A quick and simple proof of Sherman's theorem on order in C^*-algebras

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To every proper convex cone in any real linear space there is (see [1], [2]) a corresponding order relation. A cone is said to be a lattice if any two elements in it have a supremum for this ordering.

Theorem (S. Sherman). Let $\mathfrak{A}$ be a C^*-algebra with or without a unit and let $\mathfrak{A}_+$ be the real cone of all positive elements in $\mathfrak{A}$. The cone $\mathfrak{A}_+$ is a lattice if and only if $\mathfrak{A}$ is Abelian.

Proof. If $\mathfrak{A}$ is Abelian then (by the Gelfand-Naimark theorem) it is easily seen that $\mathfrak{A}_+$ is a lattice.
To prove the converse, suppose that $\mathfrak{A}$ is not Abelian. Then there exists (see [3], [5], [6], [8]) an irreducible $*$-representation $x \mapsto A_x$ of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ of dimension ≥ 2. Choose in $\mathcal{H}$ two elements ξ_1, ξ_2 such that $\|\xi_1\| = \|\xi_2\| = 1$ and $\xi_1 \perp \xi_2$, and write

$$\eta_1 = \frac{\xi_1 + \xi_2}{\sqrt{2}} \quad \text{and} \quad \eta_2 = \frac{\xi_1 - \xi_2}{\sqrt{2}}$$

Denote the corresponding positive linear forms (pure states) on $\mathfrak{A}$ by φ_i, ψ_i :

$$\varphi_i(x) = (A_x\xi_i, \xi_i) \quad \text{and} \quad \psi_i(x) = (A_x\eta_i, \eta_i) \qquad x \in \mathfrak{A}, i = 1, 2$$

Let $\mathfrak{A}_H$ be the real linear space of all Hermitian elements in $\mathfrak{A}$. Now $\mathfrak{A}_H = \mathfrak{A}_+ - \mathfrak{A}_+$; in a natural way $\mathfrak{A}_H$ is an ordered topological vector space with positive cone $\mathfrak{A}_+$ (see [3], [6], [8]). The cone of all positive linear forms on $\mathfrak{A}_H$ will be denoted by P. Restriction of a positive linear form on $\mathfrak{A}$ to $\mathfrak{A}_H$ gives an element

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in P and in this canonical way a bijective correspondence emerges between P and the cone of all positive linear forms on $\mathfrak{A}$. The representation $x \mapsto A_x$ being irreducible, it follows (see [3] p. 37/38, [5] p. 259/270) that the restrictions of $\varphi_1, \varphi_2, \psi_1, \psi_2$ to $\mathfrak{A}_H$ generate mutually different extreme rays in the cone P. Moreover, it is easily verified that

$$\varphi_1 + \varphi_2 = \psi_1 + \psi_2$$

By Riesz's decomposition lemma (see [2]) this is possible only if P is *not* a lattice. Consequently $\mathfrak{A}_H$, and therefore $\mathfrak{A}_+$, can not be a lattice (see [1], [2]).
So " $\mathfrak{A}$ not Abelian" implies " $\mathfrak{A}_+$ not a lattice"; this completes the proof of the theorem.

Remark. Another (even quicker) proof, in the case where $\mathfrak{A}$ is a von Neumann algebra, can be found in [9]. Sherman's original proof was published in [7]; see also [4].

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