Large minimal covers of $\mathrm{PG}(3, q)$

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Abstract

A *cover* of $\Sigma = \mathrm{PG}(3, q)$ is a set of lines S such that each point of Σ is incident with at least one line of S. A cover is *minimal* if no proper subset is also a cover. We study minimal covers of Σ which are 'large'; the main results being constructions of sets of this kind and an upper bound on the size of minimal covers.

1 Introduction

A *cover* of $\Sigma = \mathrm{PG}(3, q)$ is a set S of lines such that every point of Σ is on at least one line of S. A cover is said to be *minimal* if no proper subset is also a cover. Every cover of Σ contains at least $q^2 + 1$ lines, and the covers with exactly this many lines are the spreads. In [1], A. Blokhuis *et. al.* study covers of Σ (and of finite generalized quadrangles) which are 'small'. In essence, they give a structure theorem for minimal covers S with $q^2 + 1 < |S| < q^2 + q + 1$.

In this note we study 'large' minimal covers. A natural first problem is to find the maximal size of a minimal cover. We begin by finding an upper bound on the size of these sets, proceed to give some constructions for large minimal covers, and finally discuss some connections between this problem and others outstanding in the literature. Along the way we describe an interesting 'regular' cover.

Throughout, $\mathrm{star}(P)$ denotes the set of lines of Σ on a point $P \in \Sigma$ while $\mathrm{pen}(P, \pi)$ denotes the plane pencil of lines defined by the incident point-plane pair (P, π).

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2 An upper bound

If S is a minimal cover of Σ, then for each line l in S there must be at least one point on l which is on no other line of S. Therefore we can construct an injective mapping from S to the set of points of Σ, so we have immediately $|S| \leq |\Sigma| = q^3+q^2+q+1$. However, by using a method used in [4] to study the size of minimal blocking sets, it is possible to derive a better bound.

Theorem 1 *If S is a minimal cover of* $\mathrm{PG}(3,q)$ *then*

$$|S| < \frac{\sqrt{5}-1}{2}q^3 + \frac{2}{\sqrt{5}}q^2 + q + \frac{1}{2}. \tag{1}$$

Proof. In fact, we will prove the dual statement–if S is a set of lines such that each plane contains a line of S and for each line $l \in S$ there exists a plane on l containing no other line of S, then $|S|$ satisfies (1). Let S be such a set.

Call a plane π *tangent* to S if π contains exactly one line of S, and *secant* if it contains more than one. Suppose that there are γ tangent planes to S. Then by assumption there are exactly $n = q^3+q^2+q+1-\gamma$ secant planes; label them as $\pi_1, \ldots, \pi_n$ and for each i put $|\pi_i \bigcap S| = x_i$. If $|S| = t$, counting incidences gives

$$\sum_{i=1}^{n} x_i = t(q+1) - \gamma \tag{2}$$

$$\sum_{i=1}^{n} x_i(x_i-1) = \zeta, \tag{3}$$

where ζ is the number of ordered pairs (l_1, l_2) of intersecting lines of S. Now J. Eisfeld ([6]) has recently shown that ζ satisfies

$$\zeta \leq \frac{t^2(q+1)}{q^2+q+1} + (q^2-1)t < \frac{t^2}{q} + (q^2-1)t. \tag{4}$$

We remark that Eisfeld has in fact given a similar bound for line sets in $\mathrm{PG}(n,q)$ for arbitrary $n \geq 3$. The bound is sharp; the sets satisfying it are exactly the Cameron-Liebler line classes introduced in [5]. Combining (2), (3) and (4) gives

$$\sum_{i=1}^{n} x_i^2 < t(q+1) - \gamma + \frac{t^2}{q} + (q^2-1)t. \tag{5}$$

Now (2) and (5), together with the inequality

$$\frac{1}{n}(\sum_{i=1}^{n} x_i)^2 \leq \sum_{i=1}^{n} x_i^2 \tag{6}$$

(the 'variance inequality') gives, with $\theta_3 = q^3+q^2+q+1$,

$$t^2\left(\frac{\theta_3}{q} - (q+1)^2\right) + t\theta_3(q^2+q) + \gamma\left(2t(q+1) - \theta_3 - \frac{t^2}{q} - t(q^2+q)\right) > 0. \tag{7}$$

Since $t > 0$, the coefficient of γ in (7) is negative. Furthermore, $\gamma \geq t$, so we can replace γ by t and preserve the inequality. Doing this, simplifying, and solving for t gives

$$t < \frac{1}{2}\left(-q^3 + 2q + 1 + \sqrt{5q^6 + 8q^5 + 2q^3 + 4q^2 + 1}\right).$$

