Projective embedding of projective spaces

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Abstract

In this paper, embeddings $\phi : M \rightarrow P$ from a linear space $(M, \mathfrak{M})$ in a projective space $(P, \mathfrak{L})$ are studied. We give examples for $\dim M > \dim P$ and show under which conditions equality holds.

More precisely, we introduce properties ($G$) (for a line $L \in \mathfrak{L}$ and for a plane $E \subset M$ it holds that $|L \cap \phi(M)| \neq 1$) and ($E$) ($\phi(E) = \overline{\phi(E)} \cap \phi(M)$, whereby $\overline{\phi(E)}$ denotes the by $\phi(E)$ generated subspace of $P$). If ($G$) and ($E$) are satisfied then $\dim M = \dim P$. Moreover we give examples of embeddings of $m$-dimensional projective spaces in $n$-dimensional projective spaces with $m > n$ that map any $n + 1$ independent points onto $n + 1$ independent points.

This implies that for a proper subspace $T$ of $M$ it holds $\phi(T) = \overline{\phi(T)} \cap \phi(M)$ if and only if $\dim T \leq n - 1$, in particular ($E$) holds for $n \geq 3$. (cf. 4.1)

1 Introduction

An embedding $\phi : M \rightarrow P$ of a linear space $(M, \mathfrak{M})$ in a linear space $(P, \mathfrak{L})$ is an injective mapping that maps collinear points onto collinear points and noncollinear points onto noncollinear points. There are lots of papers concerning the embedding of linear spaces in projective spaces (cf. [3, Chap.6]). Important results are that every locally projective space $(M, \mathfrak{M})$ of $\dim M \geq 4$ (cf. [7, 10, 17, 19]) and every locally projective space $(M, \mathfrak{M})$ of $\dim M = 3$ satisfying the Bundle Theorem (cf. [8, 15]) is embeddable in a projective space $(P, \mathfrak{L})$. Due to the construction of the projective space the mentioned Embedding Theorems have the useful property that for every subspace $T$ of $(M, \mathfrak{M})$ there exists exactly one subspace $U$ of $(P, \mathfrak{L})$ with $\phi(T) = U \cap \phi(M)$. This property is equivalent to the two properties ($G$), ($E$);
A linear space satisfying \((G)\), \((E)\) is called \textit{locally complete} (cf. 2.4). For locally complete embeddings the dimension of \(M\) and \(P\) coincide (cf. 2.5). There are also projective embeddings of linear spaces which are not locally complete, but have the property that the dimension and order of \(M\) and \(P\) are equal (cf. [13, 14]). But there exist also embeddings which do not preserve the dimension. If \(\phi(M)\) generates \(P\), one obtains \(\dim M \geq \dim P\) (cf. 2.3), hence we have to consider only the case \(\dim M > \dim P\). For example one can embed every linear space in a projective plane \(E\) by a free construction of \(E\). (Then of course, \(E\) is not a Desarguesian plane.) Kalhoff constructed in [9] the embedding of any finite partial planes in a translation plane, and hence in a projective plane of Lenz class \(V\).

In this paper we are interested in embeddings in Desarguesian projective spaces and planes. There are some papers which give a characterisation of embeddings of projective spaces in Desarguesian projective spaces. For a field \(K\) and the \((m+1)\)-dimensional vector space \((K^{m+1}, K)\) over \(K\), let \(PG(m, K)\) denote the \(m\)-dimensional projective space over \(K\) with the 1-dimensional vector subspaces as points and the 2-dimensional vector subspaces as lines. M. Limbos [16] has shown for finite projective spaces that every embedding of \(PG(m, K)\) in \(PG(n, L)\) with \(m > n\) is a product of the trivial embedding of \(PG(m, K)\) in \(PG(m, L)\) for a field extension \(L\) of \(K\), and a projection of \(PG(m, L)\) in the subspace \(PG(n, L)\). In [16] a geometric construction of embeddings is given and the proof that every embedding can be obtained by this construction. H. Havlicek [6] and C.A. Faure, A. Froelicher [4, 5] give a similar characterisation for the infinite case, but without a construction. For an arbitrary field \(K\) an example of an embedding of \(PG(m, K)\) in \(PG(m - 1, L)\) for a field extension \(L\) of \(K\) is given by A. Brezuleanu, D.-C. Rădulescu [1, (5.8)]. For a finite field \(K\), J. Brown gives in [2] an analytic example of an embedding \(\phi : PG(m, K) \rightarrow PG(2, L)\) for a field extension \(L\) of \(K\). This example does not satisfy \((E)\).

