On Buekenhout-Metz unitals

L. Rey A. Casse   Guglielmo Lunardon *

Abstract

Let $\Sigma' = \mathrm{PG}(4,q)$, Σ be a hyperplane of Σ' and $\mathcal{F}$ be a regular spread of Σ. Denote by $\pi(\Sigma',\Sigma,\mathcal{F}) \simeq \mathrm{PG}(2,q^2)$ the projective plane constructed using $\mathcal{F}$. We give a simple proof that if U is a Buekenhout–Metz unital of the plane $\pi(\Sigma',\Sigma,\mathcal{F})$ defined by an elliptic cone $\mathcal{U}$ of Σ', then there is a regular spread $\mathcal{F}'$ of Σ such that $\mathcal{U}$ defines a hermitian curve of $\pi(\Sigma',\Sigma,\mathcal{F}') \simeq \mathrm{PG}(2,q^2)$.

A *Baer subplane* of $\mathrm{PG}(2,q^2)$ is a subplane of order q. It has the property that a line of $\mathrm{PG}(2,q^2)$ meets a Baer subplane in 1 or in $q+1$ points. A set of $q+1$ points which is the intersection of a line with a Baer subplane is a *Baer subline* .

A *unital* of $\mathrm{PG}(2,q^2)$ is a set U of q^3+1 points such that a line of $\mathrm{PG}(2,q^2)$ contains either 1 or $q+1$ points of U. If the line l of $\mathrm{PG}(2,q^2)$ contains exactly one point of U, the unital is said to be *parabolic* with respect to l. A hermitan curve is a unital, which will be called *classical*.

A *regulus* of $\Sigma = \mathrm{PG}(3,q)$ is a ruling of a non-singular hyperbolic quadric of $\Sigma = \mathrm{PG}(3,q)$. If l,m,n are three mutually disjoint lines of Σ, there is a unique regulus $\mathcal{R}(l,m,n)$ containing l, m and n. A *spread* of $\Sigma = \mathrm{PG}(3,q)$ is a set $\mathcal{F}$ of q^2+1 lines which are mutually disjoint. When the regulus $\mathcal{R}(l,m,n)$ of Σ is contained in $\mathcal{F}$ for all lines l, m and n of $\mathcal{F}$, the spread $\mathcal{F}$ is said to be *regular*.

Let $\Sigma' = \mathrm{PG}(4,q)$, Σ a hyperplane of Σ'. We **always** suppose $\mathcal{F}$ is a regular spread of Σ. Define a translation plane $\pi(\Sigma',\Sigma,\mathcal{F})$ as follows. The points are either the points of $\Sigma' \setminus \Sigma$ or the elelemts of $\mathcal{F}$. The lines are either the planes of Σ' which intersects Σ in a line of $\mathcal{F}$ or Σ. The incidence is the natural one. As $\mathcal{F}$ is regular, the plane $\pi(\Sigma',\Sigma,\mathcal{F})$ is isomorphic to the desarguesian plane $\mathrm{PG}(2,q^2)$ (see [1], [4]). A

*This research begun while the first author was a C.N.R. visiting professor at the University of Naples during May 1996.

Received by the editors September 1997.
Communicated by Albrecht Beutelspacher.
1991 *Mathematics Subject Classification.* Primary 51E20, Secondary 51E21.
Key words and phrases. Unitals, regular spreads.

Baer subline of $\mathrm{PG}(2,q^2)$ is represented in $\pi(\Sigma',\Sigma,\mathcal{F})$ either by a line or by a conic in a plane α which contains a line m of $\mathcal{F}$. In the last case the line m is external to the conic (see [10]).

Let $\mathcal{O}$ be an ovoid of a hyperplane Ω of Σ', and suppose that the plane $\Sigma\cap\Omega$ is tangent at $\mathcal{O}$ in a point p. If s is the line of $\mathcal{F}$ incident with p and r a point of s different from p, let $\mathcal{U}$ be the cone which projects $\mathcal{O}$ from r. Then the points of $\mathcal{U}\setminus\{s\}$ together with the point of $\pi(\Sigma',\Sigma,\mathcal{F})$ represented by s define a unital U of $\pi(\Sigma',\Sigma,\mathcal{F})$ ([5] §4, Remark (4)) called a *Buekenhout − Metz* unital.

