On the geometry of complex Grassmann manifold, its noncompact dual and coherent states

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Abstract

Different topics on the differential geometry of the complex Grassmann manifold are surveyed in relation to the coherent states. A calculation of the tangent conjugate locus and conjugate locus in the complex Grassmann manifold is presented. The proofs use the Jordan's stationary angles. Also various formulas for the distance on the complex Grassmann manifold are furnished.

1 Introduction

"Grassmann manifold... has been intensively studied for many years. We have not got a comprehensive knowledge of its geometry, however".[1]

Without entering into historical details, the Grassmann manifold has been intensively studied from the second half of the last century. The real euclidean geometry of linear manifolds in a multidimensional space was considered by Jordan[2] using only the methods of the analytic geometry. In the first half of our century the Grassmann manifold was the main example in many constructions as the CW-cell decomposition,[3] the Chern[4] and Pontrjagin[5] classes... The basic facts about

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the Grassmann manifold can be found in standard books.[4][6]-[9] Many recent references are based on the paper[10] of Y.-C. Wong. However, the modern reader has difficulties to follow [10] because Wong uses some notions as the stationary angles of Jordan[2] between two n-planes from an $n+m$ space. In fact, part of the results contained in the paper of Wong[10] were known and they can be found in the papers of Rosenfel'd[11] and in his books.[12, 13]

In the case of the Grassmann manifold, the cut locus can be calculated explicitly.[10] The situation with the conjugate locus is more complicated. Wong[14] has published the expression of the conjugate locus in the Grassmann manifold and usually[15] his paper is quoted as an example of a calculation of the conjugate locus in a multidimensional manifold. The calculation of Wong[14] is essentially based on a structure lemma the proof of which was published later.[16] However, Wong has not published the proof of his results on conjugate loci in Grassmann manifold. Sakai[17] has calculated the tangent conjugate locus in the tangent space to the Grassmann manifold. He has observed that Wong's result announced in his paper[14] is incomplete. Apparently,[18] this disagreement of the results of Wong referring to the conjugate locus in the Grassmann manifold with the calculation of Sakai on the tangent conjugate locus has not been pointed out.

Among many other things, in this paper we present a proof of the results of Wong in the complex Grassmann manifold and also another proof of the calculation of Sakai in the tangent space to the Grassmann manifold. The part of the conjugate locus calculated by Wong can be expressed as a Schubert variety. The rest of the conjugate locus is characterized as the subset of points of the Grassmann manifold which have at least two of the stationary angles equal. It contains as subset the subset of isoclinic spheres determined by Wong[19] in connection with the Hurwitz[20] problem.

The present paper can be considered from three points of view.

In this paper are put together many facts referring to the differential geometry of the complex Grassmann manifold. From this side, the paper has a survey character. However, all the proofs are original. The still open problem refers to the conjugate locus, as was already stressed. The proof uses also the stationary angles, which are briefly presented. A short proof of the structure lemma of Wong[16] is given. Also explicit expressions for the distance on the Grassmann manifold are deduced.

On the other side, in this paper the geometry of the complex Grassmann manifold is studied in relation to the coherent states.[21, 22] The manifold of coherent vectors is the pull-back of the dual of the tautological line bundle on a manifold identified with the Det^* bundle[4] in the case of the complex Grassmann manifold. The main observation is the fact that the parameters which characterize the coherent states are in fact the Pontrjagin's coordinates of the n-plane. The proof of the result of Wong on conjugate locus uses a parametrization that also appears in the coherent state approach.

This paper is a complete and self-contained example of some notions related to the trial to find a geometrical characterization of Perelomov's construction of the coherent state manifold as Kählerian embedding into a projective space.[23] We remember that it has been pointed out that for symmetric spaces the cut locus is equal to the polar divisor.[24] This situation is illustrated in the case of the complex Grassmann manifold. Also the equality between the dimension of the projective space in which the Plücker embedding takes place, the Euler-Poincaré characteristic of the manifold and the maximal number of orthogonal coherent vectors is true at least for flag manifolds.

The considerations below concern the geometry of the finite dimensional complex Grassmann manifold $G_n(\mathbb{C}^{m+n})$, also denoted

$$X_c = G_c/K = SU(n+m)/S(U(n) \times U(m)) \ . \tag{1.1}$$

Simultaneously, some of the considerations are made also in the case of the noncompact dual of the compact Grassmann manifold

$$X_n = G_n/K = SU(n,m)/S(U(n) \times U(m)) \ . \tag{1.2}$$

Most of the results are still true for the infinite dimensional Grassmannian.[25]

The paper is organised as follows.

In §2 some basic facts about the complex Grassmann manifold are remembered. The Cauchy formula is still true for projectively induced analytic line bundles over homogeneous Kähler manifolds. The Pontrjagin's coordinatization, the polar divisor and cell structure are considered in the Section 3. A rapid presentation of Schubert varieties is proposed in Section 4 while the stationary angles are presented in §5. The complex Grassmann manifold as symmetric space is treated in §6. In the same Section is presented the connection between the Grassmann manifold and the parametrization used in the coherent state approach. The explicit expression of the exponential map which gives geodesics in the Grassmann manifold is essential for calculating the conjugate locus in the manifold. Lemma 6 will be used for determination of the tangent conjugate locus. The expression of the diastasis function of Calabi,[26] recently used in the context of coherent states,[27] is given. The cut locus and conjugate locus are treated in §7. The main results are contained in Theorem 2, Proposition 2 and Comment 2. The last Section presents explicit expressions for the distance on the complex Grassmann manifold (noncompact Grassmann manifold) which generalize the corresponding ones from the case of the Riemann sphere (respectively, the disk $|z| < 1$).

2 The Cauchy formula

2.1 Let us denote by $D_n(\mathbf{K})$ the set of pure (decomposable) $n-$vectors of the exterior algebra $\bigwedge^n \mathbf{K}$, where $\mathbf{K}$ is a complex vector space. For every $Z \in D_n(\mathbf{K})$, there exists n vectors $\mathbf{z}_1, \ldots, \mathbf{z}_n \in \mathbf{K}$ such that

$$Z = \mathbf{z}_1 \wedge \ldots \wedge \mathbf{z}_n \ . \tag{2.1}$$

The elements $Z, Z' \in D_n(\mathbf{K})$ are equivalent iff there exists $\lambda \in \mathbb{C}^\star \equiv \mathbb{C} \setminus \{0\}$ such that $Z = \lambda Z'$, that is, the associated $n-$subspaces $M_Z = \ < \mathbf{z}_1, \ldots, \mathbf{z}_n >$, $M_{Z'} = < \mathbf{z'_1}, \ldots, \mathbf{z'_n} >$ are identical. So, there is a canonical bijection of the set of $n-$subspaces of $\mathbf{K}$ on the image $G_n(\mathbf{K})$ of $D_n(\mathbf{K})$ in the projective space $\mathbf{P}(\bigwedge^n \mathbf{K})$ associated to $\bigwedge^n \mathbf{K}$. $G_n(\mathbf{K})$ is called the Grassmannian of index n of $\mathbf{K}$. The space of holomorphic sections of Det^* on $G_n(K)$ is naturally isomorphic with $\bigwedge^n(\mathbf{K}^*)$ (see Prop 2.9.2 in [25]). Here E^* denotes the dual of the space E.

When $\mathbf{K} = \mathbb{C}^N,\ N = n + m$ and $\{\mathbf{e_i}\}_{i=1,\dots,N}$ is the canonical base of $\mathbf{K}$, $G_n(\mathbf{K})$ is the set of points of $\mathbf{P}(\bigwedge^n \mathbf{K})$ with homogeneous (Plücker) coordinates $[Z_I]$

$$Z = \sum_{I\subset \mathcal{S}(n,N)} Z_I \mathbf{e}_I \ . \tag{2.2}$$

If $I_N = \{1, \dots, N\}$, then $\sigma :\ I_N \rightarrow I_N$ is a Schubert symbol, i.e. a permutation with the property that its restrictions to I_n and $I_N \setminus I_n$ are increasing and the set of $N(n) = \binom{N}{n} = \dfrac{N!}{n!m!}$ Schubert symbols was denoted by $\mathcal{S}(n, N)$. The Plücker embedding $\iota : G_n(\mathbb{C}^{m+n}) \hookrightarrow \mathbb{CP}^{N(n)-1} = \mathbf{PL},\ \mathbf{L} = \bigwedge^n \mathbb{C}^N$,

$$\iota(Z) = [Z_I]_{I\subset \mathcal{S}(n,N)} \tag{2.3}$$

is isometric and biholomorphic.[28] We have denoted $[\omega] \equiv \xi(\omega)$, where $\xi : \mathbf{K}\backslash\{0\} \rightarrow \mathbf{PK}$ is the natural projection.

The $n-$vector $Z \neq 0$ is pure iff the (Plücker-) Grassmann (-Cayley) relations are fulfilled, i.e.

$$\sum \epsilon_{i,J,H} Z_{J\setminus\{i\}} Z_{H\cup\{i\}} = 0 \ , \tag{2.4}$$

where $J,\ H \subset I_N,\ n = \#\{J\} - 1 = \#\{H\} + 1$, and $\epsilon_{i,J,H} = +1\ (-1)$ if the number of elements of J and H less than i have the same (resp. opposite) parity (cf. Bourbaki[6]; see also [29] for the Hilbert space Grassmannian).

Let $A^{i_1\dots i_p}_{j_1\dots j_p}$ denotes the minor of order p of the matrix A whose elements are at the intersection of the rows i_k with the columns $j_k,\ k = 1, \dots, p$. If in eq. (2.1)

$$\mathbf{z}_i = \sum_{a=1}^{N} \hat{Z}_{ia} \mathbf{e}_a \ , \tag{2.5}$$

then

$$Z = \mathbf{z}_1 \wedge \dots \wedge \mathbf{z}_n = \sum_{1\le i_1 < \dots < i_n \le N} Z^{i_1\dots i_n} \mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_n} \ , \tag{2.6}$$

where $Z^{i_1\dots i_n} = \hat{Z}^{1\dots n}_{i_1\dots i_n}$ are the Plücker coordinates denoted in eq. (2.2) by Z_I and $\hat{Z}$ denotes the matrix $(\hat{Z}_{ia})_{1\le i\le n; 1\le a\le N}$.

2.2 Let $((\cdot,\cdot))$ be the application $D_n(\mathbf{K}) \times D_n(\mathbf{K}) \rightarrow \mathbb{C}$ defined by the *Cauchy formula*

$$((Z', Z)) \equiv (\iota(Z'), \iota(Z)) \ , \tag{2.7}$$

where $(\cdot,\cdot)$ is the hermitian scalar product in $\mathbf{K} \times \mathbf{K}$. The name of equation (2.7) is justified by the *Cauchy identity* (see eq. (6) p. 10 in [30]) contained in

Remark 1 *The following relation is true:*

$$(\iota(Z'), \iota(Z)) \equiv ((\mathbf{z'}_1 \wedge \dots \wedge \mathbf{z'}_n, \mathbf{z}_1 \wedge \dots \wedge \mathbf{z}_n)) = \det[(\mathbf{z'_i}, \mathbf{z}_j)]_{1\le i,j\le n} \ . \tag{2.8}$$

Proof: In fact

$$\det[(\mathbf{z}'_{\mathbf{i}}, \mathbf{z}_j)]_{1\le i,j\le n} = \det(\hat{Z}'\hat{Z}^{+}), \tag{2.9.a}$$

$$\det[(\mathbf{z}'_{\mathbf{i}}, \mathbf{z}_j)]_{1\le i,j\le n} = \det(\hat{Z}\hat{Z}'^{+}), \tag{2.9.b}$$

depending respectively on the convention of the hermitian scalar product $(\cdot,\cdot)$: $\mathbf{K}\times\ \mathbf{K}\rightarrow \mathbb{C}$

$$(\mathbf{a}, \lambda\mathbf{b}) = \bar{\lambda}(\mathbf{a}, \mathbf{b}), \tag{2.10.a}$$

$$(\mathbf{a}, \lambda\mathbf{b}) = \lambda(\mathbf{a}, \mathbf{b}). \tag{2.10.b}$$

This corresponds respectively, to

$$(\iota(Z'), \iota(Z)) = \sum_{1\le i_1<\ldots<i_n\le N} Z'^{i_1\ldots i_n}\bar{Z}^{i_1\ldots i_n}\ , \tag{2.11.a}$$

$$(\iota(Z'), \iota(Z)) = \sum_{1\le i_1<\ldots<i_n\le N} \bar{Z}'^{i_1\ldots i_n}Z^{i_1\ldots i_n}\ . \tag{2.11.b}$$

Eq. (2.8) is a consequence of eqs. (2.9), (2.11) and of the Binet-Cauchy formula: if $A,\ B,\ C$ are matrices with $m\times n,\ n\times m$, respectively $m\times m$ elements and $C = AB$, then (eq. (15) p. 9 in [30])

$$\det C = \sum_{1\le k_1<\ldots<k_m\le n} A^{1\ldots m}_{k_1\ldots k_m} B^{k_1\ldots k_m}_{1\ldots m}\ . \tag{2.12}$$

■

So, eqs. (2.7)-(2.9) justify the usual[4] definition of the *hermitian scalar product of two pure $n-$vectors* ($n-$*planes* of the Grassmannian), or, more precisely, of the *hermitian scalar product in the holomorphic line bundle Det**:

$$((Z', Z)) \equiv \det[(\mathbf{z}'_{\mathbf{i}}, \mathbf{z}_j)]_{1\le i,j\le n}\ . \tag{2.13}$$

The infinite dimensional case can be found in Prop. 7.1 of [25]; see also eq. 2.10 in [29].

2.3 If $Z, Z' \in G_n(\mathbf{K})$, let θ be the angle defined by the hermitian scalar product of two planes

$$\cos\theta(Z', Z) \equiv \frac{|((Z', Z))|}{\|Z'\|\|Z\|}\ . \tag{2.14}$$

Remark that θ in equation (2.14) *is not the angle between the two n-planes*, because θ is not invariant under the motion group on the Grassmann manifold. The quantities which are invariant under the group action are the n stationary angles $\theta_1, \ldots, \theta_n$ of Jordan[2] related to θ by the relation (5.10). The only situation in which the angle θ in relation (2.14) is the angle of the two $n-$planes occurs when the Grassmann manifold has rank 1, i.e. $r \equiv \min(m,n) = 1$.

