Elliptic spaces with the rational homotopy type of spheres

Geoffrey M.L. Powell *

1 Introduction

This paper is directed towards an understanding of those p-elliptic spaces which have the rational homotopy type of a sphere, by classifying the algebraic models which occur when the space satisfies an additional 'large prime' hypothesis, relative to the prime p. The main results of the paper are given at the end of this section.

Definition 1.1 [10] A topological space Z is p-elliptic if it has the p-local homotopy type of a finite, 1-connected CW complex and the loop space homology $H_*(\Omega Z; \mathbf{F}_p)$, with coefficients in the prime field of characteristic p, is an elliptic Hopf algebra. (That is: finitely-generated as an algebra and nilpotent as a Hopf algebra [9]). ■

The Milnor-Moore theorem shows that the $\mathbf{Q}$-elliptic spaces are precisely those spaces which have the rational homotopy type of finite, 1-connected CW complexes and have finite total rational homotopy rank. This class of spaces is important because of the *dichotomy theorem* (the subject of the book [8]) which states that a finite, 1-connected complex either has finite total rational homotopy rank or the rational homotopy groups have exponential growth when regarded as a graded vector space. Moreover, elliptic spaces are the subject of the Moore conjectures, asserting that the homotopy groups of a finite, 1-connected CW complex have finite exponent at all primes if and only if it is $\mathbf{Q}$-elliptic.

The p-elliptic spaces form a sub-class of the class of $\mathbf{Q}$-elliptic spaces. A p-elliptic space Z is known to satisfy the following important properties [10, 11].

1. $H^*(Z; \mathbf{F}_p)$ is a Poincaré duality algebra.

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2. The rationalization $Z_{\mathbf{Q}}$ is a $\mathbf{Q}$-elliptic space; in particular $\pi_*(Z) \otimes \mathbf{Q}$ is finite dimensional.
3. The formal dimension of Z over $\mathbf{F}_p$, $\mathrm{fd}(Z; \mathbf{F}_p)$, is determined by the Hilbert series of $H_*(\Omega Z; \mathbf{F}_p)$ and is equal to the formal dimension over the rationals, $\mathrm{fd}(Z; \mathbf{Q})$. (The formal dimension of a space X over a field $\mathbf{k}$ is $\mathrm{fd}(X; \mathbf{k}) = \sup\{m | H^m(X; \mathbf{k}) \neq 0\}$).
4. p^r annihilates the torsion module of $H_*(\Omega Z; \mathbf{Z}_{(p)})$ for some integer $r \geq 0$.

Many examples of p-elliptic spaces are known; for example finite (1-connected) H-spaces, spheres, the total space of a fibration in which both the base space and the fibre are elliptic. However, little is known regarding a general description or classification of these spaces, even under the 'large prime hypothesis' defined below in Definition 1.2.

The spheres may be regarded as being the simplest possible $\mathbf{Q}$-elliptic spaces, so it is natural to address the question of identifying those p-elliptic spaces which have the rational homotopy type of a sphere; such spaces lead to examples of p-elliptic spaces which do *not* have the p-local homotopy type of a finite H-space.

Application of the algebraic arguments used in this paper require the following restriction on the prime under consideration:

Definition 1.2 Suppose that Z is a CW complex with cells in degrees $(r, n]$, where $r \geq 1$. A prime p is a **large prime** for Z if $p \geq n/r$ or $p = 0$, when we understand that $\mathbf{F}_0 = \mathbf{Q}$. ■

We discuss the formality of these spaces, with coefficients in a field. This property is studied in [7], where equivalent conditions are formulated.

Definition 1.3 Suppose that X is a 1-connected space with $\mathbf{F}_p$-homology of finite type as a vector space.

1. X is p-formal if a minimal Adams-Hilton model $\mathcal{A} = T(V)$ for X over $\mathbf{F}_p$ has a quadratic differential (that is $d : V \to V \otimes V \subset T(V)$).
2. X is weakly p-formal if the Eilenberg-Moore spectral sequence collapses $\mathrm{Tor}^{H^*(X;\mathbf{F}_p)}(\mathbf{F}_p, \mathbf{F}_p) \Rightarrow H_*(\Omega X; \mathbf{F}_p)$.

■

The main result of this paper may be stated as follows; a more precise version of the second statement is given in Section 3.1. Write $X \sim_{\mathbf{Q}} Y$ to indicate that X has the rational homotopy type of Y.

Theorem 1 *Suppose that p is a large prime for the 1-connected space Z, which is p-elliptic, and that $Z \sim_{\mathbf{Q}} S^N$, for $N \geq 2$.*

1. *If $N = 2n$, then Z has the p-local homotopy type of S^N.*
2. *If $N = 2n + 1$, then Z is p-formal and has cohomology algebra $H^*(Z; \mathbf{F}_p) \cong \Lambda a(2t-1) \otimes B(2t)$, where $t \geq 1$ and $B(2t)$ is an algebra with the same Hilbert series as $\mathbf{F}_p[b(2t)]/(b^m)$ for some $m \geq 1$ and $N = 2mt - 1$, where the numbers in parentheses indicate the degrees of elements.*

The first part of this theorem is a special case of Theorem 3 of Section 2.1. The results of Section 3.1 may be summarized in the following result:

