{n} = \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{n} + (\mathbf{p}_2 - \mathbf{p}_1)N = \mathbf{m}\,\mathbf{n} + \mathbf{d}N \tag{16}$$

conveniently consists of the defining entities of the Euclidean line through $\mathbf{p}_1$ and $\mathbf{p}_2$. The Euclidean bivector $\mathbf{m}$ represents the moment and the Euclidean vector $\mathbf{d}$ the direction of the line. Using $\mathbf{m}$ and $\mathbf{d}$ we can give the parametric form of the line as

$$\mathbf{x} = (\mathbf{m} + \alpha)\mathbf{d}^{-1}, \alpha \in \mathbb{R}. \tag{17}$$

All points $X = \mathbf{x} + \frac{1}{2}x^2\mathbf{n} + \bar{\mathbf{n}}$ with the $\mathbf{x}$ as specified in (17) represent the full solution to the problem

$$V_{line} \wedge X = 0, X^2 = 0. \tag{18}$$

The one-dimensional circle representation of point pairs (11) immediately leads to a second often useful parametrization of lines as

$$P_1 \wedge P_2 \wedge \mathbf{n} = 2r\hat{\mathbf{p}} \wedge C \wedge n = 2r\{\hat{\mathbf{p}} \wedge \mathbf{c}\ \mathbf{n} - \hat{\mathbf{p}}N\}. \tag{19}$$

It is important to note that the conformal point C in eq. (19) does not need to be the midpoint of the point pair. Any conformal point on the straight line $P_1 \wedge P_2 \wedge \mathbf{n}$ can take the place of C in eq. (19). $\hat{\mathbf{p}}$ and r are defined as in eq. (12).

[2]This characterization is e.g. very useful for investigating the full (real and virtual) meet of two circles, or of a straight line and a circle.[15]

2.3 Circles

General conformal trivectors of the form

$$V_3 = P_1 \wedge P_2 \wedge P_3, \tag{20}$$

with conformal points P_1, P_2 and P_3 represent Euclidean circles through the corresponding Euclidean points $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$. The equation for all points X on such a circle is again given as

$$V_3 \wedge X = 0, X^2 = 0. \tag{21}$$

In order to clearly interpret and apply V_3 and its various components we will explicitly insert the three points

$$P_1 = \mathbf{p}_1 + \frac{1}{2}p_1^2\mathbf{n} + \bar{\mathbf{n}}, P_2 = \mathbf{p}_2 + \frac{1}{2}p_2^2\mathbf{n} + \bar{\mathbf{n}}, P_3 = \mathbf{p}_3 + \frac{1}{2}p_3^2\mathbf{n} + \bar{\mathbf{n}}. \tag{22}$$

The conformal circle trivector becomes

$$\begin{aligned} V_3 \;=\; & \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{p}_3 \\ & + \frac{1}{2}(p_1^2\mathbf{p}_2 \wedge \mathbf{p}_3 + p_2^2\mathbf{p}_3 \wedge \mathbf{p}_1 + p_3^2\mathbf{p}_1 \wedge \mathbf{p}_2)\mathbf{n} \\ & + (\mathbf{p}_2 \wedge \mathbf{p}_3 + \mathbf{p}_3 \wedge \mathbf{p}_1 + \mathbf{p}_1 \wedge \mathbf{p}_2)\bar{\mathbf{n}} \\ & + \frac{1}{2}\{\mathbf{p}_1(p_2^2 - p_3^3) + \mathbf{p}_2(p_3^2 - p_1^3) + \mathbf{p}_3(p_1^2 - p_2^3)\}N \end{aligned} \tag{23}$$