Eq. (2.7) implies

$$\cos\theta(Z',Z) = \frac{|(\iota(Z'),\iota(Z))|}{\|\iota(Z')\|\|\iota(Z)\|} , \tag{2.15}$$

and the r.h.s. of eq. (2.15) *defines*[31, 32] *the (intrinsic) distance on the geodesics joining* $\iota(Z')$, $\iota(Z)$ *in the projective space* **PL** *in which the Grassmann manifold is embedded*,

$$\cos d_c(\iota(Z'),\iota(Z)) = \frac{|(\iota(Z'),\iota(Z))|}{\|\iota(Z')\|\|\iota(Z)\|} . \tag{2.16}$$

The elliptic hermitian distance, here called the Cayley distance,[31] is

$$d_c([\omega'],[\omega]) = \arccos\frac{|(\omega',\omega)|}{||\omega'||||\omega||} . \tag{2.17}$$

The infinite dimensional case was treated by Kobayashi.[33]

Now, it follows that

Remark 2 (Rosenfel'd[11]) *The angle* θ *defined in eq. (2.14) it is related to the Cayley distance* d_c *by the relation*

$$\theta(Z',Z) = d_c(\iota(Z'),\iota(Z)) . \tag{2.18}$$

Proof: The Remark results from eq. (2.15) and eq. (2.16). ■

Some authors (e.g. Study[34]) prefer instead of the definition (2.17) of the distance d_c the definition

$$d_c([\omega'],[\omega]) = 2\arccos\frac{|(\omega',\omega)|}{||\omega'||||\omega||} , \tag{2.19}$$

which lead, instead of (2.18) to

$$\theta(Z',Z) = \frac{1}{2}d_c(\iota(Z'),\iota(Z)) . \tag{2.20}$$

With the definition (2.17), ((2.19)) the elliptic hermitian distance of two points on the Riemann sphere is one half the arc (respectively, the arc) of the great circle connecting the corresponding points of the Riemann sphere[32] (resp.[34]) (see also §8). d_c in eq. (2.17) is equal to the minimum angle between the real lines belonging to the complex lines (real 2-planes) in **K** represented by $[\omega], [\omega'] \in$ **PK**.

The Cauchy formula (2.7) is still true[23, 24] for projectively induced[36] analytic line bundles over homogeneous Kähler manifolds.

2.4 Now we briefly discuss the case of the noncompact manifold X_n.

Firstly, let us denote by

$$\mathbb{CP}^{n-1,1} = SU(n,1)/S(U(n)\times U(1)) \tag{2.21}$$

the hermitian hyperbolic space dual to $\mathbb{CP}^{n-1}$. Then the noncompact analogue of the distance (2.17) is the hyperbolic hermitian Cayley distance

$$d_c([\omega'],[\omega]) = \operatorname{arccosh} \frac{|(\omega',\omega)_n|}{||\omega'||_n ||\omega||_n} \ , \tag{2.22}$$

where the hermitian form on $\mathbb{C}^N$, antilinear in the second entry (convention (2.10.a)), in the orthonormal basis, is

$$(\omega',\omega)_n = \omega_1 \bar{\omega_1'} - \sum_{i=2}^{n} \omega_i \bar{\omega}_i \ . \tag{2.23}$$

The noncompact manifold X_n (1.2) admits the embedding $\iota' : X_n \hookrightarrow \mathbb{CP}^{N(n)-1,1}$. Eq. (2.7) becomes

$$((Z',Z))_n \equiv (\iota'(Z'),\iota'(Z))_n \ , \tag{2.24}$$

and the *Remark 1 with the r.h.s. in formula* (2.8) *replaced by* $\det[(\mathbf{z}'_\mathbf{i},\mathbf{z}_j)_n]_{1\le i,j\le n}$ *is also true in the case of the noncompact manifold* X_n.

The equation corresponding to eq. (2.14) ((2.15)) is

$$\cosh\theta(Z',Z) \equiv \frac{|((Z',Z))_n|}{\|Z'\|_n \|Z\|_n} \ , \tag{2.25}$$

(respectively)

$$\cosh\theta(Z',Z) = \frac{|(\iota'(Z'),\iota'(Z))_n|}{\|\iota'(Z')\|_n \|\iota'(Z)\|_n} \ . \tag{2.26}$$

So, *Remark 2 is also true in the noncompact case, with* eq. (2.14) *replaced by* (2.25) *and* eq. (2.17) *replaced by* (2.22).

3 Pontrjagin's coordinatization, polar divisor and cells

3.1 Let us consider $Z_0 \in G_n(\mathbf{K})$, where

$$Z_0 = \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n \ . \tag{3.1}$$

Then we have the orthogonal decomposition

$$\mathbf{K} = < Z_0 > \oplus < Z_0^\perp > \ , \tag{3.2}$$

where $Z_0^\perp$ is the $m-$plane (completely) orthogonal to Z_0 defined by the m vectors (3.22). Any $\mathbf{x} \in \mathbf{K}$ admits the decomposition

$$\mathbf{x} = \mathbf{u} \oplus \mathbf{v},\ \mathbf{u} \in < Z_0 >,\ \mathbf{v} \in < Z_0^{\perp} > \ ,$$

and let φ denotes the orthogonal projection $\mathbf{u} = \varphi(\mathbf{x})$. More precisely, let us denote by $\varphi_{Z_0}(\mathbf{x})$ the orthogonal projection $\mathbf{u}$ of the vector $\mathbf{x} \in \mathbf{K}$ on the $n-$plane Z_0 in the direction $Z_0^{\perp}$.

Let us consider the (open) neighbourhood of the fixed $n-$plane Z_0

$$\mathcal{V}_{Z_0} = \{Z \in G_n(\mathbf{K}) | \text{ projection } \varphi\ ,\ \varphi_{\mathbf{Z_0}}(\mathbf{Z}) \subseteq \mathbf{Z_0} \text{ is nondegenerate}\}\ . \tag{3.3}$$

For a fixed $Z \in G_n(\mathbf{K})$ let

$$\Sigma_Z = \{Y \in G_n(\mathbf{K}) \mid ((Z, Y)) = 0\}\ . \tag{3.4}$$

Lemma 1 *Let $Z_0,\ Z \in G_n(\mathbf{K})$. Then $Z \in \mathcal{V}_{Z_0}$ iff one of the following equivalent conditions are fulfilled*

$$\begin{array}{lll} (A1) & ((Z, Z_0)) \neq 0, & \\ (A2) & \varphi_Z(Z_0) = Z, & \text{or } (A2')\ \ \varphi_{Z_0}(Z) = Z_0\ , \\ (A3) & Z_0 \cap \Sigma_Z = 0, & \text{or } (A3')\ \ Z \cap \Sigma_{Z_0} = 0\ . \end{array}$$

Equivalently, $Z \notin \mathcal{V}_{Z_0}$ iff one of the following equivalent conditions are fulfilled

$$\begin{array}{lll} (B1) & ((Z, Z_0)) = 0\ , & \\ (B2) & \varphi_Z(Z_0) \subset Z,\ \varphi_Z(Z_0) \neq Z, & \text{or } (B2')\ \ \varphi_{Z_0}(Z) \subset Z_0,\ \varphi_{Z_0}(Z) \neq Z_0\ , \\ (B3) & Z_0 \in \Sigma_Z\ , & \text{or } (B3')\ \ Z \in \Sigma_{Z_0}, \end{array}$$

and

$$\Sigma_{Z_0} = \{Y \in G_n(\mathbf{K}) \mid \ \dim(Y \cap Z_0^{\perp}) \geq 1\}\ . \tag{3.5}$$

The complex Grassmann manifold can be represented as the disjoint union

$$G_n(\mathbf{K}) = \mathcal{V}_{Z_0} \cup \Sigma_{Z_0}\ . \tag{3.6}$$

Proof: To prove *(A)*, observe that the subspaces $Z,\ Z'$ *are related* (cf. Prop. 3.3 Ch. 7 in [8]), while (B) can be obtained using the Remarks of Ch. I §2 in [37], especially Lemma 1.3. See also Ch. 9 in [4]. For the infinite dimensional Grassmannian see Ch. 7 of [25], especially Prop. 7.5.4. ∎

Geometrically, Σ_{Z_0} is the cut locus of Z_0, as was firstly observed by Wong[10] (also cf. Proposition 1 below). The same property is true for a class of spaces which generalizes the symmetric ones.[23, 24] Σ_{Z_0} can be expressed as a Schubert variety (cf. Lemma 3).

Σ_X is called the *polar divisor* of X (cf. Wu[37]).

Lemma 1 implies that for any $Z \in \mathcal{V}_{Z_0}$, there exists the vectors $\mathbf{z}_1, \ldots, \mathbf{z}_n \in \mathbb{C}^N$ such that relation (2.1) holds, $\varphi_{Z_0}(Z) = Z_0$ and $\varphi(\mathbf{z}_i) = \mathbf{e}_i$. Then, using the Pontrjagin[5] coordinates,

$$\mathbf{z}_i = \mathbf{e}_i + \sum_{\alpha=n+1}^{N} Z_{i\alpha}\mathbf{e}_{\alpha},\ i = 1, \ldots, n\ , \tag{3.7}$$

and $\mathcal{V}_{Z_0}$ is homeomorphic to $\mathbb{C}^{n\times m}$.

Let the vectors $\mathbf{z_i^\sigma}$ be such that $\hat{Z}^\sigma \in \mathcal{V}_\sigma$, where

$$\mathbf{z_i^\sigma} = \mathbf{e}_{\sigma(i)} + \sum_{\alpha=n+1}^{N} Z_{\sigma(i)\sigma(\alpha)}\mathbf{e}_{\sigma(\alpha)} \ , \ i = 1,\dots,n \ , \tag{3.8}$$

which for σ identity was already given by eq. (3.7). Then $(Z^\sigma, \mathcal{V}_\sigma)$, $\sigma \in \mathcal{S}(n,N)$ furnish an atlas of $G_n(\mathbf{K})$, where $Z^\sigma = (Z_{\sigma(i),\sigma(\alpha)})_{1\le i\le n,\, n+1\le\alpha\le n}$.

Let $\hat{Z}^\sigma$ be the (extended) matrix attached to the n vectors (3.8). If ${}_\sigma Y$ denotes the submatrix of Y containing only the columns $\sigma(i)$, $i = 1,\dots,n$, then

$${}_\sigma(\hat{Z}^\sigma) = \mathbb{1}_n, \tag{3.9}$$

$$\mathcal{V}_\sigma = \{X \subset G_n(\mathbb{C}^{m+n}) | \det\ {}_\sigma(\hat{X}) \neq 0\}. \tag{3.10}$$

If $\hat{X}$ is the $n\times N$ matrix whose $i-$th row consists of the coordinates of the vectors $\mathbf{x}_i$, $i = 1,\dots,n$, where $X = \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_n$, then

$${}_\sigma\hat{X} = \hat{X}\Delta^\sigma, \tag{3.11}$$

$$\hat{X}^\sigma = ({}_\sigma\hat{X})^{-1}\hat{X} \ , \tag{3.12}$$

where

$$(\Delta^\sigma)_{ij} = \delta_{i\sigma(j)}, \ i = 1,\dots,N; \ j = 1,\dots,n.$$

The equations (3.9) and (3.11) imply that on $\mathcal{V}_\sigma \cap \mathcal{V}_\tau \neq \emptyset$ a change of charts is given by the homographic transformation of the extended matrices

$$\hat{Z}^\tau = (\hat{Z}^\sigma\Delta^\tau)^{-1}\hat{Z}^\sigma, \ \sigma,\tau \in \mathcal{S}(n,N). \tag{3.13}$$

The equations of the $n-$plane $\hat{Z}^\sigma$ of $\mathbb{C}^N$, generated by the n vectors (3.8), $\hat{Z}^\sigma \subset \mathcal{V}_\sigma$, are

$$x_{\sigma(\alpha)} = \sum_{i=1}^{n} x_{\sigma(i)}Z_{\sigma(i)\sigma(\alpha)} \ , \ \alpha = n+1,\dots,N \ , \tag{3.14}$$

where $(x_1,\dots,x_N)$ are the local coordinates of $\mathbb{C}^N$.

3.2 An ordering of the Schubert symbols is introduced as follows: σ proceeds τ ($\sigma \prec \tau$) if the least index i, $i \in I_n$ for which $\sigma(i) \neq \tau(i)$, has the property $\sigma(i) < \tau(i)$, where $\sigma,\ \tau \in \mathcal{S}(n,N)$.

Let

$$\mathcal{C}_\sigma = \{\hat{Z}^\sigma \subset \mathcal{V}_\sigma \mid \det({}_\tau\hat{Z}) = 0, \ \sigma \prec \tau, \ \det({}_\sigma\hat{Z}) \neq 0\} \ , \tag{3.15}$$

and the matrix $\hat{Z}^\sigma$ is brought to the reduced echelon form:[38, 9]

$$\hat{Z}^\sigma = \begin{pmatrix} Z_1^1\dots Z_1^{\sigma(1)-1} r_1 0 & \dots 0 & 0\ 0 & \dots 0 & 0\ 0\dots 0 \\ Z_2^1\dots Z_2^{\sigma(1)-1} 0\ Z_2^{\sigma(1)+1} & \dots Z_2^{\sigma(2)-1} & r_2 0 & \dots 0 & 0\ 0\dots 0 \\ \vdots\ \ddots \vdots & \vdots\ \vdots & \ddots \vdots & \vdots\ \vdots & \ddots \vdots & \vdots\ \vdots\ddots\vdots \\ Z_n^1\dots Z_n^{\sigma(1)-1} 0\ Z_n^{\sigma(1)+1} & \dots Z_n^{\sigma(2)-1} 0 & Z_n^{\sigma(2)+1} & \dots Z_n^{\sigma(n)-1} r_n 0\dots 0 \end{pmatrix} \tag{3.16}$$

with the elements $r_i = 1,\ i = 1, \ldots, n$.