Theorem 2 *Suppose that p is a large prime for the 1-connected space Z, which is p-elliptic, and that $Z \sim_{\mathbf{Q}} S^{2n+1}$. The Adams-Hilton model for a p-minimal decomposition of Z may be taken to be the cobar construction on the dual of a commutative graded differential algebra $\mathcal{A} = \mathcal{B} \otimes \Lambda x_1$ where $\mathcal{B} = \mathbf{Z}_{(p)}[y_1, \ldots, y_K]/(p^{r_{j+1}} y_{j+1} + y_j^{\alpha_j} = 0)$ with $dy_j = 0$ and $dx_1 = p^{r_1} y_1$, where $r_j \geq 1$ and $\alpha_j \geq 2$ for all relevant j; here $|x_1| = 2t - 1$ for some $t \geq 1$, $|y_1| = |x_1| + 1$ and $2n + 1 = 2t(\prod_{i=1}^{K} \alpha_i) - 1$.*

1.1 Examples

It is easy to show that there exist p-elliptic spaces with the rational homotopy type of a sphere but which do not have the p-local homotopy type of a sphere. Since a p-elliptic space is 1-connected and the cohomology algebra satisfies Poincaré duality, the fewest number of cells for which this may occur is three.

Take p an odd prime and integers $n \geq 2, k \geq 1$; let $P^{2n}(p^k)$ denote the Moore space which is the cofibre of the Brouwer degree p^k map: $S^{2n-1} \xrightarrow{p^k} S^{2n-1}$. Let ι denote the identity map on $P^{2n}(p^k)$ and $[\iota, \iota]$ be the Whitehead product of this map with itself (for details of p-primary homotopy theory, see [15]). Now let $\alpha : S^{4n-2} \to P^{2n}(p^k)$ be the restriction of $[\iota, \iota]$ to the $(4n-2)$-skeleton of $P^{4n-1}(p^k)$. Define $Z = P^{2n}(p^k) \cup_\alpha e^{4n-1}$; it may be shown that the space Z is p-elliptic (for example by calculating the Adams-Hilton model, using the knowledge of the attaching maps) and Z visibly has the rational homotopy type of the sphere S^{4n-1}, since $P^{2n}(p^k)$ is rationally acyclic when $k \geq 1$. Moreover, if $p \geq (n+2)$, then it may be shown that (up to homotopy) this is the unique three cell space having this property.

This is the first in a sequence of such p-elliptic spaces, examples which were first considered in [2]. As above, take p an odd prime and integers $k \geq 1, t \geq 1$; let $S^{2t+1}\{p^k\}$ be the fibre of the Brouwer map $S^{2t+1} \xrightarrow{p^k} S^{2t+1}$. Then, for any integer $m \geq 2$, define $V_m = V_m(p^k, t)$ to be the $(2mt-1)$-skeleton of the p-minimal CW decomposition of $\Omega S^{2t+1}\{p^k\}$. The cohomology algebra $H^*(\Omega S^{2t+1}\{p^k\}; \mathbf{F}_p)$ is well-known and the inclusion $V_m \to \Omega S^{2t+1}\{p^k\}$ induces a surjection $H^*(\Omega S^{2t+1}\{p^k\}; \mathbf{F}_p) \to H^*(V_m; \mathbf{F}_p)$ which is an isomorphism of vector spaces in degrees $\leq (2mt-1)$.

For $2 \leq m < p$, one may conclude that $H^*(V_m; \mathbf{F}_p) \cong \Lambda(a_{2t-1}) \otimes \mathbf{F}_p[b_{2t}]/(b^m)$, the tensor product of an exterior algebra by a truncated polynomial algebra. The Eilenberg-Moore spectral sequence converging to the mod-p loop space homology of V_m collapses giving:

Proposition 1.4 *The space $V_m = V_m(p^k, t)$, $2 \leq m < p$, is p-elliptic and has the rational homotopy type of S^{2mt-1}. The mod-p loop space homology is isomorphic (as a Hopf algebra) to a universal enveloping algebra, $H_*(\Omega V_m; \mathbf{F}_p) \cong UL_m$ where, for $m > 2$, L_m is the abelian Lie algebra $L_m = \langle x_{2t-2}, y_{2t-1}, z_{2mt-1} \rangle$ and, for $m = 2$, the graded Lie algebra $L_2 = \langle x_{2t-2}, y_{2t-1}, [y, y] \rangle$.*

These spaces are very well understood; for $m < p - 1$, decompositions for ΩV_m as a product of atomic factors may be given directly by using the methods of [6], thus generalizing the results of [2]. In addition, if X_m denotes the $2mt$-skeleton of $\Omega S^{2t+1}\{p^k\}$, so that $V_m = X_{m-1} \cup e^{2mt-1}$, then such decompositions may be given for X_m.

These provide very useful explicit examples of the behaviour of elliptic spaces.■

There is no reason to believe that these are the only p-elliptic spaces which have the rational homotopy type of odd spheres. Consider the following algebraic example as evidence for this; it is intended to resemble an Adams-Hilton model for a space [1]:

Example 1.5 Suppose that p is an odd prime; for fixed integers $N \geq 2, r \geq 0, k \geq 1$, define a differential graded algebra $\mathcal{A} = \mathcal{A}(r)$ over $\mathbf{Z}_{(p)}$, the integers localized at p, as the tensor algebra with generators in the degrees indicated by the subscripts:

$$\mathcal{A} = T(a_{2N-2}, b_{2N-1}, c_{4N-2}, e_{4N-1}, f_{6N-2}, g_{6N-1}, \omega_{8N-2}).$$

and with differential of degree -1 defined by $db = p^k a$, $dc = [a,b]$, $de = p^r(p^k c - b^2)$, $df = [a,e] - p^r[b,c]$, $dg = p^k f - [b,e]$, $d\omega = [a,g] - [b,f] - [c,e]$. Thus $\mathcal{A}$ is the universal enveloping algebra on a differential, free graded Lie algebra. (The reader is invited to check that the above defines a differential, so that $d^2 = 0$).