In this paper we answer the question, if there exists an embedding \(\phi : P \rightarrow P'\) of a Pappian projective space \((P, \mathcal{L})\) in a Pappian projective space \((P', \mathcal{L}')\) which does not preserve dimension, but satisfy property \((E)\). We show the corresponding statements for higher dimensions. We show that for \(\dim P' = n\) there are embeddings which map any \(n + 1\) independent points of \(P\) onto \(n + 1\) independent points of \(P'\). It follows that the image of an \((n - 1)\)-dimensional subspace \(T\) of \(P\) generates an \((n - 1)\)-dimensional subspace \(\overline{\phi(T)}\) of \(P'\) with \(\phi(T) = \overline{\phi(T)} \cap \phi(P)\). We remark that there exist also embeddings of projective spaces in projective planes satisfying property \((G)\).

2 Locally Complete Embeddings

A \textit{linear space} \((P, \mathcal{L}, I)\) will be defined as a set \(P\) of elements, called \textit{points}, a distinct set \(\mathcal{L}\) of elements, called \textit{lines}, and an incidence relation \(I\) such that any two distinct points are incident with exactly one line and every line is incident with at least two points. Usually one identifies a line \(L \in \mathcal{L}\) with the set of points incident with \(L\), hence the lines of \((P, \mathcal{L}, I) = (P, \mathcal{L})\) are subsets of \(P\).

A \textit{subspace} is a subset \(U \subset P\) such that for all distinct points \(x, y \in U\) the unique line incident with \(x, y\) is contained in \(U\). Let \(\mathfrak{U}\) denote the set of all subspaces. For
every subset $X \subset P$ we define the following closure operator:

$$\overline{\cdot} : \mathcal{P}(P) \to \mathfrak{U} : X \mapsto \overline{X} := \bigcap_{U \in \mathfrak{U}} U$$

(1)

The closure of $X$ is a subspace containing $X$. For $U \in \mathfrak{U}$ we call $\dim U := \inf\{|X| : X \subset U \text{ and } \overline{X} = U\}$ the dimension of $U$. A subspace of dimension two is a plane. A subset $X \subset P$ is independent if $x \notin \overline{X} \setminus \{x\}$ for every $x \in X$, and is a basis of a subspace $U$ if $X$ is independent and $\overline{X} = U$.

For two linear spaces $(M, \mathfrak{M})$ and $(P, \mathfrak{L})$, an injective mapping

$$\phi : M \to P, \ x \mapsto \phi(x)$$

(2)

is called an embedding, if $\phi$ maps collinear points onto collinear points and non-collinear points onto noncollinear points, i.e., $\{\phi(G) : G \in \mathfrak{M}\} = \{L \cap \phi(M) : L \in \mathfrak{L}\}$ and $|L \cap \phi(M)| \geq 2$. Hence $\left(\phi(M), \{\phi(G) : G \in \mathfrak{M}\}\right)$ is the restriction of $(P, \mathfrak{L})$ to $\phi(M)$. Clearly:

**Lemma 2.1** If $\phi$ is an embedding of $(M, \mathfrak{M})$ in $(P, \mathfrak{L})$, and $\psi$ is an embedding of $(P, \mathfrak{L})$ in $(P', \mathfrak{L}')$, then $\psi \circ \phi$ is an embedding of $(M, \mathfrak{M})$ in $(P', \mathfrak{L}')$.