Let the notation

$$\omega(i) = \sigma(i) - i,\ i = 1, \ldots, n\ . \qquad (3.17)$$

$\mathcal{C}_\sigma$ is homeomorphic to an (open) cell of complex dimension

$$d(\sigma) = \sum_{i=1}^{n} \omega(i)\ . \qquad (3.18)$$

For the complex Grassmannian the groups of cell chains coincide, due to the even dimension of the cells, with the groups of cycles, and, the group of frontiers being trivial, the homology groups are isomorphic with the groups of cell chains.

Normalizing to one the row vectors in eq. (3.16) such that the last element is positive, the reduced echelon form[9, 38] are reobtained. The open (closed) cells correspond to $r_i > 0$, (respectively, $r_i \geq 0$).

In the Theorem below use is made of some notions referring to the coherent states. The usual notation will be remembered in §6.

Theorem 1 *For the Grassmann manifold $G_n(\mathbb{C}^{m+n})$ we have the equality of the following numbers:*

1. the maximal number of orthogonal coherent vectors;

2. the number of critical points of the energy function f_H associated to a Hamiltonian H which is a linear combination with unequal coefficients of the generators of the Cartan algebra;

3. the minimal dimension $N(n)$ appearing in the Kodaira (here Plücker) embedding $\iota : G_n(\mathbb{C}^N) \hookrightarrow \mathbb{CP}^{N(n)-1}$;

4. the Euler-Poincaré characteristic of the manifold, $\chi(G_n(\mathbb{C}^N))$;

5. the number of Borel-Morse cells which appears in the CW-complex decomposition of the Grassmannian;

6. the number of global sections in the holomorphic line bundle Det^*;

7. the dimension of the fundamental representation in the Borel-Weil-Bott theorem.

Proof: The theorem is proved with the theorems 1 and 2 in [39] particularized for the Grassmann manifold and using the Cauchy formula. ■

Theorem 1 is a particular case of a theorem true for flag manifolds.[23]

3.3 If $Z = (Z_{i\alpha})_{1\leq i \leq n < \alpha \leq N}$ describes a $n-$plane $Z \in \mathcal{V}_0$, then the extended matrix $\hat{Z}$ is

$$\hat{Z} = (\mathbb{1}_n Z)\ , \qquad (3.19)$$

and the scalar product in eqs. (2.9) can be written down, respectively, as

$$((Z', Z)) = \det(\mathbb{1}_n + Z'Z^+)\ , \qquad (3.20.a)$$

$$((Z', Z)) = \det(\mathbb{1}_n + ZZ'^+)\ . \qquad (3.20.b)$$

The noncompact analogue of eq. (3.20.b) is

$$((Z',Z))_n = \det(\mathbb{1}_n - ZZ'^+) \ . \tag{3.21}$$

Given the $n-$plane $Z \in \mathcal{V}_0$ generated by the n vectors in the formula (3.7), then the $m-$plane $Z^\perp$ orthogonal to Z is generated by

$$\mathbf{z}^\perp_\alpha = \mathbf{e}_\alpha - \sum_{i=1}^{n} \bar{Z}_{i\alpha}\mathbf{e}_i \ , \ \ \alpha = n+1,\ldots,n+m \ , \tag{3.22}$$

and

$$(\mathbf{z'}^\perp_\alpha, \mathbf{z}_i) = 0 \ .$$

Note also the following relations, corresponding to the scalar product (2.10.a) (respectively (2.10.b))

$$((Z'^\perp, Z^\perp)) = \det(\mathbb{1}_m + Z'^+ Z) \ , \tag{3.23.a}$$

$$((Z'^\perp, Z^\perp)) = \det(\mathbb{1}_m + Z^+ Z') \ . \tag{3.23.b}$$

Below we give a technical remark which has a clear geometrical meaning.

Remark 3 *If $Z, Z' \in \mathcal{V}_0 \subset G_n(\mathbf{K})$, then*

$$((Z',Z)) = \overline{((Z'^\perp, Z^\perp))} \ , \tag{3.24}$$

or, explicitly,

$$\det(\mathbb{1}_n + ZZ'^+) = \overline{\det}(\mathbb{1}_m + Z^+ Z') \ . \tag{3.25}$$

Similarly, for X_n

$$\det(\mathbb{1}_n - ZZ'^+) = \overline{\det}(\mathbb{1}_m - Z^+ Z') \ . \tag{3.26}$$

Proof: We present an algebraic proof of eq. (3.25) for $\mathbf{K} = \mathbb{C}^N$. This equation is a consequence of the Schur formulas I, II (cf. [30] p. 46). Let a matrix be partitioned in 4 blocks, where the matrices A and D are non-singular. Then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B) = \det(A - BD^{-1}C)\det(D) \ . \tag{3.27}$$

The Remark 3 follows taking the matrices $A,\ B,\ C,\ D$ as , respectively $\mathbb{1}_n,\ Z,$ $-\,Z'^+,\ \mathbb{1}_m$. ■

The Remark 3 implies, via Remark 2, that

$$\cos\theta(Z',Z) = \cos\theta(Z'^\perp, Z^\perp) \ . \tag{3.28}$$

Equation (3.28) follows geometrically from Lemma 4 below and the fact that two $n-$planes and their orthogonal complements have the same invariants (cf. §48 p. 110 in [2]).

4 Schubert varieties

4.1 A Schubert variety $Z(\omega)$ associated with the monotone sequence

$$\omega = \{0 \leq \omega(1) \leq \ldots \leq \omega(n) \leq m\}, \tag{4.1}$$

is the subset of the Grassmannian

$$Z(\omega) = \left\{X \in G_n(\mathbb{C}^{n+m}) | \dim(X \cap \mathbb{C}^{\sigma(i)}) \geq i\right\} , \tag{4.2}$$

where the sequences σ and ω are related through the relation (3.17) (see Pontrjagin[5]).

Usually (cf. Ch. XIV p. 316 in [7] and [4]), a nested sequence of planes V_i of dimension $\sigma(i)$, $i = 1, \ldots, n$ is attached to the sequence (4.1), and these are the planes considered in the definition (4.2) instead of $\mathbb{C}^{\sigma(i)}$. However, because the definition of the Schubert variety is independent of the concrete sequence of planes V_i modulo a congruence, it is enough to take $V_i = \mathbb{C}^{\sigma(i)}$ (cf. Pontrjagin[5]). This definition is also adopted by Milnor and Stasheff.[9]

Instead of considering the sequence (4.2), it is enough to consider the sequence of "jumps"[5, 9]

$$I_\omega = \{0 = i_0 < i_1 < \ldots < i_{l-1} < i_l = n\} , \tag{4.3}$$

where

$$\omega(i_h) < \omega(i_{h+1}),\ \omega(i) = \omega(i_{h-1}),\ i_{h-1} < i \leq i_h,\ h = 1, \ldots, l. \tag{4.4}$$

Then

$$Z(\omega) = \left\{X \in G_n(\mathbb{C}^{n+m}) | \dim(X \cap \mathbb{C}^{\sigma(i_h)}) \geq i_h,\ i_h \in I_\omega\right\}. \tag{4.5}$$

Let us consider the set of elements in "general position"[5] in $Z(\omega)$ (in fact, the subset of *generic elements* in the sense of algebraic geometry[7]):

$$Z'(\omega) = \left\{X \in G_n(\mathbb{C}^{n+m}) | \dim(X \cap \mathbb{C}^{\sigma(i_h)}) = i_h,\ i_h \in I_\omega\right\}. \tag{4.6}$$

The element $Z \in G_n(\mathbb{C}^N)$, $Z \in \mathcal{V}_0 \cap Z(\omega) \subset Z'(\omega)$, *iff the coordinates in* eq. (3.7) *verify the condition*:[5, 3]

$$Z_{ij} = 0, \quad j > \omega(i),\ i = 1, \ldots, n. \tag{4.7}$$

Note that $Z'(\omega)$ defined by eq. (4.6) corresponds to the open cell defined by eq. (8.15) in Chern's book,[4] and its complex dimension is given by equation (3.18).

So, any $Z \in Z'(\omega)$ is locally characterized in $\mathcal{V}_0$ by a matrix Z of the type

$$Z = \begin{pmatrix} Z_{i_1,\omega(i_1)} & \mathbf{0}_{i_1,m-\omega(i_1)} \\ Z_{i_2-i_1,\omega(i_2)} & \mathbf{0}_{i_2-i_1,m-\omega(i_2)} \\ \vdots & \vdots \\ Z_{i_l-i_{l-1},\omega(i_l)} & \mathbf{0}_{i_l-i_{l-1},m-\omega(i_l)} \end{pmatrix} \tag{4.8}$$

and this representation makes very transparent the Chern[4] proof of the cell decomposition of the Grassmann manifold. In the formula above $Z_{p,q}$ denotes the $p \times q$

matrix and $\mathbf{0}$ is the matrix with all elements 0. The extended matrix (3.16) of the matrix (4.8) is obtained using the relations (3.8).

4.2 Now the subset of the Grassmann manifold

$$V_l^p = \left\{ Z \in G_n(\mathbb{C}^{n+m}) | \dim(Z \cap \mathbb{C}^p) \geq l \right\}, \tag{4.9}$$

will be expressed as a Schubert variety, where $1 < p < n+m$. Sometimes another fixed $p-$plane of $\mathbb{C}^N$, say P_p, will be considered in eq. (4.9) instead of $\mathbb{C}^p$. This situation will occur when the Theorem 2 will be reformulated in the notation of Wong,[14] where $P_n = \mathbf{O}$ and $P_m = \mathbf{O}^{\perp}$.

Let also the notation

$$W_l^p = V_l^p - V_{l+1}^p = \left\{ Z \in G_n(\mathbb{C}^{n+m}) | \dim(Z \cap \mathbb{C}^p) = l \right\} . \tag{4.10}$$

We shall prove the following structure Lemma of Wong[14, 16]

Lemma 2 *Let $l \geq 1$ and $1 < p < n+m$. Then*

$$V_l^p = Z(\omega_l^p) , \tag{4.11}$$

$$W_l^p = Z'(\omega_l^p) . \tag{4.12}$$

The following disjoint union is obtained

$$V_l^p = \begin{cases} \emptyset \text{ , if } p < l \text{ or } l > n \text{ or } p > l+m , \\ G_n(\mathbb{C}^{n+m}), \ l = p-m , \\ W_l^p \cup W_{l+1}^p \cup \cdots \cup W_{r_1-1}^p \cup W_{r_1}^p, \ \max(1, p-m+1) \leq l \leq r_1 = \min(n,p). \end{cases} \tag{4.13}$$

where

$$W_{r_1}^p = \begin{cases} \mathbb{C}^n \subset G_n(\mathbb{C}^{n+m}) , & \text{if} \quad p = n , \\ G_m(\mathbb{C}^{n+m-p}) , & \text{if} \quad p < n , \\ G_n(\mathbb{C}^p) , & \text{if} \quad p > n . \end{cases} \tag{4.14}$$

Proof: The set of jumps (4.3) for Schubert variety (4.9) is

$$0 = i_0 < i_1 = l < i_2 = n . \tag{4.15}$$

The relation $p = i_1 + \omega(i_1) = \sigma(i_1)$ obtained forcing eqs. (4.9) and (4.5) to coincide with the representations (4.7), (4.8) on the generic elements (4.6) imply that the sequence

$$\omega_l^p = (\underbrace{p-l, \ldots, p-l}_{l}, \underbrace{m, \ldots, m}_{n-l}) \tag{4.16}$$

is responsible for the Schubert variety (4.9). The conditions that the variety (4.9) to be nonvoid are $m \geq p-l$, $n-l \geq 0$. The case $p-l = m$ corresponds to $\omega = (m, \ldots, m)$ and then $V_l^{m+l} = G_n(\mathbb{C}^{n+m})$.

Eq. (4.15) implies that the matrix (4.8) characterizing the set (4.9) has in this case only one submatrix $\mathbf{0}_{l,m+l-p}$ with all elements zero. Then it follows the disjoint union

$$V_l^p = W_l^p \cup V_{l+1}^p \ . \tag{4.17}$$

In the representation (4.8) the generic elements are characterized by the fact that their first neighbours bordering the $\mathbf{0}$ matrix in equation (4.8) are all nonzero, i.e.

$$\prod_{i=1}^{l} Z_{i\,p-l} \prod_{j=0}^{m+l-p} Z_{l+1\,p-l+j} \neq 0 \ . \tag{4.18}$$

The structure lemma is proved iterating the splitting (4.17) as far as possible. ∎

Note that V_l^p *is not* a (differentiable) (sub)manifold (of the Grassmannian). In fact, W_l^p consists of simple points of V_l^p and V_{l+1}^p is the singular locus (cf. Ch. X §14 p. 87 in [7]) of V_l^p.[14, 16] V_l^p is an (irreducible) algebraic variety of dimension $l(p-l)+m(n-l)$, while W_l^p is an (analytic) submanifold of $G_n(\mathbb{C}^N)$ of the same dimension.

In particular, let Σ_0 be the polar divisor as defined by eq. (3.4) of $\mathbf{O} \in G_n(\mathbb{C}^N)$. Then

Lemma 3 (Wong,[10] Wu[37]) *The polar divisor of the point* $\mathbf{O}$ *is given by*

$$\begin{aligned}\Sigma_0 &= V_1^m = Z(\omega_1^m) = Z(m-1,m,\ldots,m)\\ &= \left\{X \subset G_n(\mathbb{C}^{n+m}) |\dim(X\cap \mathbf{O}^{\perp}) \geq 1\right\}.\end{aligned} \tag{4.19}$$

Proof: The polar divisor is expressed as in eq. (3.5). The relation (4.11) implies the Lemma. ∎

5 The stationary angles

5.1 Let Z', Z be two $n-$planes of $G_n(\mathbb{C}^{n+m})$ given as in eq. (2.6). Then the (n) *stationary angles* (see Jordan[2] for the real case), *of which at most* $r = \min(m,n)$ *are nonzero*, are defined as the *stationary* angles $\theta \in [0,\pi/2]$ between the vectors

$$\mathbf{a} = \sum_{i=1}^{n} a_i \mathbf{z}'_\mathbf{i}, \ \mathbf{b} = \sum_{i=1}^{n} b_i \mathbf{z}_i, \tag{5.1}$$

where

$$\cos\theta = \frac{|(\mathbf{a},\mathbf{b})|}{||\mathbf{a}||||\mathbf{b}||} \ . \tag{5.2}$$

We shall prove a Lemma, which is implicitly contained in Jordan[2] :

Lemma 4 *The squares* $\cos^2\theta_i$ *of the stationary angles between the* $n-$*planes* Z, Z' *with* $((Z,Z'))\neq 0$ *are given as the eigenvalues of the matrix* W

$$W = (\hat{Z}\hat{Z}^{+})^{-1}(\hat{Z}\hat{Z'}^{+})(\hat{Z'}\hat{Z'}^{+})^{-1}(\hat{Z'}\hat{Z}^{+}) \ , \tag{5.3}$$

which, for $Z, Z' \in \mathcal{V}_0$ *is*

$$W = (1\!\!1 + ZZ^{+})^{-1}(1\!\!1 + ZZ'^{+})(1\!\!1 + Z'Z'^{+})^{-1}(1\!\!1 + Z'Z^{+}) \ . \tag{5.4}$$

Proof: We adapt the Rosenfel'd's method (cf. §3.3.15 p. 106 in [12] and [13]) to the complex Grassmann manifold in the Pontrjagin's coordinates.