When $r = 0$, the algebra $\mathcal{A}$ may be taken as an Adams-Hilton model for the space $V_4(p^k, N)$ considered in the previous example. For $r \geq 1$, standard algebraic arguments may be used to show that $H_*(\mathcal{A} \otimes \mathbf{F}_p, d)$ is an elliptic Hopf algebra: the 'cohomology' corresponding to $\mathcal{A}$ may be calculated and the 'Eilenberg-Moore spectral sequence' has initial term which is of polynomial growth, which suffices by [9].

In fact, if the prime p is sufficiently large compared with N, so that the model lies within the 'tame range', the constructions of tame homotopy theory [14] may be used to show that $\mathcal{A}$ may be realized as the Adams-Hilton model of a p-elliptic space X. This space has the rational homotopy type of an odd sphere but is not homotopically equivalent to any of the V_m's. ■

The paper is organized as follows: the next section considers the algebraic model which is used and proves the first part of Theorem 1. Section 3 then proves the part concerning those spaces with the rational homotopy type of an odd sphere. Finally, Section 3.1 shows how one can use this to completely determine the $\mathbf{Z}_{(p)}$-model and indicates how this yields the Adams-Hilton model by a property of formality over the ring $\mathbf{Z}_{(p)}$.

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2 Algebraic models at large primes

The large prime restriction is of importance for the work of Anick [3], which shows that, if p is a large prime for a finite, 1-connected CW complex X, then an Adams-

Hilton model for X over $\mathbf{Z}_{(p)}$ may be taken to be a universal enveloping algebra on a differential free graded Lie algebra. Halperin proves in [13] that this implies the existence of a commutative, minimal model $(\Lambda W, d)$ over $\mathbf{Z}_{(p)}$ for the cochains on the space X, satisfying:

- $W = W^{\geq 1}$ is a free $\mathbf{Z}_{(p)}$-module and ΛW is the free graded commutative algebra on W.
- d is a differential of degree $+1$; if d_1 denotes the linear part of d then $d_1 \otimes \mathbf{F}_p$ is trivial. (The **minimality** condition).
- There exists a sequence of morphisms $\rightarrow$, $\leftarrow$ between $(\Lambda W, d)$ and $C^*(X; \mathbf{Z}_{(p)})$, each of which induces a homology equivalence. ($(\Lambda W, d)$ is a **model**).
- If (Y, ∂) denotes the chain complex $[s(W, d_1)]^*$ (the dual of the suspension of the linear part of the complex, so that $Y_i \cong (W^{i+1})^*$) then there exists an $\mathbf{F}_p$-Lie algebra $E_{\mathbf{F}_p}$ and a $\mathbf{Q}$-Lie algebra $E_{\mathbf{Q}}$ such that $E_{\mathbf{F}_p} \cong Y \otimes \mathbf{F}_p$ and $E_{\mathbf{Q}} \cong H_*(Y \otimes \mathbf{Q})$ as vector spaces *and* $H_*(\Omega X; \mathbf{F}_p) \cong UE_{\mathbf{F}_p}$, $H_*(\Omega X; \mathbf{Q}) \cong UE_{\mathbf{Q}}$ as Hopf algebras.

Notation/ Convention: Here and throughout the rest of the paper, the abbreviation CGDA denotes a commutative graded differential algebra. All CGDAs will be *connected*, so that the indecomposables are in degrees ≥ 1. ∎

It is straightforward to see that a 1-connected CW complex, Z, at a large prime p is p-elliptic if and only if it has a minimal, commutative cochain model $(\Lambda W, d)$ as above with both W and $H^*(\Lambda W, d)$ finitely-generated $\mathbf{Z}_{(p)}$-modules.

The equality between $\mathrm{fd}(Z; \mathbf{F}_p)$ and $\mathrm{fd}(Z; \mathbf{Q})$ leads to the following result; more precise restrictions may be given on the degrees of the generators of U.

Proposition 2.1 *Suppose that p is a large prime for a space Z which is p-elliptic and has minimal, commutative cochain model $(\Lambda W, d)$ over $\mathbf{Z}_{(p)}$. Then $W \cong W_0 \oplus U$ as $\mathbf{Z}_{(p)}$-modules, where $H^*(W, d_1) \otimes \mathbf{Q} \cong W_0 \otimes \mathbf{Q} \cong \pi_*(Z) \otimes \mathbf{Q}$ and the linear differential $d_1 \otimes \mathbf{Q} : U^{odd} \otimes \mathbf{Q} \to U^{even} \otimes \mathbf{Q}$ is an isomorphism of $\mathbf{Q}$-vector spaces. Moreover, W_0 is concentrated in degrees $\leq 2n_Z - 1$ and U is concentrated in degrees $\leq n_Z - 2$, where n_Z is the formal dimension of Z.*

Sketch of proof: The loop space homology of Z is determined as a graded vector space by W and, if W has generators in degrees $(2b_i - 1)$ for $1 \leq i \leq 1$ and $(2a_j)$ for $1 \leq j \leq r$, then $\mathrm{fd}(Z; \mathbf{F}_p) = \sum_{i=1}^{q}(2b_i - 1) - \sum_{j=1}^{r}(2a_i - 1)$. A similar statement holds for $\mathrm{fd}(Z; \mathbf{Q})$ in terms of the generators of W_0. Since these are equal and $W_0 \otimes \mathbf{Q} = H^*(W, d_1) \otimes \mathbf{Q}$, the action of d_1 may be deduced: a differential $d_1 : b \in U^{odd} \mapsto a \in U^{even}$ 'removes' a pair (b, a) from W which has a contribution of zero to the above sum. However, a differential $d_1 : \alpha \in U^{even} \mapsto \beta \in U^{odd}$ would remove a pair (α, β) with a contribution of two to the sum. This is impossible, so that the linear part of the differential acts as claimed.

