Parameter-dependent solutions of the classical Yang-Baxter equation on sl(n,C).

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Abstract

For any integers n and m ($m \geq 4$) such that $n+m$ is odd we exhibit triangular solutions of the classical Yang-Baxter equation on $sl((n+1)(m+2), \mathbb{C})$ parametrized by points of a quotient of complex projective space $\mathbb{P}^{\lfloor \frac{n}{2} \rfloor}(\mathbb{C})$ by the action of the symmetric group $Sym(\lfloor \frac{n+1}{2} \rfloor)$ and we prove that no two of these solutions are isomorphic.

1 Introduction

The motivation for this work is to exhibit solutions of the classical Yang-Baxter equations depending on a large number of parameters, Such solutions lead, by a construction indicated by Drinfeld [1], to quantum groups. We hope that these parameter-dependent quantum groups may have interesting geometrical applications [3].

2 The classical Yang-Baxter equation.

Let $\mathcal{G}$ be a finite-dimensional Lie algebra over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$); an element $R \in \wedge^2 \mathcal{G}$ is said to be a solution of the classical Yang-Baxter equation iff

$$[R, R] = 0$$

where $[\ ,\ ] : \wedge^2 \mathcal{G} \otimes \wedge^2 \mathcal{G} \to \wedge^3 \mathcal{G}$ is the Schouten bracket, defined on bivectors by

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$$[X_1 \wedge X_2, Y_1 \wedge Y_2] := [X_1, Y_1] \wedge X_2 \wedge Y_2 - [X_1, Y_2] \wedge X_2 \wedge Y_1 - [X_2, Y_1] \wedge X_1 \wedge Y_2 + [X_2, Y_2] \wedge X_1 \wedge Y_1$$

Lemma 2.1. (see [2] corollary 2.2.4 and proposition 2.2.6).*There is a bijective correspondence between the solutions R of the classical Yang-Baxter equation on $\mathcal{G}$ and the symplectic subalgebras $(\mathcal{H}, \omega)$ of $\mathcal{G}$; i.e. ω belongs to $\wedge^2\mathcal{H}^\star$, the radical of ω is zero and ω is a 2-cocycle of $\mathcal{H}$ with values in $\mathbb{K}$ for the trivial representation of $\mathcal{H}$ on $\mathbb{K}$.*

Let us recall the argument of this well known lemma. Denote by $\underline{R} : \mathcal{G}^\star \to \mathcal{G}$ the map defined by $\underline{R}(\alpha) := i(\alpha)R, \forall \alpha \in \mathcal{G}^\star$. The condition $[R, R] = 0$ implies that $\mathcal{H} := Im\underline{R}$ is a subalgebra of $\mathcal{G}$, and that R is an element of $\wedge^2\mathcal{H}$. Furthermore $\underline{R}$ induces an isomorphism (still denoted $\underline{R}$) $\mathcal{H}^\star \to \mathcal{H}$. The element ω of $\wedge^2\mathcal{H}^\star$ defined by

$$\omega(X, Y) = \langle R | \underline{R}^{-1} X \wedge \underline{R}^{-1} Y \rangle$$

is of maximal rank. Finally

$$\oint_{\alpha,\beta,\gamma} \omega([\underline{R}\alpha, \underline{R}\beta], \underline{R}\gamma) = \frac{1}{8} \langle [R, R] | \alpha \wedge \beta \wedge \gamma \rangle \quad \forall \alpha, \beta, \gamma \in \mathcal{H}^\star$$

and hence ω is a 2-cocycle.

3 A family of symplectic nilpotent Lie algebras.

Let $\mathcal{G}$ be the complex vector space generated by the elements $x, y, e_{i,j}; 0 \leq i \leq n, 0 \leq j \leq m$ .We assume that $n+m$ is odd and that $m \geq 4$. Define non vanishing brackets as:

$$[x, e_{i,j}] := e_{i,j+1}$$
$$[y, e_{i,j}] := a_i \; e_{i,j+3}$$

(with the convention $e_{i,m+k} := 0, k > 0$). We assume that at least one of the $(n+1)$ complex numbers a_i is different from zero. Define the element ω of $\wedge^2\mathcal{G}$ by

$$\omega(x, y) := 1 \;\text{ and }\; \omega(e_{i,j}, e_{i',j'}) = (-1)^{i+j} \; \delta_{i',n-i} \; \delta_{j',m-j}$$