Let us introduce the auxiliary function

$$U = (\mathbf{a},\mathbf{b}) + \lambda(\mathbf{a},\mathbf{a}) + \mu(\mathbf{b},\mathbf{b}) \ . \tag{5.5}$$

The calculation below is done with the condition (2.10.a) for the scalar product.

Taking the derivatives of U with respect to a_i and b_i, it follows that

$$\begin{cases} (\mathbf{z'_i},\mathbf{z}_j)\bar{b}_j + \lambda(\mathbf{z'_i},\mathbf{z'_j})\bar{a}_j = \ 0 \ , \\ a_i(\mathbf{z'_i},\mathbf{z}_j) + \mu b_i(\mathbf{z}_i,\mathbf{z}_j) = \ 0 \ , \end{cases} \tag{5.6}$$

which implies

$$\lambda\bar{\mu} = \cos^2\theta \geq 0. \tag{5.7}$$

Introducing the matrix of coordinates as in eq. (2.5), eqs. (5.6) can be written down in matricial form

$$\begin{cases} \hat{Z'}\hat{Z}^{+}\bar{b} + \lambda\hat{Z'}\hat{Z'}^{+}\bar{a} = \ 0 \ , \\ \hat{Z}\hat{Z'}^{+}\bar{a} + \bar{\mu}\hat{Z}\hat{Z}^{+}\bar{b} \ = \ 0 \ , \end{cases} \tag{5.8}$$

where $\hat{Z}, \ \hat{Z'}$ are the extended $n\times N$ matrices (3.19) attached to the $n-$planes Z, respectively Z', and $a, \ b$ are $n-$column vectors.

It results from equations (5.8) that $\bar{b}$ are the eigenvectors corresponding to the eigenvalues $\lambda\bar{\mu}$ of the operator W given by eq. (5.3) when $Z'\notin\Sigma_Z$, i.e. $((Z',Z))\neq 0$.

Similarly, the scalar product with the convention b) leads for the vector b^t to the eigenvalues $\lambda\bar{\mu}$ of the operator

$$W^{+} = (\hat{Z}\hat{Z}^{+})W(\hat{Z}\hat{Z}^{+})^{-1} = (\hat{Z}\hat{Z'}^{+})(\hat{Z'}\hat{Z'}^{+})^{-1}(\hat{Z'}\hat{Z}^{+})(\hat{Z}\hat{Z}^{+})^{-1} \ . \tag{5.9}$$

The Lemma is proved taking into account eq. (5.7). ■

Using the relations (5.6), an algebraic proof of the theorems 1-3 of Wong[10] follows. A geometrical proof of these theorems in the case of the real Grassmann manifold is given by Sommerville.[40]

We shall show

Lemma 5 *Let* θ *be the angle defined by the hermitian scalar product in eqs. (2.14)-(2.18),* d_c *the Cayley distance and* $\theta_1,\ldots,\theta_n$ *the stationary angles. Then*

$$\cos\theta(Z,Z') = \cos d_c(\iota(Z'),\iota(Z)) = \cos\theta_1\cdots\cos\theta_n \ . \tag{5.10}$$

Proof: It is observed that Lemma 4 implies

$$\det\sqrt{W}=\prod_{i=1}^{n}\cos\theta_i=\frac{|\det(\mathbb{1}+ZZ'^{+})|}{|\det(\mathbb{1}+ZZ^{+})|^{1/2}|\det(\mathbb{1}+Z'Z'^{+})|^{1/2}}\ . \tag{5.11}$$

But equations (2.13)-(2.18) implies that $\cos\theta(Z,Z')=\cos d_c(\iota(Z'),\iota(Z))$ has also the expression (5.11). ∎

Another proof of eq. (5.10) can be found in [11] or in more recent papers[41, 42] which are based on the results of Wong.[10]

Now we attach an index n to the n-plane Z given by eq. (2.6).

Comment 1 *Let the $n'(n)$-plane $Z'_{n'}$ (resp. Z_n) with $n'\le n$ such that $Z'_{n'}\cap Z_n=Z''_{n''}$. Then $n'-n''$ angles of $Z'_{n'}$ and Z_n are different from 0 and n'' angles are 0.*

Proof: Firstly we consider the case $n'=n$. Then there are n'' common eigenvalues of the matrix W for which **a** and **b** are proportional. It follows that $\theta=0$, corresponding to $\lambda\bar{\mu}=1$ for n'' eigenvalues.

Let now $n'<n$. Then *there are at most* $r_0=\min(n',n,m',m)$ *angles which are different from* 0, where $n'+m'=n+m=N$. Observing that $\cos\theta=1$ iff **a**, **b** are proportional, the considerations in the Comment are still true even in this case. ∎

The assertion contained in Comment 1 is largely discussed by Sommerville in Ch. IV p. 47 of [40] for the general case for the real Grassmann manifold and also by Jordan in [2] §49 at p. 110. Reading the paper of Jordan, caution must be paid to the fact that a n-plane in $\mathbb{C}^{n+m}$ in Jordan's terminology is in fact a m-plane in the actual terminology.

5.2 We now briefly discuss the construction presented in this Section in the case of the noncompact manifold X_n.

If in eq. (5.2) we consider instead of the hermitian scalar product $(\cdot,\cdot)$ the hermitian form $(\cdot,\cdot)_n$ defined by (2.23), then we could look for the stationary "angles" (see also Wong[43]) defined by the equation

$$\cosh\theta=\frac{|(\mathbf{a},\mathbf{b})_n|}{||\mathbf{a}||_n||\mathbf{b}||_n}\ . \tag{5.12}$$

With eq. (3.21), we find *the analogue of Lemma 5 in the case of the noncompact manifold X_n, $\cos^2\theta_i$ being substituted with $\cosh^2\theta_i$ and $W=W|_{\epsilon=-1}$, where*

$$W(\epsilon)=(\mathbb{1}+\epsilon ZZ^{+})^{-1}(\mathbb{1}+\epsilon ZZ'^{+})(\mathbb{1}+\epsilon Z'Z'^{+})^{-1}(\mathbb{1}+\epsilon Z'Z^{+})\ . \tag{5.13}$$

The noncompact analogue of eq. (5.11) is

$$\det\sqrt{W(\epsilon)}=\prod_{i=1}^{n}\cosh\theta_i=\frac{|\det(\mathbb{1}+\epsilon ZZ'^{+})|}{|\det(\mathbb{1}+\epsilon ZZ^{+})|^{1/2}|\det(\mathbb{1}+\epsilon Z'Z'^{+})|^{1/2}}\ . \tag{5.14}$$

where $\epsilon=-1$, while (5.10) becomes

$$\cosh\theta(Z,Z')=\cosh d_c(\iota'(Z'),\iota'(Z'))=\cosh\theta_1\cdots\cosh\theta_n\ . \tag{5.15}$$

6 The complex Grassmannian as symmetric space and coherent states

6.1. We remember firstly the algebraic notation used in the construction of symmetric spaces. The Grassmann manifold is considered as compact hermitian irreducible Riemannian globally symmetric space of type A III.[44] We shall also remember the relationship between the compact and noncompact Grassmann manifold. We use the conventions and notation from [45].

X_n: the symmetric space of noncompact type (1.2).

X_c: compact dual of X_n (1.1).

o: fixed base point of X_n and X_c.

K: maximal compact subgroup of G_n, equal to the isotropy group of G_n and G_c at o.

$G^{\mathbb{C}} = G_n^{\mathbb{C}} = G_c^{\mathbb{C}} = G$: complexification of G_c and G_n.

σ: Cartan involution, $\sigma = Ad\, s,\ s =$ symmetry at o.

$\mathfrak{g}_n,\ \mathfrak{g},\ \mathfrak{g}_c, \mathfrak{k}$: Lie algebras of $G_n,\ G,\ G_c,\ K$, respectively.

$\mathfrak{g}_n = \mathfrak{k} + \mathfrak{m}_n$: sum of $+1$ and -1 eigenspaces of $d\sigma$.

$\mathfrak{g} = \mathfrak{g}_n^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}_n$: complexification, where $\mathfrak{m} = \mathfrak{m}_n^{\mathbb{C}}$.

$\mathfrak{g}_c = \mathfrak{k} + \mathfrak{m}_c$: compact real form of $\mathfrak{g}$, where $\mathfrak{m}_c = i\mathfrak{m}_n$.

$\mathfrak{t}$: Cartan algebra in $\mathfrak{g}_n$ and $\mathfrak{g}_c$.

$\mathfrak{t}^{\mathbb{C}}$: Cartan subalgebra of $\mathfrak{g}$.

Δ: $\mathfrak{t}^{\mathbb{C}}$-root system of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{t}^{\mathbb{C}} + \Sigma_{\varphi\in\Delta}\ \mathfrak{g}^{\varphi}$.

Δ_k: set of compact roots, i.e. $\mathfrak{t}^{\mathbb{C}}$-root system of $\mathfrak{k}^{\mathbb{C}}$.

Δ_n: set of noncompact roots, i.e. $\mathfrak{m}$-roots.

$\Delta^{\pm}$: set of positive and negative roots.

(G_c, K): compact symmetric pair.

$(\mathfrak{g}_c, d\sigma)$: orthogonal symmetric algebra of compact type corresponding to (G, K).

$\mathfrak{a}$: Cartan subalgebra of (G_c, K), i.e. maximal commutative subset of $\mathfrak{m}_c,\ \mathfrak{a} \subset \mathfrak{t}$.

Σ: set of restricted roots, $\Sigma \equiv \Delta(\mathfrak{a}; \mathfrak{g}) = \{\gamma \in \mathfrak{a} | \mathfrak{g}_\gamma^{\mathbb{C}} \neq \{0\}\}$.

$\mathfrak{g}_\gamma^{\mathbb{C}}$: restricted root space, i.e. $\mathfrak{g}_\gamma^{\mathbb{C}} = \{X_\gamma \in \mathfrak{g}^{\mathbb{C}} | [H, X_\gamma] = \gamma(H) X_\gamma, \forall H \in \mathfrak{a}\}$.

Let $\pi : G_c \to G_c/K$ the natural projection and $o = \pi(e)$, where e is the unity element in G. Then the subspace $\mathfrak{m}_c$ is naturally identified with the tangent space $(X_c)_0$ by means of the mapping $d\pi$. The negative of the Killing form on $\mathfrak{g}_c$ defines an inner product $Q : \mathfrak{m}_c \times \mathfrak{m}_c \to \mathbb{C}$:

$$Q(X, Y) = -\frac{1}{2}\mathrm{Tr}(XY), \tag{6.1}$$

which is $Ad\ G_c$ and σ-invariant. The Riemann structure is defined by restricting Q to $\mathfrak{m}_c \times \mathfrak{m}_c$ and translating with G_c. The geodesic map $\gamma_X : t \to \mathrm{Exp}_0 tX$ which emanates from o with initial direction X in $\mathfrak{m}_c$ is given by $\mathrm{Exp}_0 tX = \pi \exp tX$, where exp is the exponential map from the Lie algebra $\mathfrak{g}_c$ to the Lie group G_c. A similar construction works in the case of X_n.

Below we specify the quantities introduced in this Section in the case of $G_n(\mathbb{C}^{m+n})$. o is taken as the n-plane $\mathbf{O}$ given by eq. (3.1). The groups are $G_c = SU(n+m),\ G_n = SU(n, m),\ G = SL(n+m, \mathbb{C}),\ K = S(U(n)\times U(m))$. The symmetry at o is $s = I_{nm}$, where $I_{nm} = I_{nm}(-1)$ and

$$I_{nm}(\epsilon) = \begin{pmatrix} \epsilon \mathbb{1}_n & \mathbf{0} \\ \mathbf{0} & \mathbb{1}_m \end{pmatrix} , \tag{6.2}$$

where $\epsilon = 1$ ($\epsilon = -1$) for X_c (resp. X_n). In fact

$$U^+ I_{nm}(\epsilon) U = I_{nm}(\epsilon), \tag{6.3}$$

where $U \in G_c$ (G_n) for $\epsilon = 1$ (resp. $\epsilon = -1$). We have also

$$\mathfrak{k} = \left\{ \begin{pmatrix} a & \mathbf{0} \\ \mathbf{0} & d \end{pmatrix} | a^+ = -a, d^+ = -d, \mathrm{Tr}(a) + \mathrm{Tr}(d) = 0 \right\}, \tag{6.4}$$

$$\mathfrak{m}_{c,n} = \left\{ \begin{pmatrix} \mathbf{o} & \mathbf{b} \\ -\epsilon b^+ & \mathbf{0} \end{pmatrix} \right\}, \tag{6.5}$$

where a, d and b are, respectively, $n \times n$, $m \times m$ and $n \times m$ matrices.

The complex structure of the Grassmann manifold is inherited from his representation as flag manifold[46] $X_c = G^{\mathbb{C}}/P$, the parabolic group P being

$$P = \left\{ \begin{pmatrix} A & \mathbf{0} \\ C & D \end{pmatrix} | \ \det A \det D = 1 \right\} \tag{6.6}$$

and A (D) is an $n \times n$ (resp. $m \times m$) matrix.