Let $Y \mapsto \overline{Y}$ denote the closure of $(P, \mathfrak{L})$ and $X \mapsto \langle X \rangle$ the closure of $(M, \mathfrak{M})$. By [12, (1.1)]:

**Lemma 2.2** If $\phi$ is an embedding of $(M, \mathfrak{M})$ in $(P, \mathfrak{L})$, and $U$ a subspace of $(P, \mathfrak{L})$ and $X \subset M$, then:

1. $\phi^{-1}(U \cap \phi(M))$ is a subspace of $M$.
2. $\phi(\langle X \rangle) \subset \overline{\phi(X)}$ and $\phi(\langle X \rangle) = \overline{\phi(X)}$.
3. If $\phi(X)$ is independent in $P$, then $X$ is independent in $M$.

**Lemma 2.3** If $\phi : M \to P$ is an embedding of a linear space $(M, \mathfrak{M})$ in a linear space $(P, \mathfrak{L})$ satisfying $\overline{\phi(M)} = P$, then $\dim M \geq \dim P$.

**Proof.** Let $X \subset M$ be a subset generating $M$, i.e. $\langle X \rangle = M$. Then $P = \overline{\phi(M)} = \overline{\phi(\langle X \rangle)} = \overline{\phi(X)}$ by 2.2. Therefore $\phi(X)$ is a generating set of $P$ with $|X| = |\phi(X)|$, hence $\dim P \leq \dim M$.

We call an embedding $\phi$ of $(M, \mathfrak{M})$ in $(P, \mathfrak{L})$ locally complete, if for every nonempty subspace $T$ of $M$, there is exactly one subspace $U$ of $P$ with $\phi(T) = U \cap \phi(M)$.

By [12, (1.5)] we have:

**Lemma 2.4** For an embedding $\phi$ of $(M, \mathfrak{M})$ in $(P, \mathfrak{L})$ the following statements are equivalent:

1. $\phi$ is locally complete.
2. For every subspace \( T \) of \((M, \mathcal{M})\) and for every subspace \( U \) of \((P, \mathcal{L})\) with \( \phi(M) \cap U \neq \emptyset \) we have

\[
U = \overline{U \cap \phi(M)} \quad \text{and} \quad \phi(T) = \overline{\phi(T) \cap \phi(M)}
\]

3. The following properties \((G), (E)\) are satisfied.

\[\text{(G) For every line } L \in \mathcal{L}, \quad |L \cap \phi(M)| \neq 1\]

\[\text{(E) For every plane } E \in M, \quad \phi(E) = \overline{\phi(E) \cap \phi(M)}\]

A linear space \((P, \mathcal{L})\) satisfies the exchange condition if

for \( S \subseteq P \) and \( x, y \in P \) with \( x \in \overline{S \cup \{y\}} \setminus S \) it follows that \( y \in \overline{S \cup \{x\}} \).

\[\text{(3)}\]

**Lemma 2.5** If \( \phi \) is a locally complete embedding of a linear space \((M, \mathcal{M})\) in a linear space \((P, \mathcal{L})\) satisfying the exchange condition, then \( \dim M = \dim P \).

**Proof.** Since \( \phi \) is locally complete, \( P = \overline{P \cap \phi(M)} = \overline{\phi(M)} \), hence, by Lemma 2.3, \( \dim P \leq \dim M \). Now let \( x \in \phi(M) \). Since \((P, \mathcal{L})\) is an exchange space, there is a basis \( C \) of \( P \) containing \( x \) (cf. \([11, \S 8]\)). By Lemma 2.4, \((G)\) holds. Moreover for every \( y \in C \setminus \{x\} \), there exists a \( y' \in \overline{\{x, y \cap \phi(M)\}} \setminus \{x\} \). Hence we obtain a basis \( C' \subseteq \phi(M) \) of \( P \) with \( |C| = |C'| \). Let \( T := \overline{\phi^{-1}(C')} \) denote the subspace of \( M \) generated by \( \phi^{-1}(C') \), i.e. \( C' \subseteq \phi(T) \) and \( P = \overline{C'} = \phi(T) \). We get \( \phi(T) = \overline{\phi(T) \cap \phi(M)} = P \cap \phi(M) = \phi(M) \), hence \( M = T \) is generated by \( \phi^{-1}(C') \) and \( \dim M \leq \dim P \).