The Euclidean bivector component factor of $\bar{\mathbf{n}}$

$$\begin{aligned} I_c &= -\{[V_3 + (V_3 * i)i] \wedge \mathbf{n}\}N \\ &= \mathbf{p}_2 \wedge \mathbf{p}_3 + \mathbf{p}_3 \wedge \mathbf{p}_1 + \mathbf{p}_1 \wedge \mathbf{p}_2 \\ &= (\mathbf{p}_1 - \mathbf{p}_2) \wedge (\mathbf{p}_2 - \mathbf{p}_3) \end{aligned} \tag{24}$$

is obviously parallel to the plane (of the Euclidean circle) through $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$. Assuming the Euclidean center vector of the circle to be $\mathbf{c}$ and the radius r, we can rewrite (22) as

$$P_k = \mathbf{c} + r\mathbf{r}_k + \frac{1}{2}(c^2 + r^2 + 2r\mathbf{c} * \mathbf{r}_k)\mathbf{n} + \bar{\mathbf{n}}, \;\; \mathbf{r}_k^2 = 1, \;\; k = 1, 2, 3. \tag{25}$$

The three vectors $\mathbf{r}_k$ are unit length vectors pointing from the circle center $\mathbf{c}$ to the three points $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$, respectively. Replacing the P_k in (23) accordingly we get after doing some algebra the simplified form

$$V_3 = \mathbf{c} \wedge I_c + [\frac{1}{2}(r^2 + c^2)I_c - \mathbf{c}(\mathbf{c} \rfloor I_c)]\mathbf{n} + I_c\bar{\mathbf{n}} - (\mathbf{c} \rfloor I_c)N. \tag{26}$$

We see that the three vectors $\mathbf{r}_k, k = 1, 2, 3$ do no longer occur explicitly. They enter equation (26) only by defining the orientation of the circle plane I_c in (24).

If we assume only to know V_3 as outer product (20) of three general conformal points we can now extract the radius r by calculating

$$r^2 = -\frac{V_3^2}{I_c^2}. \tag{27}$$

We can decompose the center vector $\mathbf{c}$ by way of projection and rejection into components parallel and perpendicular to the circle plane

$$\begin{aligned}\mathbf{c} &= \mathbf{c}_\parallel + \mathbf{c}_\perp, \qquad (28)\\ \mathbf{c}_\parallel &= (\mathbf{c} \rfloor I_c^{-1})I_c = (\mathbf{c} \rfloor I_c)I_c^{-1} \overset{(26)}{=} -[(V_3 \llcorner \mathbf{n}) \llcorner \bar{\mathbf{n}}]I_c^{-1}, \qquad (29)\\ \mathbf{c}_\perp &= (\mathbf{c} \wedge I_c^{-1})I_c = (\mathbf{c} \wedge I_c)I_c^{-1} \overset{(26)}{=} -(V_3 * i)iI_c^{-1}. \qquad (30)\end{aligned}$$

The Euclidean circle center vector can hence be extracted from any V_3 as

$$\mathbf{c} = \mathbf{c}_\parallel + \mathbf{c}_\perp \overset{(29),(30),(24)}{=} -[(V_3 \llcorner n) \llcorner \bar{n} + (V_3 * i)i]I_c^{-1} \qquad (31)$$

Inserting the decomposition $\mathbf{c} = \mathbf{c}_\parallel + \mathbf{c}_\perp$ we get the following expression for the circle trivector

$$\begin{aligned}V_3 &= \mathbf{c}_\perp I_c + [\frac{1}{2}(r^2 - c^2)I_c + \mathbf{c}\mathbf{c}_\perp I_c]\mathbf{n} + I_c\bar{\mathbf{n}} - \mathbf{c}_\parallel I_c N\\ &= \{\mathbf{c}_\perp N + [\frac{1}{2}(r^2 - c^2) + \mathbf{c}\mathbf{c}_\perp]\mathbf{n} - \bar{\mathbf{n}} - \mathbf{c}_\parallel\}I_c N\\ &= \{-\mathbf{c}_\parallel - \frac{1}{2}c_\parallel^2\mathbf{n} - \bar{\mathbf{n}} + \frac{1}{2}r^2\mathbf{n} + \mathbf{c}_\perp N + [-\frac{1}{2}c_\perp^2 + \mathbf{c}\mathbf{c}_\perp]\mathbf{n}\}I_c N \qquad (32)\end{aligned}$$