The compact roots are

$$\Delta_k = \{ e_i - e_j \mid 1 < i \neq j \leq n, \text{ or } n < i \neq j \leq m + n \} ,$$

where $e_i, \ i = 1, \ldots, N$ belong to the Cartan-Weyl basis.

The manifold X_c and his noncompact dual X_n can be parametrized as

$$X_{n,c} = \exp \begin{pmatrix} \mathbf{0} & B \\ -\epsilon B^+ & \mathbf{0} \end{pmatrix} o = \begin{pmatrix} \mathrm{co}\sqrt{BB^+} & B\dfrac{\mathrm{si}\sqrt{B^+B}}{\sqrt{B^+B}} \\ -\epsilon \dfrac{\mathrm{si}\sqrt{B^+B}}{\sqrt{B^+B}} B^+ & \mathrm{co}\sqrt{B^+B} \end{pmatrix} o \tag{6.7.a}$$

$$= \begin{pmatrix} \mathbb{1} & Z \\ \mathbf{0} & \mathbb{1} \end{pmatrix} \begin{pmatrix} (\mathbb{1} + \epsilon ZZ^+)^{1/2} & \mathbf{0} \\ \mathbf{0} & (\mathbb{1} + \epsilon Z^+ Z)^{1/2} \end{pmatrix} \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ -\epsilon Z^+ & \mathbb{1} \end{pmatrix} o \tag{6.7.b}$$

$$= \exp \begin{pmatrix} \mathbf{0} & Z \\ \mathbf{0} & \mathbf{0} \end{pmatrix} P, \tag{6.7.c}$$

where co is an abbreviation for the circular cosine cos (resp. the hyperbolic cosine coh) for X_c (resp. X_n) and similarly for si. The sign $\epsilon = +$ $(-)$ in eqs. (6.7.a), (6.7.b) corresponds to the compact (resp. noncompact) X. In eq. (6.7.c) Z is the $n \times m$ matrix of Pontrjagin coordinates in $\mathcal{V}_0$ related to B by the formula

$$Z = Z(B) = B \frac{\mathrm{ta}\sqrt{B^+B}}{\sqrt{B^+B}}, \tag{6.8}$$

and ta is an abbreviation for the hyperbolic tangent tanh (resp. the circular tangent tan) for X_n (resp. X_c) and eq. (6.8) realises the exponential map in $\mathcal{V}_0$.

The noncompact case is realised under the restriction

$$\mathbb{1}_m - Z^+ Z > 0 \; . \tag{6.9}$$

The representation (6.7.c) for the noncompact case is the Harish-Chandra embedding[48] of the noncompact dual X_n of X_c in X_c. Note that because of (6.7.c) the complex matrix Z parametrizes the Grassmann manifold.

The invariant metric on $G_n(\mathbb{C}^{m+n})$ to the group action, firstly studied by Teleman[47] and Leichtweiss[28], in the Pontrjagin coordinates reads

$$ds^2 = k\mathrm{Tr}[(\mathbb{1} + \epsilon Z Z^+)^{-1} dZ (\mathbb{1} + \epsilon Z^+ Z)^{-1} dZ^+]. \tag{6.10}$$

$k = 1$ in eq. (6.10) for $X_c = G_n(\mathbb{C}^{m+n})$ corresponds to

$$ds^2 = \iota_* ds^2|_{FS} \tag{6.11}$$

where

$$ds^2|_{FS} = d_c^2([\omega], [\omega + d\omega]) = \frac{(\omega, \omega)(d\omega, d\omega) - (\omega, d\omega)(d\omega, \omega)}{(\omega, \omega)^2}, \tag{6.12}$$

and similarly for X_n.

The equation of the geodesics for $X_{c,n}$ is

$$\frac{d^2 Z}{dt^2} - 2\epsilon \frac{dZ}{dt} Z^+ (\mathbb{1} + \epsilon Z Z^+)^{-1} \frac{dZ}{dt} = 0 \; , \tag{6.13}$$

where $\epsilon = 1$ (-1) for X_c (resp. X_n). It is easy to see that $Z = Z(tB)$ in (6.8) verifies (6.13) with the initial condition $\dot{Z}(0) = B$.

A realization of the algebra $\mathfrak{a}$ consists of vectors of the form

$$H = \sum_{i=1}^{r} h_i D_{i\, n+i}, \;\; h_i \in \mathbb{R}, \tag{6.14}$$

where r is the symmetric rank of X_c (and X_n) and we use the notation

$$D_{ij} = E_{ij} - E_{ji}, \;\; i, j = 1, \ldots, N. \tag{6.15}$$

E_{ij} is the matrix with entry 1 on the i-th line and j-th column and 0 otherwise. We shall also need the notation

$$S_{ij} = E_{ij} + E_{ji}. \tag{6.16}$$

The following Lemma will be used in order to calculate the tangent conjugate locus:

Table 1: Restricted roots for $G_n(\mathbb{C}^{m+n})$. *The lower indices of the roots vectors* X^j_{ab} *are: for* $1 \leq j \leq 8$: $1 \leq a \neq b \leq r$; *for* $9 \leq j \leq 12$: *for every fixed* $a = 1, \ldots, r$, $b = 1, \ldots, |m-n|$ *(if* $m \neq n$*); for* $1 \leq j \leq 14$: $a = 1, \ldots, r$. $(i = \sqrt{-1})$.

Root space vectors	Roots	Multiplicity
$X^1_{ab}; X^5_{ab}$	$i(h_a - h_b)$	2
$X^2_{ab}; X^6_{ab}$	$i(h_b - h_a)$	2
$X^3_{ab}; X^7_{ab}$	$-i(h_a + h_b)$	2
$X^4_{ab}; X^8_{ab}$	$i(h_a + h_b)$	2
$X^9_{ab}; X^{11}_{ab}$	ih_a	$2\|m-n\|$
$X^{10}_{ab}; X^{12}_{ab}$	$-ih_a$	$2\|m-n\|$
X^{13}_a	$2ih_a$	1
X^{14}_a	$-2ih_a$	1

Lemma 6 *The restricted roots of* (G_c, K) *are given in Table I, while the root space vectors are presented below.*

The first eight eigenvectors X^{1-8} *correspond to the eigenvalues*

$$\lambda_{ab} = \epsilon_2 i(h_a + \epsilon_1 h_b),\ \epsilon_1^2 = \epsilon_2^2 = 1, \tag{6.17}$$

of the equation

$$[H, X^j_{ab}] = \lambda_{ab} X^j_{ab},\ \forall H \in \mathfrak{a},\ X \in \mathfrak{g}^{\mathbb{C}}. \tag{6.18}$$

With the notation

$$X^j_{ab} = D^{\epsilon_1\epsilon_2}_{ab},\quad X^{j+4}_{ab} = S^{\epsilon_1\epsilon_2}_{ab}, \tag{6.19}$$

$$j = j(\epsilon_1, \epsilon_2) = \epsilon_1(1 + \epsilon_2/2) + 5/2, \tag{6.20}$$

and if F *is any of the matrices* D *and* S, *then*

$$F^{\epsilon_1\epsilon_2}_{ab} = F_{a\,n+b} + \epsilon_1 F_{n+a\,b} + i\epsilon_2(F_{n+a\,n+b} - \epsilon_1 F_{a\,b}). \tag{6.21}$$

Explicitly

$$\begin{aligned}
X^{1;5}_{ab} &= \ \ i(F_{ab} + F_{n+a\,n+b}) - F_{n+a\,b} + F_{a\,n+b},\\
X^{2;6}_{ab} &= -i(F_{ab} + F_{n+a\,n+b}) - F_{n+a\,b} + F_{a\,n+b},\\
X^{3;7}_{ab} &= \ \ i(F_{ab} - F_{n+a\,n+b}) + F_{n+a\,b} + F_{a\,n+b},\\
X^{4;8}_{ab} &= -i(F_{ab} - F_{n+a\,n+b}) + F_{n+a\,b} + F_{a\,n+b},
\end{aligned}$$

where the first (second) upper index of X *corresponds to* $F = D$ *(resp.* $F = S$*).*

The other vectors are as follows

$$X^{13}_a = \frac{1}{2}X^8_{aa},\ X^{14}_a = \frac{1}{2}X^7_{aa},$$

$$X^{9;10}_{ab} = \begin{cases} E_{n+a\,2n+b} \mp iE_{2n+b\,n+a} & \text{if } n \le m, \\ E_{n+a\,m+b} \mp iE_{a\,m+b} & \text{if } n > m; \end{cases}$$

$$X^{11;12}_{ab} = \begin{cases} E_{2n+b\,a} \pm iE_{2n+b\,n+a} & \text{if } n \le m, \\ E_{m+b\,a} \mp iE_{m+b\,n+a} & \text{if } n > m. \end{cases}$$

Proof: The simplest proof is to solve the eigenvalue equation (6.18). ∎

6.2. The manifold $\widetilde{\mathbf{M}}_{n,c}$ of coherent states[22] (in the sense of Perelomov[21]) corresponding to $X_{n,c}$ is introduced in the notation of [45]. The manifold of coherent vectors is the holomorphic line bundle associated to the character of the parabolic subgroup P, with base the manifold of coherent states taken a homogeneous Kählerian manifold. The coherent states are paramerized by a matrix Z in front of the noncompact positive roots which appear at the exponent.[45]

We shall prove the following

Remark 4 *The coherent vector $|Z, j_0>=|Z>$, where Z it is an $n \times m$ matrix, corresponds to the $n-$plane of $\mathbb{C}^{n+m}$ parametrized by the Pontrjagin coordinates Z in $\mathcal{V}_0$ leading to $\hat{Z} = (\mathbb{1}_n Z)$.*

Moreover, we have the equality of the scalar product of coherent vectors $<\cdot|\cdot>$ and the hermitian scalar product $((\cdot,\cdot))$ of the holomorphic line bundle Det^*:

$$< Z', j_0|Z, j_0 >=< Z'|Z >= ((Z', Z)) = ((Z^{\perp}, Z'^{\perp})) \ . \tag{6.22}$$

and similarly for the noncompact manifold X_n.

Proof: The scalar product of two coherent vectors is[45]

$$< Z', j|Z, j >= \prod_{k=1}^{m+n} (A^{k\,k+1\ldots m+n}_{k\,k+1\ldots m+n})^{j_k - j_{k+1}} \ , \ j_0 = 0, \ j_1 \ge j_2 \ge \cdots \ge j_{m+n} \ , \tag{6.23}$$

where A is the matrix

$$A = \begin{pmatrix} (\mathbb{1}_n + \epsilon Z Z'^{+})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbb{1}_m + \epsilon Z'^{+} Z) \end{pmatrix} , \tag{6.24}$$

the sign $\epsilon = -\ (+)$ corresponds to X_n (resp. X_c) and the coherent vectors are considered in the chart $\mathcal{V}_0$ in the case of X_c.

Using the particular dominant weight

$$j = j_0 = (\underbrace{1, \ldots, 1}_{n}, \underbrace{0, \ldots, 0}_{m}) \ , \tag{6.25}$$

it is found[45] that

$$< Z', j_0|Z, j_0 >= \det(\mathbb{1}_n + \epsilon Z Z'^{+})^{\epsilon} \ , \tag{6.26}$$

under the condition (2.10.b) of the scalar product. ∎

Note that in the convention of [45] the coherent vector $|Z>$ corresponds to the $n-$plane Z^t as a consequence of the fact that the fixed base point of the Grassmann manifold in [45] was chosen

$$Z_0 = \mathbf{e}_{m+1} \wedge \ldots \wedge \mathbf{e}_{n+m}$$

and not Z_0 given by (3.1). So, using Remark 3, it follows that eq. (6.22) should corresponds in the conventions of [45] to

$$< Z', j_0|Z, j_0 >=< Z'|Z >= ((Z'^t, Z^t)) = ((Z^{t\perp}, Z'^{t\perp})) \ . \tag{6.27}$$

Finally, we remember that Calabi's diastasis function[26] $D(Z', Z)$ has been used in the context of coherent states,[27] observing that $D(Z', Z) = -2\log < \underline{Z'}|\underline{Z} >$, where $|\underline{Z} >=< Z|Z >^{-1/2} |Z >$.

The noncompact Grassmann manifold X_n admits the embedding in an infinite dimensional projective space $\iota_n : X_n \hookrightarrow \mathbf{PK}$ and also the embedding $\iota' : X_n \hookrightarrow \mathbb{CP}^{N(n)-1,1}$. Let δ_n (θ_n) be the length of the geodesic joining $\iota'(Z'), \iota'(Z)$ (resp. $\iota_n(Z'), \iota_n(Z)$). Then we have the

Remark 5 *For the noncompact Grassmann manifold, δ_n, θ_n and D_n are related through the relation*

$$\cos\theta_n = \cosh^{-1}\delta_n = e^{-D_n/2} = \frac{|\det(\mathbb{1} - ZZ'^+)|}{\det[(\mathbb{1} - ZZ^+)(\mathbb{1} - Z'Z'^+)]^{1/2}} \ . \tag{6.28}$$

Proof: Due to eq. (6.22), the diastasis is related to the geodesic distance $\theta(Z', Z)$ given by eq. (2.18) by the relation $D(Z', Z) = -2\log\cos(\theta(Z', Z))$. So, the diastasis for X_c if $Z, Z' \in \mathcal{V}_0$ is

$$D(Z', Z) = \log\frac{\det[(\mathbb{1} + ZZ^+)(\mathbb{1} + Z'Z'^+)]}{|\det(\mathbb{1} + ZZ'^+)|^2}, \tag{6.29}$$

and similarly for the noncompact case. ∎

The equation (6.29) is still valid for the infinite dimensional Grassmann manifold.[29]

7 Cut locus and conjugate locus

7.1 Preliminaries

We begin remembering some definitions referring to the cut locus and conjugate locus.

Let V be a compact Riemannian manifold of dimension n, $p \in V$ and let Exp_p be the (geodesic) exponential map at the point p. Let C_p denote the set of vectors $X \in V_p$ (the tangent space at $p \in V$) for which $\mathrm{Exp}_p X$ is singular. A point q in V (V_p) *is conjugate to* p if it is in $\mathbf{C}_p = \mathrm{Exp}\, C_p$ (C_p)[44] and $\mathbf{C}_p$ (C_p) is called *the conjugate locus* (resp. *tangent conjugate locus*) of the point p.