Since $5q^6 + 8q^5 + 2q^3 + 4q^2 + 1 < (\sqrt{5}q^3 + (4/\sqrt{5})q^2)^2$ for $q \geq 2$, (1) is a weakening of this last inequality, so the theorem holds. ■

3 Some examples of large minimal covers

3.1 Hyperbolic quadrics

Let $\mathcal{H}_1, \ldots, \mathcal{H}_q$ be q hyperbolic quadrics forming a spread corresponding to a flock of a quadratic cone (see [7]). Then the $\mathcal{H}_i$ are skew apart from one line l which is contained in each of them. Denote by $\mathcal{R}_i$ the regulus of $\mathcal{H}_i$ which contains l. The following lemma describes some of the structure of this set of quadrics.

Lemma 1 *For each plane π containing l, there exists a (unique) $P \in l$ such that the lines of $\mathcal{R}_i^{opp}$ on P, together with l, form the pencil defined by P and π. Furthermore, any plane not containing l is tangent to exactly one of the $\mathcal{H}_i$ (and hence meets each of the others in a non-singular conic).*

Proof. Let $P \in l$. Each of $\mathcal{R}_1^{opp}, \ldots, \mathcal{R}_q^{opp}$ has a unique line l_i on P. Put $\pi_i = l \vee l_i$ and assume that for some $i \neq j$, $\pi_i \neq \pi_j$. Since $l \subset \mathcal{H}_i$ and $l \subset \pi_j$, π_j is a tangent plane to $\mathcal{H}_i$ and therefore contains a line of $\mathcal{H}_i$, m say, apart from l. Furthermore, m cannot be incident with P, since the two lines of $\mathcal{H}_i$ on P are l and l_i. Therefore m intersects l_j in a point off l, contradicting $\mathcal{H}_i \bigcap \mathcal{H}_j = l$. Therefore the tangent planes to $\mathcal{H}_i$, $i = 1, \ldots, q$ all coincide, so the set of lines of the $\mathcal{R}_i^{opp}$ on P together with l form a plane pencil. This proves the first assertion. As for the second, any plane ρ not containing l hits l in a single point P; therefore the intersections of the $\mathcal{H}_i$ with ρ partition the points of ρ apart from P. Now a counting argument using the fact that the intersection of each $\mathcal{H}_i$ with ρ is either two lines or a non-singular conic (see [8]) gives the required result. ■

Let $P \in l$ be a fixed point; let l_i be the line of $\mathcal{R}_i^{opp}$ on P $(i = 1, \ldots, q)$, and let π be the plane containing l such that $\mathrm{pen}(P, \pi)$ contains all the l_i. Define a set S of lines by

$$S = (\mathrm{star}(P) \setminus \{l_1, \ldots, l_{q-1}\}) \bigcup \left(\bigcup_{i=1}^{q-1} \mathcal{R}_i\right). \tag{8}$$

Theorem 2 *The set S defined by* (8) *is a minimal cover of Σ with $2q^2 - q + 2$ lines.*

Proof. The size of S is easily calculated. Since the lines of $\mathrm{star}(P)$ are a cover of Σ and the points on l_i are covered by the lines of $\mathcal{R}_i$, S is a cover of Σ. Now we prove that S is minimal. The two lines of $\mathrm{pen}(P, \pi)$ in S, namely l and l_q, cannot be removed from S since each of their points, except P, is on no other line of S

(each line of S not on P is in some $\mathcal{R}_i$ with $i < q$ and is therefore skew to both l and l_q). The lemma above implies that any line m of star(P) not contained in π is secant to each of the $\mathcal{H}_i$, and hence intersects $\mathcal{H}_q$ in P and one further point Q. Since the only points of $\mathcal{H}_q$ which are covered by lines of S are in π, Q is covered by no other line of S, and therefore m cannot be removed from S. Therefore no line of star$(P) \bigcap S$ can be removed from S. The lines of S not on P, of which there are $q(q-1)$, collectively cover all the points of π not on l or l_q, of which there are $q(q-1)$. Therefore each of these lines intersects π in a point not covered by any other line of S, so S is minimal. ∎

Note that all but two of the lines in this set have only one point which is not on any other line of S. Note also that while S covers every point of Σ, not every plane of Σ contains a line of S–if ρ is a plane not containing P which is tangent to $\mathcal{H}_q$, then no line of S is in ρ. Thus S misses a rather large number of planes, namely q^2.