The statement regarding the degrees of the generators of W_0 follows from standard results in rational homotopy theory; details may be found in [8, Chapter 5]. The condition on U may be derived from arguments similar to those given below. ∎

Notation: Recall some of the details relating to the construction of the 'odd spectral sequence' (see [8, Chapter 5] for the rational version), generalized to the study of free CGDAs over $\mathbf{Z}_{(p)}$.

Suppose that $(A, d) = (\Lambda Z, d)$ is a free CGDA over $\mathbf{Z}_{(p)}$, where Z is some choice of the module of indecomposables of A, so that $Z \cong A^+/(A^+.A^+)$ as a $\mathbf{Z}_{(p)}$-module. Then Z has a direct sum decomposition as $Z = Z^{odd} \oplus Z^{even}$ into odd and even degree parts.

Let $\langle Z^{odd}\rangle$ denote the two-sided ideal of A generated by Z^{odd}; this ideal does not depend on the choice of Z. Correspondingly, by an abuse of notation, let $\Lambda(Z^{even})$ denote the quotient algebra $A/\langle Z^{odd}\rangle$. This quotient is independent of the choice of Z but there is an isomorphism $\Lambda(Z^{even}) \cong A/\langle Z^{odd}\rangle$ for any choice of Z, induced from the inclusion of algebras $\Lambda(Z^{even}) \hookrightarrow A$.

For a given choice of Z, let $\hat{d} : Z^{odd} \to \Lambda(Z^{even})$ denote the composite $d : Z^{odd} \to (\Lambda Z)^{even} \to \Lambda(Z^{even})$. The ideal $\mathcal{I} := \hat{d}(Z^{odd})\Lambda(Z^{even})$ is independent of the choice of Z and is generated by elements $\hat{d}(z_i)$ as z_i ranges through a basis of some Z^{odd}. ■

Given a free CGDA (A, d) over $\mathbf{Z}_{(p)}$, one may consider $A \otimes \mathbf{F}_p$ as a free CGDA over $\mathbf{F}_p$; the following lemmas are then standard.

Lemma 2.2 *Suppose that $(\Lambda Z, d)$ is a free CGDA over $\mathbf{F}_p$ and that $y \in \Lambda(Z^{even})$. If $dy \neq 0$ then $dy^n = 0$ if and only if $n \equiv 0 \mod p$.*

Lemma 2.3 *Suppose that $(\Lambda Z, d)$ is a free, minimal CGDA over $\mathbf{F}_p$ and that $Y \in Z^{even}$. If $H^*(\Lambda Z, d)$ is finite dimensional, there exists $x \in \Lambda(Z^{even}) \otimes Z^{odd}$ and an integer $\alpha \geq 2$ such that $dx \equiv y^\alpha \mod \langle Z^{odd}\rangle$.*

Proof: y^{pk} is a cocycle for all $k \geq 1$; thus there exists a minimal integer K such that it is a coboundary (since the cohomology is in bounded degree). Therefore, there exists $z \in (\Lambda Z)^{odd}$ such that $dz = y^{pK}$. Write $z = x + \Phi$, where $x \in \Lambda(Z^{even}) \otimes Z^{odd}$ and $\Phi \in \Lambda Z^{even} \otimes \Lambda^{>1}(Z^{odd})$, then x will suffice. The condition $\alpha \geq 2$ follows by the minimality hypothesis. ■

Proposition 2.4 *Suppose that $(\Lambda Z, d)$ is a free, minimal $\mathbf{Z}_{(p)}$-CGDA which is elliptic (so that Z and $H^*(\Lambda Z)$ are finitely generated modules). Let B denote the algebra $B := \Lambda(Z^{even}) \otimes \mathbf{F}_p$ and $\mathcal{J}$ denote the ideal $\mathcal{J} := \mathcal{I} \otimes \mathbf{F}_p$. Then $B/\mathcal{J}$ is a finite-dimensional $\mathbf{F}_p$-algebra, generated by elements in the image of $Z^{even} \otimes \mathbf{F}_p \to B/\mathcal{J}$.*

Proof: The statement concerning the generators of $B/\mathcal{J}$ is clear. Since $B/\mathcal{J}$ is a finitely-generated, commutative algebra, it suffices to show that the algebra generators are nilpotent. Take $u \in Z^{even} \otimes \mathbf{F}_p$; by Lemma 2.3, there exists a minimal n such that u^{pn} is a coboundary in $\Lambda Z \otimes \mathbf{F}_p$, with $u^{pn} \equiv dx \mod \langle Z^{odd}\rangle$, where x may be taken in $\Lambda(Z^{even}) \otimes Z^{odd}$. In particular, extending $\hat{d}$ as a map of $\Lambda(Z^{even})$-modules, $u^{pn} = \hat{d}(x) \in \mathcal{J}$, the ideal generated by $\hat{d}(Z^{odd}) \otimes \mathbf{F}_p$. This shows that $u^{pn} = 0$ in $B/\mathcal{J}$. ■

Remark 2.5 These results are an important part of the consideration of the odd spectral sequence for elliptic $\mathbf{Z}_{(p)}$-CGDAs. ■

In order to prove the results of this paper, basic results concerning complete intersections of Krull dimension zero are considered.