If β is a fixed isomorphism from $\mathrm{PG}(2,q^2)$ to $\pi(\Sigma',\Sigma,\mathcal{F})$, denote by l_∞ the line of $\mathrm{PG}(2,q^2)$ mapped by β in the line represented by Σ. A classical unital, which is parabolic with respect to the line l_∞, is represented in $\pi(\Sigma',\Sigma,\mathcal{F})$ by an elliptic cone $\mathcal{U}$ of Σ' such that $\Sigma\cap\mathcal{U}=s$ is a line of $\mathcal{F}$, and each plane of Σ' which contains a line of $\mathcal{F}$ either is tangent to $\mathcal{U}$ or intersects $\mathcal{U}$ in a conic which represents a Baer subline of of $\pi(\Sigma',\Sigma,\mathcal{F})$ (see [5])

Let α be a plane of Σ' which represents a line of $\pi(\Sigma',\Sigma,\mathcal{F})$. Then α intersects Σ in a line m of $\mathcal{F}$. It has been proved in [7] that there is a conic C of α disjoint from m which is not a Baer-subline of $\pi(\Sigma',\Sigma,\mathcal{F})$. Let s be a fixed line of $\mathcal{F}$ different from m. If p is a fixed point of s, then there is an elliptic quadric $Q^-(3,q)$ of $<p,\alpha>$ containing C and p. If $\mathcal{M}$ is the elliptic cone which projects $Q^-(3,q)$ from a point of s different from p, then $\mathcal{M}$ defines a non-classical Buekenhout-Metz unital of $\pi(\Sigma',\Sigma,\mathcal{F})$ (see [7]). In this paper we give a simple proof the following theorem proved in [6]

Theorem *Let U be a Buekenhout-Metz unital of $\pi(\Sigma',\Sigma,\mathcal{F})$ defined by an elliptic cone $\mathcal{U}$ of Σ'. Then there is a regular spread $\mathcal{F}'$ of Σ, such that $\mathcal{U}$ defines a classical unital of the plane $\pi(\Sigma',\Sigma,\mathcal{F}')$ which is parabolic with respect to the line of $\pi(\Sigma',\Sigma,\mathcal{F}')$ represented by Σ.*

Proof. Let $\Lambda^*=\mathrm{PG}(5,q^2)$ and let $(x_0,x_1,x_2,x_3,x_4,x_5)$ be the homogeneous coordinates of a point of Λ^*. Denote by σ the involutory collineation of Λ^* defined by $(x_0,x_1,x_2,x_3,x_4,x_5)^\sigma=(\bar{x_3},\bar{x_4},\bar{x_5},\bar{x_0},\bar{x_1},\bar{x_2})$ where $\bar{a}=a^q$ for all a in $\mathrm{GF}(q^2)$. The points fixed by σ belong to $\Lambda=\{(x_0,x_1,x_2,\bar{x_0},\bar{x_1},\bar{x_2})\mid x_0,x_1,x_2\in\mathrm{GF}(q^2)\}$ which is a subgeometry of Λ^* isomorphic to $\mathrm{PG}(5,q)$.

Let π be the plane of Λ^* with equations $x_3=x_4=x_5=0$.

Then π is disjoint from Λ and the plane $\bar{\pi}=\pi^\sigma$ has equations $x_0=x_1=x_2=0$. For each point x of π, let $l(x)$ be the line joining the points x and x^σ. Then $l(x)\cap\Lambda$ is a line of Λ, i.e. $l(x)$ contains exactly $q+1$ point of Λ. If $l(x)$ and $l(y)$ are not disjoint, then $<x,y,x^\sigma,y^\sigma)>$ is a plane. This is impossible because the line $<x,y>$ of π and the line $<x^\sigma,y^\sigma>$ of π^σ are disjoint, we conclude that $\mathcal{S}=\{l(x)\mid x\in\pi\}$ is a line-spread of $\Lambda=\mathrm{PG}(5,q)$.

For each line m of π let $\mathcal{S}_m=\{l(x)\mid x\in m\}$. Then $\mathcal{S}_m$ is a regular spread of the $3-$dimensional subspace $<m,m^\sigma>\cap\Lambda$ of Λ (see [3]). If a $3-$dimensional subspace Σ of Λ contains two lines $l(x)$ and $l(y)$ of $\mathcal{S}$, and m is the line of π joining the points x and y, then $\Sigma=<m,m^\sigma>\cap\Lambda$ and

$$\mathcal{S}_\Sigma=\{n\in\mathcal{S}\mid n\cap\Sigma\neq\emptyset\}=\mathcal{S}_m$$

is a regular spread of Σ. Hence the incidence structure

$$\Pi=(\mathcal{S},\{\mathcal{S}_m\mid m\ is\ a\ line\ of\ \pi\})$$

is isomorphic to $\pi = \mathrm{PG}(2, q^2)$ via the map $\tau : x \mapsto l(x)$ (see [2]).

If Σ' is a hyperplane of Λ, then there is exactly one 3−dimensional subspace Σ of Σ' such that $\mathcal{S}_\Sigma$ is a (regular) spread of Σ. Then the map ρ from Π to $\pi(\Sigma', \Sigma, \mathcal{S}_\Sigma)$, which maps the line $l(x)$ of $\mathcal{S}$ into $l(x) \cap \Sigma'$, is an isomorphism.

Let $Q^+(5, q^2)$ be the hyperbolic quadric of $\Lambda^* = \mathrm{PG}(5, q^2)$ defined by the equation $x_0x_5 + x_1x_4 + x_2x_3 = 0$. Then the plane π and π^σ are contained in $Q^+(5, q^2)$, and $Q^+(5, q^2) \cap \Lambda = Q^-(5, q)$ is the elliptic quadric of Λ defined by the equation $x_0x_2^q + x_1^{1+q} + x_2x_0^q = 0$, which is quadratic over $\mathrm{GF}(q)$.