The Lemma 2.5 applies in particular, if \((P, \mathcal{L})\) is a projective space.

**Theorem 2.6** Let \((P, \mathcal{L}), (M, \mathcal{M})\) be linear spaces satisfying the exchange condition and \( \dim M > \dim P \). If \( \phi : M \to P \) is an embedding satisfying \((G)\), then there exist subspaces \( M' \subset M, P' \subset P \) with \( \dim M' > \dim P' = 2 \) such that \( \phi|_{M'} : M' \to P' \) is an embedding satisfying \((G)\).

**Proof.** By Lemma 2.5 \((E)\) is not satisfied, since \( \dim M > \dim P \). Hence there exists a plane \( E \subset M \) with \( \phi(E) \neq \overline{(\phi(E) \cap \phi(M))} \). Therefore \( M' := \phi^{-1}(\phi(E) \cap \phi(M)) \) is a subspace with \( E \subset M' \) and \( E \neq M' \), i.e. \( \dim M' > 2 \). Since \( E \) is a plane, also \( P' := \overline{\phi(E)} = \overline{\phi(M')} \) is a plane, and the restriction of \( \phi \) to \( M' \) is an embedding. For a line \( L \subset P' \) we have \( L \cap \phi(M) = L \cap \phi(M') \). Hence if \( x \in L \cap \phi(M') \) we have \( G := \phi^{-1}(L \cap \phi(M')) \in \mathcal{M} \), since \( \phi \) satisfies \((G)\). Because \( G \subset M' \), also \( \phi|_{M'} \) satisfies \((G)\).

**Theorem 2.7** Let \((P, \mathcal{L}) = \text{PG}(m, K)\) and \((P', \mathcal{L}') = \text{PG}(n, L)\) be projective spaces and \( \phi : P \to P' \) an embedding, then \( K \) is isomorphic to a subfield of \( L \).

**Proof.** Let \( E \) be a plane of \( P \). Then \( \phi(E) \simeq \text{PG}(2, K) \) is a subplane of the Desarguesian projective plane \( \phi(E) \simeq \text{PG}(2, L) \), hence \( K \) is isomorphic to a subfield of \( L \) (cf. \([18, (8.2)], [6, (3.6.1)]\)).
3  A mapping of a vector space in a vector space over a field extension

In this section let $n, s \in \mathbb{N}$ be integers with $n \geq 2$, let $K$ be a commutative field, and $L = K(t)$ an extension field of $K$ with a transcendental or algebraic element $t$ of degree at least $2^{s(n+1)}$ over $K$. We consider the two left vector spaces $(K^{n+s+1}, K)$ and $(L^{n+1}, L)$. For $i = 0, 1, \ldots, n$ let $\xi_i \in K^{n+s+1}$, more precisely
\[ \xi_i = (x_{i,0}, x_{i,1}, \ldots, x_{i,n+s}) \] (4)
with elements $x_{i,k} \in K$. We denote the rows of the matrix
\[ X := \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} x_{0,0} & \cdots & x_{0,n+s} \\ \vdots & \vdots & \vdots \\ x_{n,0} & \cdots & x_{n,n+s} \end{pmatrix} = \begin{pmatrix} a^T_0, \ldots, a^T_{n+s} \end{pmatrix}, \] (5)
where $a^T_k = \begin{pmatrix} x_{0,k} \\ \vdots \\ x_{n,k} \end{pmatrix}$ for $k = 0, 1, \ldots, n+s$. (6)

Since the column rank and the row rank of $X$ are equal, we have:

**Lemma 3.1**  The following statements are equivalent:

1. The vectors $\xi_0, \xi_1, \ldots, \xi_n$ are linearly independent in $(K^{n+s+1}, K)$.
2. The matrix $X = (a^T_0, a^T_1, \ldots, a^T_{n+s})$ has rank $n+1$.
3. There exist distinct integers $i_0, i_1, \ldots, i_n \in \{0, 1, \ldots, n+s\}$ such that $a_{i_0}, a_{i_1}, \ldots, a_{i_n}$ are linearly independent in $(K^{n+1}, K)$.