Proposition 2.6 *Suppose that $A := \mathbf{F}_p[y_1, \ldots, y_n]/(\phi_1, \ldots, \phi_k)$ is a graded commutative algebra, with the generators and relations in even degrees ≥ 2.*

1. *Suppose that A is finite dimensional as an $\mathbf{F}_p$-vector space, then $k \geq n$.*
2. *If $k = n$, then A is finite dimensional over $\mathbf{F}_p$ if and only if $(\phi_1, \ldots, \phi_n)$ is a regular sequence, when A is a complete intersection of dimension zero.*

This is a standard result for the non-graded case; for the graded commutative case, the requisite material is covered fairly briefly in [5, Chapter 4]. In particular, note that a finite dimensional, graded, connected algebra has Krull dimension zero. For algebraic topologists, [4, Section 3] may be a familiar reference.

Corollary 2.7 *Suppose that A is as in Proposition 2.6; there does not exist an ordering of the generators and relations and an integer $2 \leq m \leq n$ such that $\phi_1, \ldots, \phi_m \in \mathbf{F}_p[y_1, \ldots, y_{m-1}]$.*

Proof: Suppose that $\phi_1, \ldots, \phi_m \in \mathbf{F}_p[y_1, \ldots, y_{m-1}]$ and pass to the quotient by the ideal generated by $(\mathbf{F}_p[y_1, \ldots, y_{m-1}])^+$. This gives a commutative diagram:

$$\begin{array}{ccc} \mathbf{F}_p[y_1, \ldots, y_n] & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathbf{F}_p[y_m, \ldots, y_n] & \longrightarrow & \mathbf{F}_p[y_m, \ldots, y_n]/(\overline{\phi}_{m+1}, \ldots, \overline{\phi}_n) \end{array}$$

(where $\overline{\phi}_*$ denotes the image of ϕ_*), in which all the arrow are surjections. However, A is finite dimensional so that $\mathbf{F}_p[y_m, \ldots, y_n]/(\overline{\phi}_{m+1}, \ldots, \overline{\phi}_n)$ must be as well. This contradicts Proposition 2.6. ■

In particular, for the application, the following hypothesis is valid;

Hypothesis 2.8 The elements ϕ_k lie in the sub-algebra of $\mathbf{F}_p[y_1, \ldots, y_n]$ generated by the elements of degree $< |\phi_k|$.

Corollary 2.9 *Suppose that A is as in Proposition 2.6 and that Hypothesis 2.8 applies. Then, for all k, $|\phi_k| > |y_k|$.*

2.1 The Euler-Poincaré characteristic

To state the main result of this section, recall the following definition:

Definition 2.10 Suppose that $(\Lambda Z, d)$ is a minimal, free $\mathbf{Z}_{(p)}$-CGDA such that Z is a finitely-generated $\mathbf{Z}_{(p)}$-module. The Euler-Poincaré characteristic of $(\Lambda Z, d)$ over $\mathbf{F}_p$ is defined as $\chi_\pi(\Lambda Z; \mathbf{F}_p) := \dim(Z^{even}) - \dim(Z^{odd})$.
The rational Euler-Poincaré characteristic is defined as $\chi_\pi(\Lambda Z; \mathbf{Q}) := \dim(W^{even}) - \dim(W^{odd})$, where $(\Lambda W, d)$ is a $\mathbf{Q}$-minimal model for $(\Lambda Z, d) \otimes \mathbf{Q}$. ■

A standard result for minimal models in rational homotopy theory implies the following result:

Proposition 2.11 *Suppose that $(\Lambda Z, d)$ is a minimal, free $\mathbf{Z}_{(p)}$-CGDA such that Z is finite dimensional, then $\chi_\pi(\Lambda Z; \mathbf{F}_p) = \chi_\pi(\Lambda Z; \mathbf{Q})$.*

As an application of the previous theory, we have the following result:

Theorem 3 *Suppose that $(\Lambda Z, d)$ is a minimal, elliptic $\mathbf{Z}_{(p)}$-CGDA with Euler-Poincaré characteristic $\chi_\pi(\Lambda Z; \mathbf{F}_p) = 0$. The lowest degree elements of Z are in even degree and are cocycles. In particular, $\tilde{H}^*(\Lambda Z; \mathbf{F}_p)$ has lowest degree elements in even degree.*

Proof: By Proposition 2.4, the algebra $B/\mathcal{J}$ is finite dimensional. Suppose that $\dim Z^{even} = n = \dim Z^{odd}$ (equality by the hypothesis on the Euler-Poincaré characteristic), then $B/\mathcal{J} \cong \mathbf{F}_p[y_1, \ldots, y_n]/(\phi_1, \ldots, \phi_n)$ where $\{y_1, \ldots, y_n\}$ are in degree-preserving bijection with a basis of $Z^{even} \otimes \mathbf{F}_p$ and ϕ_j represents $\hat{d}(z^j)$, as z_j ranges through a basis of $Z^{odd} \otimes \mathbf{F}_p$, so that $|\phi_j| = |z_j| + 1$; we may order the bases by increasing degree. The minimality condition on $(\Lambda Z, d)$ shows that Hypothesis 2.8 holds, so that Corollary 2.9 implies that $|\phi_1| > |y_1|$, which proves the result. ∎

The proof of Theorem 1, part 1 appears as a corollary.