If a line $l(x)$ of $\mathcal{S}$ contains a point of $Q^-(5, q)$, then $l(x)$ is contained in $Q^-(5, q)$ because it is incident with three points of $Q^+(5, q^2)$. This implies that $\mathcal{H} = \{l(x) \mid l(x) \cap Q^-(5, q) \neq \emptyset\}$ is a spread of $Q^-(5, q)$ [1]. If a 3−dimensional subspace Σ of Λ contains two lines of $\mathcal{H}$, then $\mathcal{S}_\Sigma$ is a regular spread of Σ, and $Q^-(5, q)$ intersects Σ in a non-singular hyperbolic quadric $Q^+(3, q)$. Thus there are exactly $q + 1$ lines of the spread $\mathcal{S}_\Sigma$ contained in $\mathcal{H}$ and these lines form a regulus of Σ (see [8]). Moreover $H(3, q^2) = \{x = (a_0, a_1, a_2, 0, 0, 0) \in \pi \mid l(x) \in \mathcal{H}\}$ is the hermitian curve of π defined by the equation $a_0a_2^q + a_1^{1+q} + a_2a_0^q = 0$ (see [8]).

Let $s = \mathcal{U} \cap \Sigma$ be the line of $\mathcal{F}$ contained in Σ. Embed Σ' in Λ in such a way that Σ' is the tangent hyperplane of $Q^-(5, q)$ at the vertex of $\mathcal{U}$, and $\Sigma' \cap Q^-(5, q) = \mathcal{U}$. Then s belongs to $Q^-(5, q)$, and Σ is the polar of s with respect to $Q^-(5, q)$.

If $s = l(x)$ and m is the line of π tangent to $H(2, q^2)$ at x, then $\Gamma =< m, m^\sigma > \cap \Lambda$ is a 3−dimensional subspace of Σ' such that $\mathcal{S}_m$ is a regular spread of Γ. Therefore s is the unique line of $\mathcal{H}$ contained in Γ, and $< m, m^\sigma >$ intersects $Q^+(5, q^2)$ in the planes $< m, s >$ and $< m^\sigma, s >$. Hence Γ is the polar of the line s with respect to $Q^-(5, q)$, and $\Sigma = \Gamma$. This implies that $\mathcal{H}$ is mapped into $\mathcal{U}$ by the isomorphism ρ from Π and $\pi(\Sigma', \Sigma, \mathcal{S}_m)$. Hence we have proved that $\mathcal{U}$ defines a classical unital of $\pi(\Sigma', \Sigma, \mathcal{S}_m)$. ∎

References

[1] J. André, Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe. *Math. Z.* 60 (1954), 156-186.

[2] R. C. Bose, On a representation of the Baer subplane of the desarguesian plane $\mathrm{PG}(2, q^2)$ in a projective five dimensional space $\mathrm{PG}(5, q)$. *Colloquio Internazionale Teorie Combinatorie*, Roma 1973, Tomo I, 381-391.

[3] R. H. Bruck, Construction problems in finite projective spaces. *Combinatorial mathematics and its applications*, Chapel Hill, 1969, 426-514.

[4] R. H. Bruck and R. C. Bose, The construction of translation planes from projective spaces. *J. Algebra*, 1 (1964), 85-102.

[5] F. Buekenhout, Existence of unitals in finite translation planes of order q^2 with a kernel of order q. *Geom. Dedicata* 5 (1976), 189-194.

[6] L. R. A. Casse and C. T. Quinn, The Bruck-Bose map and Buekenhout-Metz unitals. *R.C. Bose Memorial Conference, Fort Collins 1995*, to appear.

[1] A *spread* of $Q^-(5, q)$ is a partition in lines of the points of $Q^-(5, q)$.

[7] R. Metz, On a class of unitals. *Geom. Dedicata*, 8 (1979), 125-126.

[8] J. A. Thas, Semi-partial geometries and spreads of classical polar spaces. *J. Combin. Theory Ser. A*, 35 (1983), 58-66.

[9] J. A. Thas: Old and new results on spreads ond ovoids of finite classical polar spaces *Ann. Discrete Math.* 52 (1992), 529-544.

[10] R. Vincenti, Alcuni tipi di varietà $\mathcal{V}_2^3$ di $S_{4,q}$ e sottopiani di Baer. *Suppl. BUMI* 2(1980), 31-44.

L. R. A. Casse
Department of Pure Mathematics
The University of Adelaide
G.P.O. Box 498, Adelaide, S.A. 5001
Australia
e-mail rcasse@maths.adelaide.edu.au

G. Lunardon
Dip. di Matematica e Applicazioni
Complesso di Monte S. Angelo - Edificio T
Via Cintia
80126 Napoli, Italy