The 2-form ω is antisymmetric as $n+m$ is odd; it is a 2-cocycle provided $a_i = a_{n-i}$. The p-th element of the central descending series of $\mathcal{G}$, $\mathcal{G}^{(p)}(1 \leq p \leq m)$ is $\rangle e_{i,j}; 0 \leq i \leq n, p \leq j \leq m\langle$, the centralizer of $\mathcal{G}^{(m-3)}$ is $\rangle e_{i,j}; 0 \leq i \leq n, 0 \leq j \leq m\langle$ and the centralizer of $\mathcal{G}^{(m-2)}$ is $\rangle y, e_{i,j}; 0 \leq i \leq n, 0 \leq j \leq m\langle$. The linear map $ad(y)$ [resp. $ad(x)^3$] induces a map $A(y)$ [resp. $A(x)$] from $\mathcal{G}^{(m-3)}/\mathcal{G}^{(m-2)} \to \mathcal{G}^{(m)}$ which is invariantly defined up to a non zero complex multiple. Denote by $\{\lambda_0, \ldots, \lambda_q\}$ the solutions of the equation $rank(\lambda A(x) - A(y)) < n+1$ and for each of these solutions λ_i define m_i its multiplicity $m_i := n + 1 - rank(\lambda_i A(x) - A(y))$. All λ_i's have even multipliciy if n is odd and if n is even there is exactly one λ_i which has odd multiplicity ; we shall denote it λ_q. The point $[a_0, \ldots, a_n]$ of $\mathbb{P}^{\lfloor \frac{n}{2} \rfloor}(\mathbb{C})$ and the point $[\lambda_0, \ldots, \lambda_0, \ldots, \lambda_q, \ldots, \lambda_q]$ (λ_i appear with multiplicity $\lfloor \frac{m_i+1}{2} \rfloor$) belong to the same orbit of $Sym(\lfloor \frac{n+1}{2} \rfloor)$. The action of an element of $Sym(\lfloor \frac{n+1}{2} \rfloor)$ on a point x

of $\mathbb{P}^{\lfloor \frac{n}{2} \rfloor}(\mathbb{C})$ with homogeneous coordinates x_i $(o \leq i \leq \lfloor \frac{n}{2} \rfloor)$ is given by the standard action on the $\lfloor \frac{n+1}{2} \rfloor$ first coordinates. Hence

Proposition 3.1. *The isomorphism class of the nilpotent algebra* $\mathcal{G}_{(a)}, (a := (a_0, \ldots, a_n) \in \mathbb{C}^{n+1} \setminus \{0\})$ *is determined by a point of complex projective space* $\mathbb{P}^{\lfloor \frac{n}{2} \rfloor}(\mathbb{C})$*. Two points give rise to isomorphic algebras if and only if they belong to the same orbit of* $Sym(\lfloor \frac{n+1}{2} \rfloor)$ *on* $\mathbb{P}^{\lfloor \frac{n}{2} \rfloor}(\mathbb{C})$.

4 A family of solutions of the classical Yang-Baxter equation for sl((n+1)(m+2),C).

Let $\{f_{i,j}; 0 \leq i \leq n, 0 \leq j \leq m+1\}$ be a basis of $\mathbb{C}^{(n+1)(m+2)}$ and let ρ be the linear, faithful representation of $\mathcal{G}_{(a)}$ on $\mathbb{C}^{(n+1)(m+2)}$ given by

$$\begin{aligned} \rho(x) f_{i,j} &:= f_{i,j+1} \\ \rho(y) f_{i,j} &:= a_i \; f_{i,j+3} \\ \rho(e_{i',j'}) f_{i,j} &:= \delta_{i',i} \; \delta_{0,j} \; f_{i,j+j'+1} \end{aligned}$$

(with the convention $f_{i,m+1+k} := 0, k > 0$). The algebra $\rho(\mathcal{G}_{(a)})$ is a subalgebra of $sl((n+1)(m+2), \mathbb{C})$ isomorphic to $\mathcal{G}_{(a)}$. It is symplectic with respect to the transported symplectic form ω of $\mathcal{G}_{(a)}$. The corresponding solutions of the classical Yang-Baxter equation are given by

$$R_a = \rho(x) \wedge \rho(y) + \sum_{i,j} (-1)^{i+j} \rho(e_{i,j}) \wedge \rho(e_{n-i,m-j}) \tag{1}$$

Observe that a bialgebra structure on $sl(N, \mathbb{C})$ determines uniquely a element $r \in \Lambda^2(sl(N, \mathbb{C}))$. If r is a solution of the classical Yang-Baxter equation (cf
2) it is well known that there exists a unique minimal subalgebra $\mathcal{H}$ of $sl(N, \mathbb{C})$ such that $r \in \Lambda^2(\mathcal{H})$; we shall call $\mathcal{H}$ the support of r.

Proposition 4.1. *Let* R_a *the solution of the classical Yang Baxter equation given by formula (1)*

1. *Let* $K := \lfloor \frac{n+2}{2} \rfloor$*. The support of* R_a *are isomorphic if and only if the projection of* $(a_0, \ldots, a_{K-1}) \in \mathbb{C}^K \setminus \{0\}$ *on* $\mathbb{P}^{K-1}(\mathbb{C})/Sym(K)$ *are identical. In particular if the projections are distinct, the corresponding bialgebras are non isomorphic.*

2. *Let* Ω *be the open dense subset of* $\mathbb{C}^K$ *defined as follows. If* $\phi : \mathbb{C}^K \to \mathbb{C}^{K(K-1)} : (a_0, \ldots, a_{K-1}) \to ((1 - \frac{a_i}{a_j}) \mid i \neq j)$ *then* $\Omega = \phi^{-1}(\mathbb{C}^{K(K-1)} \setminus (\cup_{i,j} \{(z_0, \ldots z_{K-1}) \mid z_i = z_j\} \cup_i \{(z_0, \ldots z_{K-1}) \mid z_i = 0\})$*. Let* $a, a' \in \Omega$ *then the bialgebra structures definied by* R_a *an d* $R_{a'}$ *are isomorphic if and only if the projection of* $(a_0, \ldots, a_{K-1}) \in \mathbb{C}^K \setminus \{0\}$ *on* $(\mathbb{C}^K \setminus \{0\})/Sym(K)$ *are identical.*

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References

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