Corollary 2.12 *Suppose that Z is a 1-connected, p-elliptic space for which p is a large prime. If Z has the rational homotopy type of a sphere S^{2n} then it has the p-local homotopy type of S^{2n}.*

Proof: It suffices to show that $\tilde{H}^*(Z; \mathbf{F}_p)$ is concentrated in degree $2n$. Proposition 2.1 shows that Z has a minimal, commutative cochain model $(\Lambda W, d)$ over $\mathbf{Z}_{(p)}$ with $W = U \oplus \langle w_{2n}, z_{4n-1} \rangle$, with U concentrated in degrees $\leq 2n-2$ and d acts as stated in the Proposition (the subscripts indicate the degrees of the elements). The form of W_0 follows from the well-known rational homotopy groups for an even sphere.

In particular, $\chi_\pi(\Lambda Z; \mathbf{F}_p) = 0$, so that the Theorem may be applied. In particular, if U were non-trivial, then the lowest degree element of W lies in U and would be in odd degree, contradicting the Theorem. Conclude that U is trivial; thus $(\Lambda W, d) \cong (\Lambda(w, z), dz = w^2)$, so that $\tilde{H}^*(Z; \mathbf{F}_p)$ is one dimensional in degree $2n$.

3 The odd sphere case

A significant step of the proof of part 2 of Theorem 1 is in showing that no two generators of W lie in the same degree. This is done by considering the model $(\Lambda W, d) \otimes \mathbf{F}_p$, with coefficients in the prime field.

Proposition 3.1 *Suppose that Z is a p-elliptic space for which p is a large prime, and that $Z \sim_{\mathbf{Q}} S^{2n+1}$, for some $n \geq 1$. Then Z is p-formal and $H^*(Z; \mathbf{F}_p) \cong \Lambda x(2t-1) \otimes B(2t)$ (as stated in Theorem 1) and the minimal model $(\Lambda W, d)$ over $\mathbf{Z}_{(p)}$ has at most one generator in each degree.*

Proof: By Proposition 2.1, Z has a minimal cochain model of the form $(\Lambda W, d)$ with $W \cong \langle w_0 \rangle \oplus U$, where $|w_0| = 2n+1$ and U is concentrated in degrees $\leq 2n-2$; d_1 induces an isomorphism $U^{odd} \otimes \mathbf{Q} \to U^{even} \otimes \mathbf{Q}$.

To commence, one shows that W is at most one-dimensional in each degree, using the previous theory. Choose bases for U^{odd} and U^{even} in order of increasing degree, $\{x_1, \dots, x_K\}$, $\{y_1, \dots, y_K\}$, respectively, so that $d_1 x_i = y_i$ over $\mathbf{Q}$ and $|x_i| = |y_i| - 1$. For notational purposes, w_0 may be denoted by x_{K+1}.

Consider the algebra $B/\mathcal{J}$, as in Proposition 2.4; by minimality of $(\Lambda W, d)$, x_1 is a cocycle over $\mathbf{F}_p$, so that $B/\mathcal{J} \cong \mathbf{F}_p[y_1, \dots, y_K]/(\phi_2, \dots, \phi_{K+1})$ where $\phi_i = \hat{d}x_i, (2 \leq i \leq K+1)$ in $\Lambda(W^{even}) \otimes \mathbf{F}_p$. Now, since $(\Lambda W, d)$ is elliptic, $B/\mathcal{J}$ is finite dimensional; moreover the minimality condition implies that Hypothesis 2.8 holds, so that $\phi_2, \dots, \phi_{K+1}$ is a regular sequence, with $|\phi_{i+1}| > |y_i|$. Since the elements y_i, ϕ_j are in even degrees, this implies that the bases are in order of strictly increasing degree.

Claim: *The algebra $\mathbf{F}_p[y_1, \dots, y_k]/(\phi_2, \dots, \phi_{k+1})$ is finite dimensional for each k.*

It suffices to show that $\phi_2, \dots, \phi_{k+1}$ is a regular sequence for $\mathbf{F}_p[y_1, \dots, y_k]$ for each k. Suppose that ϕ_{m+1} is a zero divisor in the ring $\Gamma(m) := \mathbf{F}_p[y_1, \dots, y_m]/(\phi_2, \dots, \phi_m)$, so that there exists an element $\zeta \in \mathbf{F}_p[y_1, \dots, y_m]$ representing a non-zero element in $\Gamma(m)$ such that $\phi_{m+1}\zeta$ is zero in $\Gamma(m)$. To derive a contradiction to the fact that $\phi_2, \dots, \phi_{K+1}$ is a regular sequence for $\mathbf{F}_p[y_1, \dots, y_K]$, it suffices to show that $\zeta \not\equiv 0$ in $\mathbf{F}_p[y_1, \dots, y_K]/(\phi_2, \dots, \phi_m)$. This is immediate: the algebra map $\mathbf{F}_p[y_1, \dots, y_K] \to \mathbf{F}_p[y_1, \dots, y_m]$ sending $y_j, (j > m)$ to zero and $y_i \mapsto y_i$, for $i \leq m$, passes to a map $\mathbf{F}_p[y_1, \dots, y_K]/(\phi_2, \dots, \phi_m) \to \mathbf{F}_p[y_1, \dots, y_m]/(\phi_2, \dots, \phi_m)$ which is split by the canonical inclusion. Consider the image of ζ under these maps. ∎

In particular, this implies that $\phi_k (k \geq 2)$ is in degree $\alpha_k |y_k|$, for some integer $\alpha_k \geq 2$. (If not, y_k would be an element of infinite height in $\mathbf{F}_p[y_1, \dots, y_k]/(\phi_2, \dots, \phi_{k+1})$, contradicting the claim).