Now we consider arbitrary vectors $a_0, a_1, \ldots, a_{n+s} \in K^{n+1} \subset L^{n+1}$ and define
\[ b^T_i := a^T_i + \sum_{j=1}^{s} t^{2(j_i-1)(n+1)+i} a^T_{n+j_i}, \quad a^T_{n+j_i} \in L^{n+1} \quad \text{for} \quad i = 0, 1, \ldots, n. \] (7)

For example, for $s = 2$ we obtain:
\[ b^T_i := a^T_i + t^{2} a^T_{n+1} + t^{2(n+1)+i} a^T_{n+2}. \]

**Lemma 3.2**  $\det(b^T_0, b^T_1, \ldots, b^T_n) \neq 0$ if and only if $\text{rank}(a^T_0, a^T_1, \ldots, a^T_{n+s}) = n+1$.

**Proof.** (i). First we introduce some notation to get a shorter representation. For $i \in \{0, \ldots, n\}$ and $j_i \in \{0, \ldots, s\}$ we define
\[ \lambda_{i,j_i} := \begin{cases} 0 & \text{if } j_i = 0 \\ 2(j_i - 1)(n+1) + i & \text{if } j_i \neq 0 \end{cases} \] and
\[ a^T_{i,j_i} := \begin{cases} a^T_i & \text{if } j_i = 0 \\ a^T_{n+j_i} & \text{if } j_i \neq 0 \end{cases}, \]
so $b^T_i := t^0a^T_i + \sum_{j_i=1} t^{2(j_i-1)(n+1)+i} a_{n+j_i}^T = \sum_{j_i=0} t^{\lambda_{i,j_i}} a_{i,j_i}^T$. 

(ii). We recall that we can write every integer $k \in \{1, 2, \ldots, 2^s(n+1) - 1\}$ as a sum of elements of $\{2^r : r = 0, 1, \ldots, s(n+1) - 1\} = \{2^{(j_i-1)(n+1)+i} : j_i = 1, 2, \ldots, s, \ i = 0, 1, \ldots, n\}$ in a unique way. Hence we have for $j_i, k_i \in \{0, 1, \ldots, s\}$

$$\sum_{i=0}^n \lambda_{i,j_i} = \sum_{i=0}^n \lambda_{i,k_i} \text{ if and only if } j_i = k_i \text{ for all } i \in \{0, \ldots, n\} \quad (9)$$

and $\prod_{i=0}^n t^{\lambda_{i,j_i}} = \prod_{i=0}^n t^{\lambda_{i,k_i}} \text{ if and only if } j_i = k_i \text{ for all } i \in \{0, \ldots, n\} \quad (10)$

(iii). By (i) we get

$$d := \det(b_0^T, \ldots, b_n^T) = \det\left(\sum_{j_0=0}^s t^{\lambda_{0,j_0}} a_{0,j_0}^T, \ldots, \sum_{j_n=0}^s t^{\lambda_{n,j_n}} a_{n,j_n}^T\right) = \sum_{j_0, \ldots, j_n=0}^s (t^{\lambda_{0,j_0}} \cdot \ldots \cdot t^{\lambda_{n,j_n}}) \det(a_{0,j_0}^T, \ldots, a_{n,j_n}^T) = \sum_{k \leq m} t^k \det A_k \quad (11)$$

with $k = \sum \lambda_{0,j_0} + \ldots + \lambda_{n,j_n}, \ m = 2^s(n+1) - 1$ and $A_k := (a_{0,j_0}^T, a_{i,j_1}^T, \ldots, a_{n,j_n}^T)$ with not necessarily distinct integers $i_j \in \{0, 1, \ldots, n+s\}, j = 0, 1, \ldots, n$.