Now if in the above construction we proceed a little further, and form

$$T = (\text{star}(P) \setminus \{l_1, \ldots, l_q\}) \bigcup \left(\bigcup_{i=1}^{q} \mathcal{R}_i \right), \tag{9}$$

we have a set of $2q^2 + 1$ lines which, while not a minimal cover of S, nevertheless has an interesting property, as follows:

Theorem 3 *Every plane contains one or $q+1$ lines of T.*

Proof. The plane π contains only one line, l, of T. Any other plane on L contains $q+1$ lines of T, namely the pencil given by lemma 1. A plane ρ on P, not containing l is a tangent to precisely one $\mathcal{H}_i$; so there exists precisely one line of $\mathcal{R}_i$ in ρ, and no line of any other $\mathcal{R}_j$, $j \neq i$. T also contains the q lines of pen(P, ρ) which are not in π, for $q+1$ in total. Finally, any plane off P is tangent to precisely one $\mathcal{H}_i$, hence contains precisely one line of $\mathcal{R}_i$, no lines of $\mathcal{R}_j$ for $j \neq i$ and thus no other lines of T. ∎

Beginning with a set of $q+1$ hyperbolic quadrics partitioning the points of space apart from two common lines, or a regular spread decomposed (as in [2]) into $q-1$ disjoint reguli and two additional lines, and proceeding as in the above construction gives other examples of minimal covers of size approximately $2q^2$.

3.2 Unitals

A *unital* U in a projective plane PG$(2, q)$ is a set of $q^{3/2}+1$ points with the property that every line of PG$(2, q)$ is on either exactly one or exactly $q^{1/2} + 1$ points of U. Unitals exist in PG$(2, q)$ if and only if q is a square. It is shown in [4] that unitals are the unique largest reduced blocking sets in PG$(2, q)$ (a *blocking set* in a projective plane is a set of points containing no line, but intersecting every line; such a set is *reduced* if it has a tangent line at every point). In [3] it is shown that the blocking set hypothesis is unnecessary here; that is, that unitals are the largest sets of points in PG$(2, q)$ having a tangent line at each point. If S is a minimal cover of PG$(3, q)$ then for any plane π, the lines of S in π form a set of lines with the property that each contains a point on no other–the dual to the situation studied in [3]. Therefore is is natural to try to construct large minimal covers using unitals. We proceed to do this now. Let q be a square, l a line of PG$(3, q)$ and $P \in l$. Let π_i, $i = 1, \ldots, q+1$ be the planes on l and for each i, let U_i be a unital in π_i such that $P \in U_i$ and l is the (unique) tangent line to U_i at P. Let S be the set of tangent lines to the unitals: each line of S lies in one of the planes π_i, $1 \leq i \leq q+1$.

Theorem 4 *S is a minimal cover of* PG$(3, q)$ *with $q^{5/2} + q^{3/2} + 1$ lines.*

Proof. Since the set of tangents to each U_i cover the points of π_i, S is a cover. Since there is only one tangent line to a unital at each of its points, each line of S hits one point which is on no other line of S. Thus S is minimal. ∎

Note that each point of Σ is on one, $\sqrt{q}+1$ or $(q+1)(\sqrt{q})+1$ lines of S, and that each line of S has exactly one point on no other lines of S. We also note that all lines of S intersect l; by [3], S is maximal among all covers having this property, and any cover with this property must have the structure of S for some line l. It seems possible that in fact S is maximal amongst all minimal covers of PG$(3, q)$, that is, that the bound (1) of theorem 1 can be improved to $q^{5/2} + q^{3/2} + 1$.

4 Remarks

The relevant property of unitals used in the above proof is that the set of tangent lines to a unital forms a dual reduced blocking set of size $q^{3/2}+1$. In fact, an identical proof shows that if there exists a reduced blocking set of size b in PG$(2, q)$, then there exists a minimal cover of PG$(3, q)$ of size $(q+1)(b-1)+1$ (constructed using dual reduced blocking sets of size b). Except when q is a square or very small, the size of a largest reduced blocking set in PG$(2, q)$ is not known. We note that using ovals instead of unitals in the proof of theorem 4 gives, when q is even, a minimal cover of size $q^2 + q + 1$ which may or may not coincide with star(P) for some P. (If all of the ovals have the same nucleus P, the constructed set will be star(P); otherwise it will be a set of i, $2 \leq i \leq q$ plane pencils sharing a common line.)

The problem under consideration in this paper is also linked with the problem of finding semi-ovals in finite projective planes, where by *semi-oval* we mean a set of points having precisely one tangent line at each point (see [9]). Excepting conics and unitals, the only semi-ovals known seem to be the deleted triangles. A *deleted triangle* is a set of three non-concurrent lines with the points of intersection removed, a semi-oval of size $3q-3$. Using a dual deleted triangle in the construction of theorem 4 gives a minimal cover of size $(q+1)(3q-4)+1$ for all q. It can be shown that the conics and the deleted triangles are the only semiovals in PG$(2, 3)$.

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