Taking coefficients in $\mathbf{F}_p$, the minimality condition implies that one may define sub-differential graded algebras $A_m \subset (\Lambda W, d) \otimes \mathbf{F}_p$ by:

$$A_m = (\Lambda(x_1, y_1, \dots, x_{m-1}, y_{m-1}, x_m), d) \otimes \mathbf{F}_p.$$

Here $A_1 = (\Lambda(x_1, 0) \otimes \mathbf{F}_p)$, $A_{K+1} = (\Lambda W, d) \otimes \mathbf{F}_p$ and $A_{m+1} = (A_m \otimes \Lambda(y_m, x_{m+1}), d)$. ($A_m$ cannot be defined as a sub-differential graded algebra over $\mathbf{Z}_{(p)}$, since the linear part of dx_m involves y_m).

Claim: *For each m, $1 \leq m \leq K+1$, $H^*(A_m, d)$ is finite dimensional, concentrated in degrees $\leq |x_m| < |y_m|$.*

Proof by induction: the statement is true for A_1, since $A_1 = \Lambda x_1 \otimes \mathbf{F}_p$.

Suppose that the statement is true for A_j with $j \leq m$; consider $A_{m+1} = (A_m \otimes \Lambda(y_m, x_{m+1}), d)$. dy_m is a cocycle in A_m of degree $> |x_m|$; thus it is cohomologous to zero in $H^*(A_m)$, by the inductive hypothesis, so is the boundary of a decomposable element in A_m. Making a new choice of space of indecomposables of $((\Lambda W, d) \otimes \mathbf{F}_p)$, we may assume that y_m is a cocycle, so that $H^*(A_m \otimes \Lambda y_m) \cong H^*(A_m) \otimes \Lambda(y_m)$.

The inductive hypothesis on the degrees of the cohomology algebras shows that $y_m^{\alpha_m}$ is the generator of the unique cohomology class in degree $\alpha_m |y_m|$. The regular sequence argument requires that x_{m+1} is not a cocycle, hence (again by choice of space of indecomposables) we may assume that $dx_{m+1} = y_m^{\alpha_m}$ with coefficients in $\mathbf{F}_p$.

Thus $H^*(A_{m+1}) \cong H^*(A_m) \otimes \mathbf{F}_p[y_m]/(y^\alpha)$, so that the induction hypothesis on the degrees of the cohomology is satisfied. ■

This argument calculates the cohomology algebra $H^*(\Lambda W; \mathbf{F}_p)$. This is:

$$H^*(Z; \mathbf{F}_p) \cong \Lambda(x_1) \otimes \bigotimes_{i=1}^{K} \mathbf{F}_p[y_i]/(y_i^{\alpha_i}).$$

To complete the proof of the proposition and the second statement of Theorem 1, it remains to show the statement concerning p-formality. This follows from the form of the model constructed over $\mathbf{F}_p$, which is the tensor product of an exterior algebra by factors of the form $\Lambda(x, y)$ with $dy = 0$ and $dx = y^\alpha$. It may be seen that these factors correspond to p-formal spaces, using the techniques of [7], so that the tensor product does as well.

3.1 The model over $\mathbf{Z}_{(p)}$

Using the above, it is possible to give the model $(\Lambda W, d)$ with coefficients in $\mathbf{Z}_{(p)}$. Let B_m be the sub-CGDA over $\mathbf{Z}_{(p)}$ defined by $B_m = (\Lambda(x_1, y_1, \ldots, x_m, y_m), d)$, where the elements x_i, y_i are basis elements as in Proposition 3.1. (There is no ambiguity here, since W has at most one element in each degree). Thus $B_{m+1} = (B_m \otimes \Lambda(x_{m+1}, y_{m+1}), d)$, with B_m as a sub-CGDA. Observe that W is in degrees ≥ 3 since W is connected and the lowest degree element of W must be in odd degree. This shows that the differential of y_{m+1} cannot involve x_{m+1}; that is

$$\begin{cases} dx_{m+1} = p^{r_{m+1}} y_{m+1} + (B_m) \\ dy_{m+1} = (B_m), \end{cases}$$

where (B_m) indicates decomposable elements of B_m.

Proposition 3.2 *For all $1 \leq m \leq K$ the algebra $H^*(B_m; \mathbf{Z}_{(p)})$ is concentrated in even degrees $k|y_1|$ for $k \geq 0$. As an algebra it is generated by elements: $\{y_1, \ldots, y_m\}$ subject to the relations*

$$\begin{cases} p^{r_1} y_1 & = & 0 \\ p^{r_{j+1}} y_{j+1} + y_j^{\alpha_j} & = & 0 \end{cases}$$

for some integers $r_j \geq 1$; there is a choice of indecomposables of B_m so that the differential is:

$$\begin{cases} dx_1 & = & p^{r_1} y_1 \\ dx_{j+1} & = & p^{r_{j+1}} y_{j+1} + y_j^{\alpha_j} \text{ for } j < K. \end{cases}$$

Proof: The proof is by induction on m.

$B_1 = (\Lambda(x_1, y_1), dx_1 = p^{r_1} y_1)$, where the differential is forced to act as given, for degree reasons.

Suppose that the result is true for $m \leq M$ and consider $B_{M+1} = (B_M \otimes \Lambda(y_{M+1}) \otimes \Lambda(x_{M+1}), d)$. Now dy_m is a cocycle of odd degree in B_M, so it is the coboundary of a decomposable element in B_M, by the hypothesis on the cohomology of B_M over $\mathbf{Z}_{(p)}$. By changing the space of indecomposables if necessary, one may suppose that $dy_{M+1} = 0$.