Let $q \in V$. The point q is in the *cut locus* $\mathbf{CL}_p$ of $p \in V$ if it is nearest point to $p \in V$ on the geodesic joining p with q, beyond which the geodesic ceases to minimize its arc length.[15] More precisely, let $\gamma_X(t) = \operatorname{Exp} tX$ be a geodesic emanating from $\gamma_X(0) = p \in V$, where X is a unit vector from the unit sphere S_p in V_p. t_0X (resp. $\operatorname{Exp} t_0X$) is called a *tangential cut point* (*cut point*) of p along $t \to \operatorname{Exp} tX$ $(0 \le t \le s)$ if the geodesic segment joining $\gamma_X(0)$ and $\gamma_X(t)$ is a minimal geodesic for any $s \le t_0$ but not for any $s > t_0$.

Let us define the function $\mu : S_p \to \mathbb{R}^+ \cup \infty$, $\mu(X) = r$, if $q = \operatorname{Exp} rX \in \mathbf{CL}_p$, and $\mu(X) = \infty$ if there is no cut point of p along $\gamma_X(t)$. Setting $I_p = \{tX,\ 0 < t < \mu(X)\}$, then $\mathbf{I}_p = \operatorname{Exp} I_p$ is called the *interior set at p*. Then[15]

1) $\mathbf{I}_p \cap \mathbf{CL}_p = \emptyset$, $V = \mathbf{I}_p \cup \mathbf{CL}_p$, the closure $\bar{\mathbf{I}}_p = V$, and $\dim \mathbf{CL}_p \le n - 1$.

2) I_p is a maximal domain containing $0 = 0_p \in V_p$ on which Exp_p is a diffeomorphism and $\mathbf{I}_p$ is the largest open subset of V on which a normal coordinate system around p can be defined.

This theorem will be used below in the proof of Proposition 1.

The importance of the cut loci lies in the fact they inherit topological properties of the manifold V.

The relative position of $\mathbf{CL}_0$ and $\mathbf{C}_0$ is given by Theorem 7.1 p. 97 in [15] :

Let the notation $\gamma_t = \gamma_X(t)$. Let γ_r be the cut point of γ_0 along a geodesic $\gamma = \gamma_t$, $0 \le t < \infty$. Then, at least one (possibly both) of the following statements holds:

(1) γ_r is the first conjugate point of γ_0 along γ;

(2) there exists, at least, two minimising geodesics from γ_0 to γ_r.

Crittenden[49] has shown that for the case of simply connected symmetric spaces, the cut locus is identified with the first conjugate locus. This result will be illustrated on the case of the complex Grassmann manifold. Generally, the situation is more complicated.[50, 51]

For $\mathbb{CP}^n$, CL is the sphere of radius π with centre at the origin of the tangent space to $\mathbb{CP}^n$ at the given point, while $\mathbf{CL}$ is the hyperplane at infinity $\mathbb{CP}^{n-1}$. Except few situations, e. g. the ellipsoid, even for low dimensional manifolds, $\mathbf{CL}$ is not known explicitly. Helgason[44] has shown that the cut locus of a compact connected Lie group, endowed with a bi-invariant Riemannian metric is stratified, i.e. it is the disjoint union of smooth submanifolds of V. This situation will be illustrated on the case of the complex Grassmann manifold. Using a geometrical method, based on the Jordan's stationary angles, Wong[10, 14, 16] has studied conjugate loci and cut loci of the Grassmann manifolds. Calculating the tangent conjugate locus on the Grassmann manifold, Sakai[17] observed that the results of Wong[14] referring to conjugate locus in Grassmann manifold are incomplete. This problem will be largely discussed in the present Section. By refining the results of Ch. VII, §5 from Helgason's book,[44] Sakai[17, 52] studied the cut locus on a general symmetric space and showed that it is determined by the cut locus of a maximal totally geodesic flat submanifold of V. However, the expression of the conjugate locus as subset of the Grassmann manifold is not known explicitly. We give a geometric characterization of the part of the conjugate locus different from those found by Wong in terms of the stationary angles.

7.2 Cut locus

Coming back to eqs. (6.7.a)-(6.7.c), it is observed that B *are normal coordinates around* $Z=0$ on the Grassmann manifold. So we have

$$< \mathbf{O}|Y >= 0 \text{ iff } ((\mathbf{O},Y)) = 0 \text{ or iff } Y \in \Sigma_0 \ . \tag{7.1}$$

The following two assertions are particular situations true for symmetric or generalized symmetric spaces[23, 24]

Proposition 1 (Wong[10]) *The cut locus, the polar divisor of* $\mathbf{O} \in \mathcal{V}_0 \subset G_n(\mathbf{K})$ *and the interior set are related by the relations*

$$\mathbf{CL}_0 = \Sigma_0 \ , \tag{7.2}$$

$$\mathcal{V}_0 = \mathbf{I}_0, \tag{7.3}$$

and Σ_0 *is given by Lemma 3.*

Remark 6 *The solution of the equation*

$$< 0|\psi >= 0 \ , \tag{7.4}$$

where $|\psi>$ *is a coherent vector, is given by the points on the Grassmann manifold corresponding to the cut locus* $\mathbf{CL}_0 = \Sigma_0$.

Proof: The dependence $Z(t) = Z(tB)$ expressed by (6.8) gives geodesics starting at $Z=0$ in the chart $\mathcal{V}_0$ and $\mathcal{V}_0$ *is the maximal normal neighbourhood.* The Proposition follows due to Thm. 7.4 of Kobayashi and Nomizu[15] and the subsequent remark at p. 102 reproduced earlier. ■

7.3 The conjugate locus in the complex Grassmann manifold

Now the conjugate locus of the point $Z=0$ in $G_n(\mathbb{C}^{m+n})$ is calculated. The Jacobian of the transformation (6.8) has to be computed. We shall prove the following theorem and remark

Theorem 2 *The conjugate locus of* $\mathbf{O}$ *in* $G_n(\mathbb{C}^{m+n})$ *is given by the union*

$$\mathbf{C}_0 = \mathbf{C}_0^W \cup \mathbf{C}_0^I. \tag{7.5}$$

$\mathbf{C}_0^W$ *consists of those points of the Grassmann manifold which have at least one of the stationary angles with the* $\mathbf{O}$ *plane* 0 *or* $\pi/2$. $\mathbf{C}_0^I$ *consists of those points of* $G_n(\mathbb{C}^{m+n})$ *for which at least two of the stationary angles with* $\mathbf{O}$ *are equal, that is at least two of the eigenvalues of the matrix (5.4) are equal.*

The $\mathbf{C}_0^W$ *part of the conjugate locus is given by the disjoint union*

$$\mathbf{C}_0^W = \begin{cases} V_1^m \cup V_1^n, & n \le m, \\ V_1^m \cup V_{n-m+1}^n, & n > m, \end{cases} \tag{7.6}$$

where

$$V_1^m = \begin{cases} \mathbb{CP}^{m-1}, & n=1, \\ W_1^m \cup W_2^m \cup \ldots W_{r-1}^m \cup W_r^m, & 1<n, \end{cases} \quad (7.7)$$

$$W_r^m = \begin{cases} G_r(\mathbb{C}^{\max(m,n)}), & n \neq m, \\ \mathbf{O}^{\perp}, & n=m, \end{cases} \quad (7.8)$$

$$V_1^n = \begin{cases} W_1^n \cup \ldots \cup W_{r-1}^n \cup \mathbf{O}, & 1<n \leq m, \\ \mathbf{O}, & n=1, \end{cases} \quad (7.9)$$

$$V_{n-m+1}^n = W_{n-m+1}^n \cup W_{n-m+2}^n \cup \ldots \cup W_{n-1}^n \cup \mathbf{O}\ ,\ n>m\ . \quad (7.10)$$

Remark 7 *The cut locus in $G_n(\mathbb{C}^{m+n})$ is given by those n-planes which have at least one angle $\pi/2$ with the plane* $\mathbf{O}$.

Proof: The proof is done in 4 steps. *a*) Firstly, a diagonalization is performed. *b*) After this, the Jacobian of a transformation of complex dimension one is computed. At *c*) the cut locus is reobtained. d) Finally, the property of the stationary angles given in Comment 1 is used in order to get the conjugate locus in $G_n(\mathbb{C}^{m+n})$.

At *b*) we argue that the proceeding gives all the conjugate locus. See also the proof of the Proposition 2, where it is stressed the equivalence of the decomposition (7.11), (7.12) with the representation given by eq. (7.28).

a) Every $n\times m$ matrix Y can be put in the form[53]

$$Y = U\Lambda V, \quad (7.11)$$

where U (V) is a unitary $n\times n$ (resp. $m\times m$) matrix,

$$\Lambda = \begin{cases} (D,0) & \text{if} \quad n\leq m\ , \\ \begin{pmatrix} D \\ 0 \end{pmatrix} & \text{if} \quad n>m\ , \end{cases} \quad (7.12)$$

and D is the $r\times r$ diagonal matrix ($r=\min(m,n)$), with diagonal elements $\lambda_i \geq 0,\ i=1,\ldots,r$. If the rank of the matrix Y is r_1, then Λ has four block form with the diagonal elements $\lambda_i>0,\ i=1,\ldots,r_1$, the other elements being 0.[54]

We shall apply a decomposition of the type (7.11)-(7.12) taking the diagonal elements of the matrix D as complex numbers. This implies an overall phase $e^{i\varphi}$ for the matrix Λ in the decomposition (7.11). This phase can be included in the matrix U such that the matrix $U'=e^{-i\varphi}U$ is unitary in the decomposition (7.11).

Applying to the $n\times m$ matrix B the diagonalization technique here presented, the $n\times m$ matrix $Z=Z(tB)$ corresponding to eq. (6.8) is also of the same form, where the diagonal elements are

$$Z_i = \frac{B_i}{|B_i|}\tan t|B_i|,\ i=1,\ldots,r\ . \quad (7.13)$$

b) In order to calculate the Jacobian J when B is diagonalized, let us firstly calculate

$$\Delta_1 = \Delta_1(X,Y) = \begin{vmatrix} \dfrac{\partial X}{\partial B_x} & \dfrac{\partial X}{\partial B_y} \\ \dfrac{\partial Y}{\partial B_x} & \dfrac{\partial Y}{\partial B_y} \end{vmatrix} , \tag{7.14}$$

where $Z = X + iY,\ B = B_x + iB_y$, and $X, Y, B_x, B_y \in \mathbb{R}$.

With eq. (7.13) we get

$$\Delta_1 = \frac{t}{|B|}\frac{\sin t|B|}{\cos^3 t|B|} . \tag{7.15}$$

Now there are two possibilities. α) Firstly, we consider the case when all the $|B_i|$ in eq. (7.13) are distinct. Then the Jacobian J corresponding to the transformation (7.11), (7.12) of Z is

$$\Delta = \prod_{i=1}^{r} \frac{t}{|B_i|}\frac{\sin t|B_i|}{\cos^3 t|B_i|} . \tag{7.16}$$

β) Otherwise, at least two of the eigenvalues $|B_i|$ in the transformation (7.13) are equal. Then the Jacobian J is zero and the points are in the tangent conjugate locus.

c) Let us now take in eq. (3.20.b) (and (6.26)) $Z' = 0$ and let us introduce Z in the diagonal form (7.12), (7.13) in eqs. (5.4), (5.11) and (5.10). It results that

$$\cos\theta_i = |\cos\, t|B_i||,\ i = 1, \ldots, r , \tag{7.17}$$

and then formula (5.10) becomes

$$\cos\theta = \prod_{i=1}^{r} |\cos t|B_i|| . \tag{7.18}$$

If for some $i \neq j$, $|B_i|$ and $|B_j|$ are identical or

$$|B_i \pm B_j| = \pi, \tag{7.19}$$

then they correspond to the same stationary angles $\theta_i = \theta_j$, cf. eq. (7.17). In other words, if at least two of the eigenvalues of the matrix (5.4) are identical, then they correspond to the same stationary angles $\theta_i = \theta_j$.

If for some i, $t|B_i| = \pi/2$ in the Z matrix put in the diagonalized form, we have to change the chart. As a consequence of the fact that the change of charts has the homographic form (3.13), a change of those coordinates which are not finite in one chart has the form $Z \to 1/Z$, the matrix B being diagonalized. So, we have to calculate instead of eq. (7.14), the Jacobian $\Delta_1' = \Delta_1'(X', Y')$, where $Z' = 1/Z = X' + iY', X',\ Y' \in \mathbb{R}$. It is easily found out that

$$\Delta_1' = \frac{t}{|B|}\frac{\cos t|B|}{\sin^3 t|B|} . \tag{7.20}$$

With equation (7.20) we get that $\Delta' = 0$ iff $\cos\theta = 0$ in eq. (7.18), where Δ' is the expression corresponding to Δ in the new chart. In fact, $\cos\theta = 0$ iff for at least one i, $t|B_i| = \pi/2,\ i = 1,\dots,r$, or, equivalently, iff at least one of the angles of $\mathbf{O}^\perp$ with Z is zero, i.e. $\dim(\mathbf{O}^\perp \cap Z) \geq 1$. The results of Proposition 1 and Remark 6 for the cut locus are reobtained, i.e. $\mathbf{CL}_0 = \Sigma_0 = V_1^m$ and the Remark 7 is proved.

From formulas (7.16), (7.18) it follows also that *in the tangent space the cut locus = first conjugate locus*, a result true for any symmetric simply connected space[49] as has been already remarked.

d) Further we look for the other points Z in the conjugate locus $\mathbf{C}_0$ but not in $\mathbf{CL}_0$, i.e. we look for the other points where $J = 0$.

Once the cut locus was gone beyond, the same chart as before the cut locus has been reached can be used. Then at lest one of the $t|B_i|$ is zero (modulo π), corresponding to at least one of the angles between $\mathbf{O}$ and Z zero.