(vi). Since $t \in L \setminus K$ has at least degree $m+1 = 2^{s(n+1)}$ over $K$, we have $d = \sum_{k=0}^m t^k \det A_k = 0$ if and only if $\det A_k = 0$ for every $k \in \{0, 1, \ldots, 2^{s(n+1)} - 1\}$. This means in particular that for distinct elements $i_0, i_1, \ldots, i_n \in \{0, 1, \ldots, n+s\}$ the vectors $a_{i_0}^T, a_{i_1}^T, \ldots, a_{i_n}^T$ are linearly dependent. By Lemma 3.1 it follows that $\text{rank}(a_{0,j_0}^T, a_{i_1}^T, \ldots, a_{n,j_n}^T) < n+1$.

On the other hand, if $d \neq 0$, then there exist integers $i_0, i_1, \ldots, i_n \in \{0, 1, \ldots, n+s\}$ with $\det(a_{i_0}^T, a_{i_1}^T, \ldots, a_{i_n}^T) \neq 0$, hence $i_0, i_1, \ldots, i_n$ are distinct and $a_{i_0}^T, a_{i_1}^T, \ldots, a_{i_n}^T$ are linearly independent. By Lemma 3.1 we get $\text{rank}(a_{0,j_0}^T, a_{i_1}^T, \ldots, a_{n,j_n}^T) = n+1$. \hfill $\blacksquare$

Now we define the map

$$f : K^{n+s+1} \rightarrow L^{n+1}, \quad \mathbf{r} = (x_0, \ldots, x_{n+s+1}) \mapsto \mathbf{r}' = (x'_0, \ldots, x'_n)$$

by $x'_i = x_i + \sum_{j=1}^s t^{2(j-1)(n+1)+i} x_{n+j}, \text{ for } i = 0, 1, \ldots, n. \quad (12)$

**Lemma 3.3**

1. $f(K\mathbf{r}) = Kf(\mathbf{r}) \subset Lf(\mathbf{r})$.

2. The vectors $\mathbf{r}_0, \ldots, \mathbf{r}_n$ are linearly independent in $(K^{n+s+1}, K)$ if and only if $f(\mathbf{r}_0), \ldots, f(\mathbf{r}_n)$ are linearly independent in $(L^{n+1}, L)$.

3. In particular for three vectors $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2 \in K^{n+s+1}$, $\text{rank}(\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2) = 3$ if and only if $\text{rank}(f(\mathbf{r}_0), f(\mathbf{r}_1), f(\mathbf{r}_2)) = 3$.

**Proof.** 1. By definition $f(\lambda \mathbf{r}) = \lambda f(\mathbf{r})$ for $\lambda \in K$. Clearly $Kf(\mathbf{r}) \subset Lf(\mathbf{r})$. 


2. For \( r_0, \ldots, r_n \in K^{n+s+1} \) with \( r_i = (x_{i,0}, x_{i,1}, \ldots, x_{i,n+s}) \), we consider the matrix

\[
X' := \begin{pmatrix}
  f(r_0) \\
  \vdots \\
  f(r_n)
\end{pmatrix}
= \begin{pmatrix}
  x_{0,0} + \sum_{j=1}^s t^{2(j-1)(n+1)} x_{0,n+j} & \cdots & x_{0,n} + \sum_{j=1}^s t^{2(j-1)(n+1)+n} x_{0,n+j} \\
  \vdots & \ddots & \vdots \\
  x_{n,0} + \sum_{j=1}^s t^{2(j-1)(n+1)} x_{n,n+j} & \cdots & x_{n,n} + \sum_{j=1}^s t^{2(j-1)(n+1)+n} x_{n,n+j}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  a_0^T + \sum_{j=1}^s t^{2(j-1)(n+1)} a_{n+j}^T, \ldots, a_n^T + \sum_{j=1}^s t^{2(j-1)(n+1)+n} a_{n+j}^T
\end{pmatrix}
= (b_0^T, \ldots, b_n^T).
\]

Hence by Lemma 3.2 we have \( \det(X') = \det(b_0^T, b_1^T, \ldots, b_n^T) \neq 0 \) iff \( \text{rank}(a_0^T, a_1^T, \ldots, a_n^T) = n + 1 \), i.e. by Lemma 3.1, iff \( r_0, \ldots, r_n \) are linearly independent. Since \( n \geq 2 \), 3. is a consequence of 2.