Knowledge of the structure of $(\Lambda Z, d) \otimes \mathbf{F}_p$ shows that $|x_{M+1}| + 1 = \alpha_M |y_M|$ for some $\alpha_M \geq 2$. Now, by the inductive hypothesis on the structure of the cohomology algebra, the cohomology $H^*(B_M)$ is generated in degree $(|x_{M+1}| + 1)$ by the class represented by the cocycle $y_M^{\alpha_M}$, so that $H^*(B_M) \otimes \Lambda(y_{M+1})$ is generated as a $\mathbf{Z}_{(p)}$-module by $y_M^{\alpha_M}$ and y_{M+1} in that degree.

Thus, again by changing the choice of indecomposable if necessary and absorbing any unit multiples (in $\mathbf{Z}_{(p)}$) into the choice of generators, one may suppose that

$$dx_{M+1} = p^{r_{M+1}} y_{M+1} + (y_M)^{\alpha_M}.$$

for some $r_{M+1} \geq 1$. This proves the inductive step of the argument, since the homology of $H^*(B_{M+1})$ may be calculated and it satisfies the statement of the Proposition. ■

To complete the determination of the model $(\Lambda W, d)$, one may show via the same arguments that there is a choice of indecomposable representing w_0 with differential $dw_0 = y_K^{\alpha_K}$. Thus, the minimal model $\mathcal{M} = (\Lambda W, d)$ has a choice of space of indecomposables over $\mathbf{Z}_{(p)}$ for which W has a free basis: $\{x_1, \dots, x_K, \}, \{y_1, \dots, y_K\}, w_0$ with respect to which the differential is:

$$\begin{cases} dx_1 &= p^{r_1} y_1 \\ dx_{j+1} &= p^{r_{j+1}} y_{j+1} + y_j^{\alpha_j} \text{ for } j < K \\ dw_0 &= y_K^{\alpha_K} \end{cases}$$

where $r_j \geq 1$ and $\alpha_j \geq 2$ for all j.

3.2 Formality of the space Z over $\mathbf{Z}_{(p)}$

It remains to determine an Adams-Hilton model for Z from the commutative cochain model $\mathcal{M} = (\Lambda W, d)$. To do this, one may exploit a property of formality *over* $\mathbf{Z}_{(p)}$.

Write $\mathcal{M}$ as an extension of commutative differential graded algebras: $\mathcal{M} = (\Gamma \otimes \Lambda x_1, d)$ where Γ is the subalgebra of $\mathcal{M}$ generated by all elements of W except x_1, and the differential makes Γ a sub differential algebra, $\Gamma \hookrightarrow \mathcal{M}$. The cohomology of Γ may be calculated directly; it has underlying module which is torsion-free:

$$H^*(\Gamma, d) = \mathbf{Z}_{(p)}[y_1, \dots, y_K] / \{(p^{r_{j+1}} y_{j+1} + y_j^{\alpha_j} = 0)_{1 \leq j < K}, y_K^{\alpha_K} = 0\}$$

and there is a morphism of commutative differential graded algebras $(\Gamma, d) \to H^*(\Gamma, d)$, defined by $y_i \mapsto [y_i], x_i \mapsto 0$, which induces an isomorphism in $\mathbf{Z}_{(p)}$-cohomology.

This extends to a map of CGDAs: $\mathcal{M} \to \mathcal{N} = (H^*(\Gamma, d) \otimes \Lambda x_1, d)$, where the differential is zero in $H^*(\Gamma, d)$ and $dx_1 = p^{r_1}[y_1]$. This map induces an isomorphism in cohomology.

If $\mathcal{B}$ is a $\mathbf{Z}_{(p)}$-free differential graded algebra of finite type, write $\Omega(\mathcal{B}^\vee)$ for the cobar construction on the dual of the algebra $\mathcal{B}$. Here, by results derived from Adams' cobar equivalence (see [12]), $\Omega(\mathcal{M}^\vee)$ gives a model for the $\mathbf{Z}_{(p)}$-chains on ΩZ. Again, standard techniques in differential homological algebra imply that $\Omega(\mathcal{N}^\vee)$ serves as a model for the $\mathbf{Z}_{(p)}$-chains on ΩZ, since the cohomology equivalence becomes a homology equivalence when applying the functor $\Omega(-^\vee)$.

Now $\mathcal{N}$ has generators (as a free $\mathbf{Z}_{(p)}$-module) in one-one correspondence with the $\mathbf{F}_p$-vector space generators of the mod-p homology of Z. Thus the algebra $A = \Omega(\mathcal{N}^\vee)$ is a tensor algebra on a free $\mathbf{Z}_{(p)}$-module, with generators in one-one correspondence with the cells of a p-minimal decomposition of Z. Thus it has the appearance of a classical Adams-Hilton model for the p-minimal CW decomposition of Z. Moreover it has a quadratic differential, which is a property of $\mathbf{Z}_{(p)}$-formality generalizing that of Definition 1.3.

Example 3.3 This behaviour is exhibited by the differential graded algebra $\mathcal{A} = \mathcal{A}(r)$ of Example 1.5, in which the commutative differential graded algebra $\mathcal{N}$ is as follows:

$$\mathcal{N} = (\mathbf{Z}_{(p)}[\hat{b}, \hat{e}]/(p^r\hat{e} + \hat{b}^2 = 0) \otimes \Lambda(\hat{a}), d)$$

where $|\hat{a}| = 2N-1, |\hat{b}| = 2N, |\hat{E}| = 4N$ and $d\hat{a} = p^k\hat{b}$. The naming of the generators indicates a correspondence with the generators of $\mathcal{A}(r)$.

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L'INSTITUT GALILEE, UNIVERSITE DE PARIS-NORD,
VILLETANEUSE, FRANCE.