Let $n \leq m$. If $Z \in G_n(\mathbb{C}^{n+m})$ is such that $\dim(Z \cap \mathbf{O}) = i,\ i = 1,\dots n$, then i stationary angles between $\mathbf{O}$ and Z are zero ($n-i$ are different from 0) and $Z \in W_i^n$. Finally $Z \in \mathbf{C}_0 \setminus \mathbf{CL}_0$ iff

$$Z \in \bigcup_{i=1}^{n} W_i^n = V_1^n \ .$$

Let now $n > m$. We look for points for which $J = 0$ different from the points of $\mathbf{CL}_0$ where $\Delta = 0$. So eq. (7.16) with $r = m$ does not include the $n - m$ stationary angles which are already zero. Then $J = 0$ when at least $n - m + 1$ angles are zero. In fact, if $\mathbf{O}$ and Z are two $n-$planes such that $n - m + i$ angles are zero (and $m - i$ angles are not zero), then $\dim(\mathbf{O} \cap Z) = n - m + i,\ i = 1,\dots,m$. So, in order that the equation $J = 0$ to be satisfied in the points of $\mathbf{C}_0 \setminus \mathbf{CL}_0$, it is necessary and sufficient that

$$Z \in \bigcup_{i=1}^{m} W_{n-m+i}^n = V_{n-m+1}^n \ .$$

The representations (7.7), (7.9), (7.10) follow particularizing the third eq. (4.13) and the last term is obtained as particular case of eq. (4.14).

To see that the union (7.6) is disjoint, it is observed that $V_1^m = Z(m-1, m,\dots,m)$ and $V_1^n = Z(n-1, m,\dots,m)$. The condition to have nonvoid intersection of the Schubert varieties $Z(\omega)$, $Z(\omega')$ is that $\omega_i + \omega'_{n-i} \geq n + m,\ i = 1,\dots,n$ (cf. p. 326 in [7]). ■

Wong's[14] notation is

$$V_l = \{Z \in G_n(\mathbf{C}^{n+m}) |\dim(Z \cap \mathbf{O}^\perp) \geq l\},\ \widetilde{V}_l = \{Z \in G_n(\mathbf{C}^{n+m}) |\dim(Z \cap \mathbf{O}) \geq l\} \ , \tag{7.21}$$

i.e. V_l ($\widetilde{V}_l$) from Wong corresponds to our V_l^m (resp. V_l^n) and

$$\mathbf{C}_0^W = \begin{cases} V_1 \cup \widetilde{V}_1 & ,\quad n \leq m \ , \\ V_1 \cup \widetilde{V}_{n-m+1} & ,\quad n > m \ . \end{cases} \tag{7.22}$$

7.4 The tangent conjugate locus

The tangent conjugate locus C_0 for $G_n(\mathbb{C}^{m+n})$ in the case $n \leq m$ was obtained by Sakai.[17] Sakai has observed that Wong's result on the conjugate locus in the manifold is incomplete, i.e. $\mathbf{C}_0^W \subset \mathbf{C}_0$ but $\mathbf{C}_0^W \subsetneqq \mathbf{C}_0 = \exp C_0$. The proof of Sakai consists in solving the eigenvalue equation $R(X, Y^i)X = e_i Y^i$ which appears when solving the Jacobi equation, where the curvature for the symmetric space $X_c = G_c/K$ at o is simply $R(X,Y)Z = [[X,Y],Z],\ X,Y,Z \in \mathfrak{m}_c$. Then $q = \mathrm{Exp}_0 tX$ is conjugate to o if $t = \pi\lambda/\sqrt{e_i},\ \lambda \in \mathbb{Z}^\star \equiv \mathbb{Z}\backslash\{0\}$. The solution of the same problem in terms of $\alpha(H)$ is given by Lemma 2.9 at page 288 in [44]. In §7.3 we have calculated $\mathbf{C}_0$ using directly the form (6.8) of the exponential map in $\mathcal{V}_0$. Below we present another calculation of C_0 and compare these results with those proved in Theorem 2 referring to the conjugate locus in $G_n(\mathbb{C}^{m+n})$.

Proposition 2 *The tangent conjugate locus C_0 of the point $\mathbf{O} \in G_n(\mathbb{C}^{m+n})$ is given by*

$$C_0 = \bigcup_{k,p,q,i} ad\, k(t_i H)\ ,\ i = 1,2,3;\ 1 \leq p < q \leq r,\ k \in K, \tag{7.23}$$

where the vector $H \in \mathfrak{a}$ (eq. 6.14) is normalized,

$$H = \sum_{i=1}^{r} h_i D_{i\,n+i},\ h_i \in \mathbb{R},\ \sum h_i^2 = 1\ . \tag{7.24}$$

The parameters $t_i,\ i = 1,2,3$ in eq. (7.23) are

$$\begin{aligned} t_1 &= \frac{\lambda\pi}{|h_p \pm h_q|}\ ,\ \text{multiplicity } 2; \\ t_2 &= \frac{\lambda\pi}{2|h_p|}\ ,\ \text{multiplicity } 1; \\ t_3 &= \frac{\lambda\pi}{|h_p|}\ ,\ \text{multiplicity } 2|m-n|;\ \lambda \in \mathbb{Z}^\star\ . \end{aligned} \tag{7.25}$$

The following relations are true

$$\mathbf{C}_0^I = \exp \bigcup_{k,p,q} Ad\, k(t_1 H)\ , \tag{7.26}$$

$$\mathbf{C}_0^W = \exp \bigcup_{k,p} Ad\, k(t_2 H)\ , \tag{7.27}$$

i.e. exponentiating the vectors of the type $t_1 H$ we get the points of $\mathbf{C}_0^I$ for which at least two of the stationary angles with $\mathbf{O}$ are equal, while the vectors of the type $t_2 H$ are sent to the points of $\mathbf{C}_0^W$ for which at least one of the stationary angles with $\mathbf{O}$ is 0 or $\pi/2$.

Proof: Any vector $X \in \mathfrak{m}_c$ can be put[44] in the form

$$X = Ad\,(k)H,\ k \in K,\ H \in \mathfrak{a}. \tag{7.28}$$

So, in order to find out the tangent conjugate locus it is sufficient to solve this problem for $H \in \mathfrak{a}$. But then X in eq. (7.28) is conjugate with o iff

$$\alpha(H) \in i\pi\mathbb{Z}^\star \tag{7.29}$$

for some root α which do not vanishes identically on $\mathfrak{a}$ (cf. i.e. Prop. 3.1 p. 294 in the book of Helgason[44]). In fact, what we have to find out is the diagram of the pair (G_c, K).

But according to Lemma 6, the space of restricted root vectors consists of three types of vectors, corresponding respectively to the restricted roots: $\pm i(h_a - h_b), \pm ih_a$ and $\pm 2ih_a$, where $1 \leq a < b \leq r$, with the vector H of the form (7.24). So, imposing to the vectors tH the condition (7.29), the values (7.25) are obtained for the parameters t_i.

To compare the results on C_0 with those on $\mathbf{C}_0$, let us observe that a diagonal matrix B as in §7.3 corresponds to the representation (7.24). When expressed in stationary angles, the "singular value decomposition" (7.11) is nothing else than the representation (7.28) expressed matricially. In eq. (7.12) Λ corresponds to B while D corresponds to H, where B is in $\mathfrak{m}_c$ as in eq. (6.7.a) and H has the form (7.24). This implies that for the vector

$$t_1 H = \frac{\pi\lambda}{|h_p \pm h_q|}\sum_{i=1}^{r} h_i D_{i\,n+i},\ \ \lambda \in \mathbb{Z}^\star\ , \tag{7.30}$$

the p^{th} (q^{th}) coordinate in the tangent space $\mathfrak{m}_c$ to $G_n(\mathbb{C}^{m+n})$ at o is $B_p = \lambda\pi h_p|h_p \pm h_q|^{-1}$ (resp. $B_q = \lambda\pi h_q|h_p \pm h_q|^{-1}$). Consequently, the relation (7.19) is fulfilled. So, due to eq. (7.17), the corresponding stationary angles are equal, $\theta_p = \theta_q$ and eq. (7.26) is proved.

Similarly, the vector

$$t_2 H = \frac{\pi\lambda}{2|h_p|}\sum_{i=1}^{r} h_i D_{i\,n+i},\ \ \lambda \in \mathbb{Z}^*\ , \tag{7.31}$$

corresponds for λ even (odd) to points on $G_n(\mathbb{C}^{m+n})$ which have at least one of the stationary angles with $\mathbf{O}$ equal to 0 (resp. $\pi/2$). This fact and also the representation (7.6) can be seen with eq. (6.7.a) with diagonal B-matrix. Then in eq. (6.7.a)

$$\mathrm{diag}(\sqrt{BB^+}) = (|h_1|, \ldots, |h_n|),\ \text{if}\ n \leq m\ , \tag{7.32.a}$$

$$\mathrm{diag}(\sqrt{B^+B}) = (|h_1|, \ldots, |h_m|),\ \text{if}\ n > m\ , \tag{7.32.b}$$

where $\mathrm{diag}(X)$ denotes the diagonal elements of the matrix X.

Choosing $o \in G_n(\mathbb{C}^{m+n})$ to correspond to $\mathbf{O}$ given by (3.1), then (cf. eq. (3.7)) a point of $\mathbf{O}$ has the coordinates

$$(x_1, \ldots, x_n, \underbrace{0, \ldots, 0}_{m})\ .$$

So, a point of the Grassmann manifold X_c is

$$(x_1\cos|h_1|,\ldots,x_n\cos|h_n|,-x_1\frac{h_1}{|h_1|}\sin|h_1|,\ldots,-x_n\frac{h_n}{|h_n|}\sin|h_n|,\underbrace{0,\ldots,0}_{m-n}), \qquad (7.33.a)$$

$$(x_1\cos|h_1|,\ldots,x_m\cos|h_m|,x_{m+1},\ldots,x_n,-x_1\frac{h_1}{|h_1|}\sin|h_1|,\ldots,-x_m\frac{h_m}{|h_m|}\sin|h_m|), \quad (7.33.b)$$

where eq. (7.33.a) ((7.33.b)) corresponds to the case $n \leq m$ (resp. to $n > m$). ■

Note that $\mathbf{C}_0^I$ is not a Schubert variety, because in general $\mathbf{C}_0^I \cap \mathbf{O} = \emptyset$ and $\mathbf{C}_0^I \cap \mathbf{O}^\perp = \emptyset$. However, the representation (7.30) with $h_p + h_q$ at the denominator gives V_2 ($\widetilde{V_2}$) for λ odd (resp. even) cf. eq. (7.33.a) or Sakai.[17] But $V_2 \subset V_1$ (in fact V_2 is the singular locus of V_1) and the union in eq. (7.23) is not disjoint even for the parts which correspond to t_1H and t_2H.

Comment 2 $\mathbf{C_0}^I$ *contains as subset the maximal set of mutually isoclinic subspaces of the Grassmann manifold, which are the isoclinic spheres, with dimension given by the solution of the Hurwitz problem.*

Proof: Wong[19] has found out the locus of isoclines in $G_n(\mathbb{R}^{2n})$, i.e. the maximal subset B of the Grassmann manifold containing $\mathbf{O}$ consisting of points with the property that every two n-planes of B have *all* the stationary angles equal. Two mutually isoclinic n-planes correspond to the situation where the matrix (5.3) is a multiple of $\mathbb{1}$. The results of Wong were generalized by Wolf,[55] who has considered also the complex and quaternionic Grassmann manifolds. The problem of maximal mutually isoclinic subspaces is related with the Hurwitz problem.[20] Any maximal set of mutually isoclinic n-planes is analytically homeomorphic to a sphere (cf. Thm 8.1 in Wong[19] and Wolf[55]), the dimension of the isoclinic spheres being given by the solution to the Hurwitz problem. ■

8 The distance

In this Chapter Z is an $n \times m$ matrix characterizing a point in the complex Grassmann manifold X_c (1.1) (resp. the noncompact dual (1.2) X_n of X_c). In the case of the compact Grassmann manifold X_c the $n-$plane Z is taken in $\mathcal{V}_0$, while in the case of the noncompact manifold X_n, the matrix Z is restricted by the condition (6.9). In formulas below $\epsilon = 1$ (-1) and arcta is an abbreviation for the inverse of the circular tangent function, arctan (hyperbolic function arctanh) for X_c (resp. X_n) and analogously for arcco and arcsi.

Let us denote by $\lambda_i(A),\ i = 1, \ldots, p$ the eigenvalues of the $p \times p$ matrix A and let $\eta = \sqrt{-\epsilon}$.

We shall prove the following

Proposition 3 *The square of the distance between two points $Z_1,\ Z_2$ is given by the formulas*

$$d^2(Z_1, Z_2) = \mathrm{Tr}\left[\mathrm{arcta}(ZZ^+)^{1/2}\right]^2, \tag{8.1}$$

$$= \sum_{j=1}^{n} \theta_j^2\ , \tag{8.2}$$

where

$$Z = (\mathbb{1} + \epsilon Z_1Z_1^+)^{-1/2}(Z_2 - Z_1)(\mathbb{1} + \epsilon Z_1^+Z_2)^{-1}(\mathbb{1} + \epsilon Z_1^+Z_1)^{1/2}, \tag{8.3}$$

$$\theta_j = \mathrm{arcta}\,\lambda_j(ZZ^+)^{1/2} = \mathrm{arcco}\,\lambda_j(V)^{1/2} = \mathrm{arcsi}\,\lambda_j(VZZ^+)^{1/2} \tag{8.4}$$

$$= \frac{1}{2\eta}\log\frac{\mathbb{1} + \eta\lambda_j(ZZ^+)^{1/2}}{\mathbb{1} - \eta\lambda_j(ZZ^+)^{1/2}} = \frac{1}{\eta}\log\lambda_j[V^{1/2}(\mathbb{1} + \eta(ZZ^+)^{1/2})], \tag{8.5}$$

$$V \equiv (\mathbb{1} + \epsilon ZZ^+)^{-1}, \tag{8.6}$$

$$V = (\mathbb{1}+\epsilon Z_1Z_1^+)^{-1/2}(\mathbb{1}+\epsilon Z_1Z_2^+)(\mathbb{1}+\epsilon Z_2Z_2^+)^{-1}(\mathbb{1}+\epsilon Z_2Z_1^+)(\mathbb{1}+\epsilon Z_1Z_1^+)^{-1/2}. \tag{8.7}$$

The matrices V and W given by eq. (5.13) have the same eigenvalues.