4 Embeddings satisfying (E)

Using the map \( f \) introduced in the preceding section, we now construct projective embeddings.

Let \((P, \mathcal{L})\) be a Pappian projective space with \( \dim P = n + s \) for \( n, s \in \mathbb{N} \) with \( n \geq 2 \). Then we can represent \((P, \mathcal{L}) = \text{PG}(n + s, K)\) by an \((n + s + 1)\)-dimensional vector space \((K^{n+s+1}, K)\) over a commutative field \( K \). Let us denote by \((P', \mathcal{L}') := \text{PG}(n, L)\) the \( n \)-dimensional projective space with the underlying vector space \((L^{n+1}, L)\) where \( L \) is the field extension of \( K \) introduced in the preceding section. We recall that three points \( a = Ka, b = Kb, c = Kc \) are noncollinear if and only if \( \text{rank}(a, b, c) = 3 \) for vectors \( a, b, c \in K^{n+s+1} \).

**Theorem 4.1**  
1. For every \( n, s \in \mathbb{N} \) with \( n \geq 2 \) and every Pappian projective space \((P, \mathcal{L})\) of dimension \( n + s \), there exists an embedding \( \phi : P \to P' \) in an \( n \)-dimensional projective Pappian space \((P', \mathcal{L}')\) such that any \( n + 1 \) points \( x_0, \ldots, x_n \in P \) are independent in \((P, \mathcal{L})\) if and only if \( \phi(x_0), \ldots, \phi(x_n) \) are independent in \((P', \mathcal{L}')\).

2. For a proper subspace \( T \) of \( P \) it holds that \( \phi(T) = \overline{T} \cap \phi(P) \) if and only if \( \dim T \leq n - 1 \).

3. For \( n \geq 3 \), \( \phi \) satisfies (E).

**Proof.** 1. Using the map \( f \) of Lemma 3.3, we define

\[
\phi : P \to P', x = Kx \mapsto \phi(x) := Lf(x)
\]  

By Lemma 3.3(1), \( \phi \) is well defined, and by Lemma 3.3(3), \( \phi \) maps collinear points onto collinear point and noncollinear points onto noncollinear points, hence \( \phi \) is
an embedding. Since $x_0 = K\tau_0, \ldots, x_n = K\tau_n$ are independent iff $\tau_0, \ldots, \tau_n$ are linearly independent, and $\phi(x_0) = Lf(\tau_0), \ldots, \phi(x_n) = Lf(\tau_n)$ are independent iff $f(\tau_0), \ldots, f(\tau_n)$ are linearly independent, one obtain by Lemma 3.3(2) that $x_0, \ldots, x_n \in P$ are independent iff $\phi(x_0), \ldots, \phi(x_n)$ are independent.

2. For $r \leq n - 1$, let $T$ be an $r$-dimensional subspace of $(P, \mathcal{L})$ with a basis $a_0, \ldots, a_r$. Assume that $\phi(T) \neq \phi(T) \cap \phi(P)$. Then there exists a point $b \in P$ with $\phi(b) \in \left(\phi(T) \cap \phi(T)\right)$, i.e. $b \not\in T$ and $a_0, \ldots, a_r, b$ are independent in $(P, \mathcal{L})$. Since $\phi(b) \in \phi(T) = \phi(a_0), \ldots, \phi(a_r)$ (cf. Lemma 2.2(2)), it follows that $\phi(a_0), \ldots, \phi(a_r), \phi(b)$ are dependent in $(P, \mathcal{L})$.

Since $r + 2 \leq n + 1$, by 1., the points $\phi(a_0), \ldots, \phi(a_r), \phi(b)$ are independent since $a_0, \ldots, a_r, b$ are independent, a contradiction to the assumption $\phi(T) \neq \phi(T) \cap \phi(P)$. Hence $\phi(T) = \phi(T) \cap \phi(P)$ for $\dim T \leq n - 1$. For every proper subspace $T$ of $P$ with $\dim T \geq n$, there are $n + 1$ independent points $a_0, \ldots, a_n \in T$. By 1. $\phi(a_0), \ldots, \phi(a_n)$ are independent in $P'$, hence $P' = \phi(a_0), \ldots, \phi(a_n) \in \phi(T)$ and $\phi(T) = \phi(T) \cap \phi(P) = P' \cup \phi(P)$, since $T$ is a proper subspace of $P$.