In the case that the circle plane includes the origin ($\mathbf{c}_\perp = 0$) we are left with

$$V_3 = -[C - \frac{1}{2}r^2\mathbf{n}]I_c N, \qquad (33)$$

and can extract the conformal center

$$C = \mathbf{c} + \frac{1}{2}c^2\mathbf{n} + \bar{\mathbf{n}} \qquad (34)$$

simply as

$$C = -V_3 N I_c^{-1} + \frac{1}{2}r^2\mathbf{n}. \qquad (35)$$

2.4 Planes

Given three conformal points P_1, P_2 and P_3 as in (22) the conformal 4-vector

$$\begin{aligned}V_{plane} &= P_1 \wedge P_2 \wedge P_3 \wedge \mathbf{n}\\ &= \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{p}_3 \wedge \mathbf{n}\\ &\quad -(\mathbf{p}_2 \wedge \mathbf{p}_3 + \mathbf{p}_3 \wedge \mathbf{p}_1 + \mathbf{p}_1 \wedge \mathbf{p}_2)N \qquad (36)\end{aligned}$$

represents the plane through the Euclidean points $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$. The Euclidean bivector component factor of N

$$I_p = -(V_{plane}\mathbf{n}) \llcorner \bar{\mathbf{n}} = \mathbf{p}_2 \wedge \mathbf{p}_3 + \mathbf{p}_3 \wedge \mathbf{p}_1 + \mathbf{p}_1 \wedge \mathbf{p}_2 = (\mathbf{p}_1 - \mathbf{p}_2) \wedge (\mathbf{p}_2 - \mathbf{p}_3) \qquad (37)$$

gives the orientation of the plane in the Euclidean space. This allows us to rewrite V_{plane} as

$$V_{plane} = (\mathbf{d} \wedge I_p)\mathbf{n} - I_p N = \mathbf{d}I_p\mathbf{n} - I_p N, \qquad (38)$$

where $\mathbf{d}$ represents the Euclidean distance vector from the origin to the plane, itself perpendicular to the plane. The Euclidean distance vector can be extracted from V_{plane} by

$$\mathbf{d} = (V_{plane} \wedge \bar{\mathbf{n}}) I_p^{-1} N. \tag{39}$$

The equation for all points X on the plane is again given as

$$V_{plane} \wedge X = 0, X^2 = 0. \tag{40}$$

Somewhat in analogy of the relation of point pairs as one-dimensional circles (11) and the resulting alternative parametrization of lines (19), an alternative parametrization of planes by means of a general conformal point C on the plane is possible

$$P_1 \wedge P_2 \wedge P_3 \wedge \mathbf{n} = C \wedge I_c \wedge \mathbf{n} = \mathbf{c} \wedge I_c \mathbf{n} - I_c N \tag{41}$$

For $\mathbf{c} \wedge I_c = 0$ (origin $\bar{\mathbf{n}}$ in plane) we get

$$P_1 \wedge P_2 \wedge P_3 \wedge \mathbf{n} = -I_c N. \tag{42}$$

2.5 Spheres

General conformal 4-vectors of the form

$$V_4 = P_1 \wedge P_2 \wedge P_3 \wedge P_4 \tag{43}$$

with conformal points

$$P_k = \mathbf{p}_k + \frac{1}{2} p_k^2 \mathbf{n} + \bar{\mathbf{n}}, k = 1, 2, 3, 4 \tag{44}$$

represent Euclidean spheres through the corresponding Euclidean points $\mathbf{p}_k, k = 1, 2, 3, 4$. The equation for all points X on the sphere is again given as

$$V_4 \wedge X = 0, X^2 = 0. \tag{45}$$

Inserting (44) explicitly in V_4 yields

$$\begin{aligned} V_4 \;=\; & -\frac{1}{2}(p_1^2 \mathbf{p}_{234} + p_2^2 \mathbf{p}_{314} + p_3^2 \mathbf{p}_{124} + p_4^2 \mathbf{p}_{132})\mathbf{n} \\ & -(\mathbf{p}_{234} + \mathbf{p}_{314} + \mathbf{p}_{124} + \mathbf{p}_{132})\bar{\mathbf{n}} \\ & +\frac{1}{2}\{(p_2^2 - p_3^2)\mathbf{p}_{14} + (p_3^2 - p_1^2)\mathbf{p}_{24} + (p_1^2 - p_2^2)\mathbf{p}_{34} \\ & \quad +(p_1^2 - p_4^2)\mathbf{p}_{23} + (p_2^2 - p_4^2)\mathbf{p}_{31} + (p_3^2 - p_4^2)\mathbf{p}_{12}\}N, \end{aligned} \tag{46}$$