Proof: The proof is done in three steps. *a)* Firstly, a homographic transformation $Z' = Z'(Z)$ for which $Z'(Z_1) = 0$ is obtained. *b)* Further on, the distance between $Z = 0$ and Z, where $Z \in \mathcal{V}_0$ for X_c is found. *c)* Finally, the transformation $Z' = Z'(Z_2)$ gives eq. (8.3), while (8.4)-(8.5) are obtained as $d^2(0, Z'(Z_2))$. The representation (8.7) of the matrix (8.6) is furnished by a matrix calculation.

a) The transitive action of an element from the group $G_c = SU(n + m)$ ($G_n = SU(n, m)$) on X_c (resp. X_n) is given by the linear fractional transformation

$$Z' = Z'(Z) = U \cdot Z = (AZ + B)(CZ + D)^{-1},\ U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_c\ (G_n)\ . \tag{8.8}$$

So, we have to find a matrix $U \in G_c$ (G_n) such that eq. (6.3) is satisfied, i.e.

$$\begin{cases} A^+A + \epsilon C^+C = \mathbb{1}_n, \\ \epsilon B^+B + D^+D = \mathbb{1}_m, \\ \epsilon B^+A + D^+C = \mathbf{0}. \end{cases} \tag{8.9}$$

It can be shown that the equations (8.9) also imply the equivalent relations

$$\begin{cases} AA^+ + \epsilon BB^+ = \mathbb{1}_n, \\ \epsilon CC^+ + DD^+ = \mathbb{1}_m, \\ \epsilon AC^+ + BD^+ = \mathbf{0}. \end{cases} \tag{8.10}$$

Now we find the matrix U with the property $Z'(Z_1) = \mathbf{0}$, i.e.

$$AZ_1 + B = \mathbf{0}.$$

With the first eq. (8.10), it is obtained

$$A^+A = (\mathbb{1} + \epsilon Z_1 Z_1^+)^{-1} \ .$$

A polar decomposition of the matrix A is used

$$A = XH \ ,$$

where H is hermitian and positive definite, while X is unitary.

The matrix U has as subblocks the submatrices

$$\begin{cases} A = X(\mathbb{1} + \epsilon Z_1 Z_1^+)^{-1/2}, \\ B = -X(\mathbb{1} + \epsilon Z_1 Z_1^+)^{-1/2} Z_1, \\ C = \epsilon X'(\mathbb{1} + \epsilon Z_1^+ Z_1)^{-1/2} Z_1^+, \\ D = X'(\mathbb{1} + \epsilon Z_1^+ Z_1)^{-1/2}, \end{cases} \tag{8.11}$$

where X (X') is a unitary $n \times n$ (resp. $m \times m$) matrix, irrelevant for the calculation of the distance.

Note that the representation (8.11) of the matrix U coincides with the one given by eq. (6.7.b), with Z replaced by $-Z$. This is a consequence of the fact that the representation (6.7.b) expresses the transformation $0 \to Z$. The inverse transformation is given by equation (8.13) below.

The condition $\operatorname{Det} U = 1$ is verified with the Schur formulas (3.27) and the representation (8.3) is obtained by $Z' = Z'(Z_2)$.

Note also that the linear fractional transformation can also be written down as

$$Z' = (AZ + B)(CZ + D)^{-1} = (ZB^+ - \epsilon A^+)^{-1}(C^+ - \epsilon ZD^+), \tag{8.12}$$

$$Z = (A - Z'C)^{-1}(Z'D - B) = (C^+ + \epsilon A^+ Z')(B^+ Z' + \epsilon D^+)^{-1}. \tag{8.13}$$

With these relations it is easy to show that the homographic transformations (8.8) leave invariant the equation (6.13) of geodesics. To verify the last assertion, the following relations are also needed

$$\begin{aligned} &\mathbb{1} + \epsilon Z^+ Z = (Z'^+ B + \epsilon D)^{-1}(\mathbb{1} + \epsilon Z'^+ Z')(B^+ Z' + \epsilon D^+)^{-1}, \\ &dZ = -(\epsilon Z'C - \epsilon A)^{-1} dZ'(B^+ Z' + \epsilon D^+)^{-1}, \\ &d^2 Z = (\epsilon Z'C - \epsilon A)^{-1}[2\epsilon dZ'C(\epsilon Z'C - \epsilon A)^{-1} dZ' - d^2 Z'](B^+ Z' + \epsilon D^+)^{-1}. \end{aligned} \tag{8.14}$$

b) Now the distance between the points $Z_1 = 0, \ Z_2 = Z$ is calculated, where in the compact case $Z \in \mathcal{V}_0$.

The starting point is the formula (6.10). Using eqs. (8.14) and also the equation

$$\mathbb{1} + \epsilon ZZ^{+} = (Z'C - A)^{-1}(\mathbb{1} + \epsilon Z'Z'^{+})(C^{+}Z'^{+} - A^{+})^{-1},$$

it is easy to verify that the infinitesimal element (6.10) is invariant under the homographic transformations (8.8).

We fix $k = 1$ in eq. (6.10). The expression (6.8) of the geodesics with $B \to Bt$ implies

$$(\mathbb{1} + \epsilon ZZ^{+})^{-1} = \mathrm{co}^2(t\sqrt{BB^{+}}); \ (\mathbb{1} + \epsilon Z^{+}Z)^{-1} = \mathrm{co}^2(t\sqrt{B^{+}B}), \tag{8.15}$$

and the equation (6.10) can be written down as

$$ds^2 = \mathrm{Tr}\,(B^{+}B)dt^2 = \sum |B_{ij}|^2 dt^2, \tag{8.16}$$

which shows that indeed $B_{ij} \in \mathfrak{m}$ are normal coordinates.

So, we find eq. (8.1) and the first eq. (8.4):

$$d^2(0, Bt) = \mathrm{Tr}\,(B^{+}B)t = \mathrm{Tr}\,[\mathrm{arcta}(ZZ^{+})^{1/2}]^2,$$

which implies also the other two relations in eqs. (8.4) and (8.5).

c) Finally, the representation (8.7) for the matrix V is obtained after a tedious matrix calculation. We only point out the main steps.

If in eq. (8.3) it is substituted

$$Z_2 - Z_1 = (\mathbb{1} + \epsilon Z_1 Z_1^{+})Z_2 - Z_1(\mathbb{1} + \epsilon Z_1^{+} Z_2),$$

then it is obtained

$$\begin{aligned} Z &= (\mathbb{1} + \epsilon Z_1 Z_1^{+})^{1/2}(\mathbb{1} + \epsilon Z_2 Z_1^{+})^{-1} Z_2 (\mathbb{1} + \epsilon Z_1^{+} Z_1)^{1/2} - Z_1, \\ (\mathbb{1} + \epsilon Z_1 Z_1^{+})^{-1/2} Z &= (\mathbb{1} + \epsilon Z_2 Z_1^{+})^{-1} Z_2 (\mathbb{1} + \epsilon Z_1^{+} Z_2)^{1/2} - (\mathbb{1} + \epsilon Z_1 Z_1^{+})^{-1/2} Z_1. \end{aligned} \tag{8.17}$$

The representation (8.17) is introduced in the auxiliary expression

$$E = (\mathbb{1} + \epsilon Z_1 Z_1^{+})^{-1/2} ZZ^{+} (\mathbb{1} + \epsilon Z_1 Z_1^{+})^{-1/2}.$$

It is obtained

$$E = F + (\mathbb{1} + \epsilon Z_1 Z_1^{+})^{-1} Z_1 Z_1^{+},$$

where, finally, F is brought to the form

$$F = (\mathbb{1} + \epsilon Z_2 Z_1^{+})^{-1}(\epsilon \mathbb{1} + Z_2 Z_2^{+})(\mathbb{1} + \epsilon Z_1 Z_2^{+})^{-1} - \epsilon \mathbb{1}.$$

So, E can be written down as

$$E = (\mathbb{1} + \epsilon Z_2 Z_1^{+})^{-1}(\epsilon \mathbb{1} + Z_2 Z_2^{+})(\mathbb{1} + \epsilon Z_1 Z_2^{+})^{-1} - \epsilon(\mathbb{1} + \epsilon Z_1 Z_1^{+})^{-1},$$

which implies

$$\mathbb{1} + \epsilon ZZ^{+} = (\mathbb{1} + \epsilon Z_1 Z_1^{+})^{1/2}(\mathbb{1} + \epsilon Z_2 Z_1^{+})^{-1}(\mathbb{1} + \epsilon Z_2 Z_2^{+})(\mathbb{1} + \epsilon Z_1 Z_2^{+})^{-1}(\mathbb{1} + \epsilon Z_1 Z_1^{+})^{1/2},$$

i.e. the representation (8.7). ∎

Now we comment the expressions given by Proposition 3:

Comment 3 *The formulas giving the distance on the complex Grassmann manifold (his noncompact dual) generalize the corresponding ones for the Riemann sphere (resp., the disk $|z| < 1$).*

Formula (8.2) was used by Rosenfel'd,[11] θ_i being the stationary angles here determined in Lemma 4, while the first expression (8.5) appears in Ch. II §6 p. 69 of [56] or in [57] in the case of symplectic group. In the last case the factor $k = 4$ in formula (6.10) eliminates the factor 1/2 in the first eq. (8.5).

Now we particularize the formulas in Proposition 3 to the case of the Riemann sphere.

If $Z_1,\ Z_2$ belong to the same chart, then

$$Z = \frac{Z_1 - Z_2}{1 + \bar{Z}_1 Z_2},\ V = \frac{|1 + Z_1\bar{Z}_2|^2}{(1+|Z_1|^2)(1+|Z_2|^2)}\ , \tag{8.18}$$

and formulas (8.4) become

$$\begin{aligned} d(Z_1, Z_2) &= \arctan\frac{|Z_1 - Z_2|}{|1 + \bar{Z}_1 Z_2|} = \arccos\frac{|1 + Z_1\bar{Z}_2|}{(1+|Z_1|^2)^{1/2}(1+|Z_2|^2)^{1/2}} \\ &= \arcsin\frac{|Z_1 - Z_2|}{(1+|Z_1|^2)^{1/2}(1+|Z_2|^2)^{1/2}}\ . \end{aligned} \tag{8.19}$$

Note that

$$d(Z_1, Z_2) = d_c(Z_1, Z_2) = \frac{1}{2}\theta(Z_1, Z_2) = \arcsin\frac{\Delta(Z_1, Z_2)}{2}\ , \tag{8.20}$$

where $\theta(Z_1, Z_2)$ is the length of the arc of the great circle (the geodesic) joining the points $Z_1,\ Z_2 \in \mathbb{CP}^1$, while $\Delta(Z_1, Z_2)$ is the chord length.

Eq. (8.5) in the case of Riemann sphere (respectively the disk $|Z| < 1$) ($\epsilon = 1,\ \eta = i$ ($\epsilon = -1,\ \eta = 1$)) reads

$$d(Z_1, Z_2) = \frac{1}{2\eta}\log\frac{1 + \eta\lambda}{1 - \eta\lambda}\ ,\ \lambda = \frac{|Z_1 - Z_2|}{|1 + \epsilon Z_1\bar{Z}_2|}\ . \tag{8.21}$$

The expression under the logarithm represents the cross-ratio $\{\mathrm{Z_1Z_2, MN}\}$. In the compact (noncompact) case M and N represents the points where the line $\mathrm{Z_1, Z_2}$ meets the absolute (Laguerre-Cayley-Klein) (resp. the frontier $|Z| = 1$).[58]

In the case of $\mathbb{CP}^n$, the second relation in eq. (8.4) is the elliptic hermitian Cayley distance (2.17) expressed in non-homogeneous coordinates

$$d(Z, Z') = d_c(Z, Z') = \arccos\frac{|1 + \sum Z_i\bar{Z'}_i|}{(1+\sum|Z_i|^2)^{1/2}(1+\sum|Z'_i|^2)^{1/2}}\ , \tag{8.22}$$

while in the case of the hermitian hyperbolic space $SU(n,1)/S(U(n)\times U(1))$ the corresponding distance is the hyperbolic hermitian distance (2.22)

$$d(Z, Z') = \operatorname{arccosh}\frac{|1 - \sum Z_i\bar{Z'}_i|}{(1-\sum|Z_i|^2)^{1/2}(1-\sum|Z'_i|^2)^{1/2}}\ . \tag{8.23}$$

■

Finally, let us denote by δ_n the distance (8.2) on $G_n(\mathbb{C}^{m+n})$ and his noncompact dual (1.2) and by s_n the distance (5.10) of the images of the points through the Plücker embedding (resp. (5.15)), where both points belong to $\mathcal{V}_0$ for X_c. We present an elementary inequality which has a simple geometrical meaning in the following

Comment 4 *Let δ_n and s_n be defined by*

$$\delta_n^2 = \theta_1^2 + \ldots + \theta_n^2 \ , \tag{8.24}$$

$$\text{co}\, s_n = \text{co}\, \theta_1 \cdots \text{co}\, \theta_n \ , \tag{8.25}$$

where for X_c $s_n, \ \theta_i \in [0, \pi/2]$. Then

$$\delta_n \geq s_n \ . \tag{8.26}$$

Proof: We indicate the proof on $X_c = G_n(\mathbb{C}^{m+n})$. Geometrically, eq. (8.26) expresses the fact that the distance on the manifold (here the Grassmannian) is greater than the distance between the images of the points through the embedding (here the Plückerian one). Infinitesimally

$$d\,\delta_n = d\,s_n, \tag{8.27}$$

as can be verified with eq. (8.25) at small stationary angles.

We also indicate an elementary algebraic proof of (8.26). Firstly, eq. (8.26) is proved for $n = 2$, i.e. if $\delta^2 = x^2 + y^2$ and $\cos s = \cos x \cos y$, then $\delta \geq s$ for $x, y \in [0, \pi/2]$. Indeed, let $x = \delta \cos\theta, \ y = \delta \sin\theta$ and let us consider the function $F(\theta) \equiv \cos s$. Then the equation $dF/d\theta = 0$ has as unique solution in $[0, \pi/2]$ the angle $\theta = \pi/4$, which is a maximum. But $F(0) = F(\pi/2) = \cos\delta$ and the inequality $\cos\delta \leq \cos s$ follows. Further the mathematical induction on n in eqs. (8.24)-(8.26) is applied. ■

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