3. By 2., (E) is satisfied for $n \geq 3$.

**Corollary 4.2** For every $n, s \in \mathbb{N}$ with $n \geq 2$ and every finite projective space $(P, \mathcal{L})$ of dimension $n + s$, there exists an embedding $\phi : P \to P'$ in an $n$-dimensional finite projective Desarguesian space $(P', \mathcal{L}')$ such that any $n + 1$ points $x_0, \ldots, x_n \in P$ are independent in $(P, \mathcal{L})$ if and only if $\phi(x_0), \ldots, \phi(x_n)$ are independent in $(P', \mathcal{L}')$. For a proper subspace $T$ of $P$ it holds that $\phi(T) = \phi(T) \cap \phi(P)$ if and only if $\dim T \leq n - 1$, and for $n \geq 3$, $\phi$ satisfies (E).

**Proof.** If $P$ is finite, then $\text{ord} P$ is finite and $(P, \mathcal{L}) = \text{PG}(n+s, K)$ for a commutative field $K$. There exists a finite field extension $L = K(t)$ of finite degree $t$ at least $2^{s(n+1)}$, hence $L$, and therefore also $P'$ are finite and the assertion follows with 4.1.

If we set $n = 2$ we obtain:

**Corollary 4.3** Every Pappian projective space is embeddable in a Pappian projective plane.

**Proof.** For a Pappian projective space $(P, \mathcal{L})$ of finite dimension, Corollary 4.3 is a direct consequence of Theorem 4.1 with $n = 2$. For $\dim P = \infty$ we modify the construction of the last section, by taking a transcendental element $t_b$ for every element $b$ of a basis $B$ of $P$. Then for $T = \{t_b : b \in B\}$ and $L := K(T)$ we get the result analogous to the proofs of Lemma 3.1 to 3.3.

Let $(M, \mathcal{M})$ be a linear space. Two lines $G, L \in \mathcal{L}$ are called *parallel* if $G = L$, or if $G, L$ are contained in a common plane and $G \cap L = \emptyset$. For $x \in M \setminus L$ let

$$\pi(x, L) := |\{G \in \mathcal{M} : x \in G \text{ and } G, L \text{ parallel }\}|$$

(14)
denote the number of all parallel lines of $L$ passing $x$. For $m \in \mathbb{N}$, $(M, \mathcal{M})$ is called an $[0, m]$-space, if for each non-incident point-line pair $(x, L)$ we have that $\pi(x, L) \in [0, m] = \{0, 1, \ldots, m\}$. Let $\pi(L) := \max\{\pi(y, L) : y \in M \setminus L\}$. If $|L| + \pi(L) - 1 \geq 3m + 1$ and $\dim M \geq 3$, then by [14, Theorem (2.10)], ord$M := |L| + \pi(L) - 1$ is constant for every line $L \in \mathcal{M}$. If ord$M \geq 3m + 2$ and $\dim M \geq 3$, then by [14, Embedding Theorem (4.5)], $(M, \mathcal{M})$ is embeddable in a projective space $(P, \mathcal{L})$ with $\dim M = \dim P$ and ord$M = \text{ord} P$. Hence:
Corollary 4.4 Every finite \([0, m]-space\) \((M,\mathcal{M})\) with \(\dim M \geq 3\) and \(\text{ord} M \geq 3m + 2\) is embeddable in a finite Pappian projective plane.

Proof. Since \(M\) is finite, also \(\text{ord} M = \text{ord} P\) and \(\dim M = \dim P\) is finite and \((M,\mathcal{M})\) is embeddable in a finite projective space \((P,\mathcal{L})\). Now by 4.2 for \(n = 2\), \((P,\mathcal{L})\) is embeddable in a finite Pappian plane \((P',\mathcal{L}')\) and by 2.1 the assertion follows.

References


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