with the abbreviations

$$\mathbf{p}_{kl} = \mathbf{p}_k \wedge \mathbf{p}_l, \;\; \mathbf{p}_{klm} = \mathbf{p}_k \wedge \mathbf{p}_l \wedge \mathbf{p}_m, \;\; k, l, m \in \{1, 2, 3, 4\}. \tag{47}$$

The $\bar{\mathbf{n}}$ factor component

$$\begin{aligned} i_s &= -(V_4 \wedge \mathbf{n})N \\ &= -(\mathbf{p}_{234} + \mathbf{p}_{314} + \mathbf{p}_{124} + \mathbf{p}_{132}) \\ &= (\mathbf{p}_1 - \mathbf{p}_2) \wedge (\mathbf{p}_2 - \mathbf{p}_3) \wedge (\mathbf{p}_3 - \mathbf{p}_4) \end{aligned} \tag{48}$$

is a Euclidean pseudoscalar, i.e. proportional to i. Similar to the discussion of the circle, assuming the Euclidean center vector of the sphere to be $\mathbf{c}$ and the radius r, we can rewrite (44) as

$$P_k = \mathbf{c} + r\mathbf{r}_k + \frac{1}{2}(c^2 + r^2 + 2r\mathbf{c} * \mathbf{r}_k)\mathbf{n} + \bar{\mathbf{n}}, \;\; \mathbf{r}_k^2 = 1, \;\; k = 1,2,3,4. \tag{49}$$

Replacing the $P_k, k = 1,2,3,4$ in (46) accordingly we get after doing lots of algebra

$$\begin{aligned} V_4 &= \frac{1}{2}(r^2 - c^2)i_s\mathbf{n} + i_s\bar{\mathbf{n}} + \mathbf{c}i_sN \\ &= (\mathbf{c} + \frac{1}{2}c^2\mathbf{n} + \bar{\mathbf{n}} - \frac{1}{2}r^2\mathbf{n})i_sN \\ &= (C - \frac{1}{2}r^2\mathbf{n})i_sN, \end{aligned} \tag{50}$$

where C represents the conformal center of the sphere. An important relationship used in the derivation of (50) is

$$i_s = (\mathbf{p}_1 - \mathbf{p}_2) \wedge (\mathbf{p}_2 - \mathbf{p}_3) \wedge (\mathbf{p}_3 - \mathbf{p}_4) = r^3(\mathbf{r}_1 - \mathbf{r}_2) \wedge (\mathbf{r}_2 - \mathbf{r}_3) \wedge (\mathbf{r}_3 - \mathbf{r}_4). \tag{51}$$

The elegant form (50) of V_4 makes it easy to extract the radius and the center from any general conformal (sphere) 4-vector:

$$r^2 = \frac{V_4^2}{(V_4 \wedge \mathbf{n})^2}, \qquad C = \frac{1}{2}r^2\mathbf{n} + \frac{V_4}{-V_4 \wedge \mathbf{n}}. \tag{52}$$

3 Conclusions

We explained how to algebraically construct conformal (homogeneous) subspaces with very intuitive Euclidean interpretations.[3]

We then analyzed in detail how the joining of conformal points yields explicit expressions for points, pairs of points, lines, circles, planes and spheres. After that we showed how the Euclidean 3D information of positions, orientations and radii, etc. can be extracted.[4] In some cases useful alternative parametrizations were given. Applications of these alternative parametrizations can e.g. be found in [15].

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[3] They are e.g. implemented, together with algebraic expressions for arbitrary translations and rotations, and for subspace operations of union (join), intersection (meet), projection and rejection as methods in the GeometricAlgebra Java package.[8, 9, 16]

[4] These formulas precisely yield the optimal mathematical structure of the related Java methods each geometric object is to have e.g. in the GeometricAlgebra Java package implementation. [9]

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