Domains of analytic existence in inductive limits of real separable normed spaces

Murielle Mauer

Abstract

Every non void, open and convex subset of a countable inductive limit of real separable normed spaces is a domain of analytic existence

1 Introduction and statement of the result

Definitions

Let Ω be an open subset of a real locally convex space E. Let us denote by $C_\infty(\Omega)$ the set of the C_∞-functions on Ω for the strong Fréchet-differentiation (cf. [5]).

A function f defined on Ω is *analytic on* Ω if the following two conditions are fulfilled

1) $f \in C_\infty(\Omega)$,

2) for every $x_0 \in \Omega$, the equality $f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$ holds on a neighbourhood of x_0.

Let us denote by $A(\Omega)$ the set of the analytic functions on Ω.

A *domain of analyticity in* E is a non void domain Ω of E such that, for every domain Ω_1 of E verifying $\Omega_1 \not\subset \Omega \not\subset E \setminus \Omega_1$ and for every connected component Ω_0 of $\Omega \cap \Omega_1$, there is $f \in A(\Omega)$ such that $f|_{\Omega_0}$ has no analytic extension onto Ω_1.

A *domain of analytic existence in* E is a non void domain Ω of E for which there is $f \in A(\Omega)$ such that, for every domain Ω_1 of E verifying $\Omega_1 \not\subset \Omega \not\subset E \setminus \Omega_1$ and every connected component Ω_0 of $\Omega \cap \Omega_1$, $f|_{\Omega_0}$ has no analytic extension onto Ω_1.

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Of course, every domain of analytic existence is a domain of analyticity.

Results

Let us recall that, in [4], J. Schmets and M. Valdivia have obtained the following three results. The first one extends a result of [3]; the last two make use of [1] and [2].

Theorem 1.1. *For every non void domain Ω of a separable real normed space E, there is a C_∞-function f on E which is analytic on Ω and has Ω as domain of analytic existence.*

In particular, every non void domain of a separable real normed space is a domain of analytic existence.

Proposition 1.2. *Every non void, open and convex subset Ω of the real locally convex space E is a domain of analyticity.*

Example 1.3. *If A is an uncountable set, then the open unit ball of $c_{0,\mathbb{R}}(A)$ is a domain of analyticity but not a domain of analytic existence.*

In this paper, we are going to extend partially the Theorem 1.1 to the case of the countable inductive limits of separable real normed spaces. Our result reads as follows.

Proposition 1.4. *Every non void, open and convex subset Ω of the inductive limit $E = \mathrm{ind}E_m$, where all the E_m's are separable real normed spaces is a domain of analytic existence.*

In fact, if Ω is a non void, open and convex subset of E such that $0 \in \Omega$, there is a C_∞-function f on E for the semi-norm $p_{\Omega\cap(-\Omega)}$ which is analytic on Ω for the semi-norm $p_{\Omega\cap(-\Omega)}$ and has Ω as domain of analytic existence.

2 Construction of a dense subset of $\partial\Omega$

Lemma 2.1. *Let Ω be a proper open subset of the inductive limit $E = \mathrm{ind}E_m$, where all the E_m's are separable normed spaces.*

For every $m \in \mathbb{N}$ such that $\Omega \cap E_m \neq \emptyset$ and $\Omega \cap E_m \neq E_m$, there is a countable subset $\{x_{m,j} : j \in \mathbb{N}\}$ of $\partial_{E_m}(\Omega \cap E_m) \subset \partial_E\Omega$ with the following property: if Ω_1 is a domain of $E = \mathrm{ind}E_m$ such that $\Omega_1 \not\subset \Omega \not\subset E \setminus \Omega_1$ and if Ω_0 is a connected component of $\Omega_1 \cap \Omega$, then there is $m \in \mathbb{N}$ such that $\Omega_1 \cap \partial_{E_m}(\Omega_0 \cap E_m)$ contains one of the $x_{m,j}$'s.

Proof. The proof goes in two steps.

a) Construction of $\{x_{m,j} : j \in \mathbb{N}\}$.

Let us fix $m \in \mathbb{N}$ such that $\Omega \cap E_m \neq \emptyset$ and $\Omega \cap E_m \neq E_m$ and let $\{y_{m,n} : n \in \mathbb{N}\}$ be a countable dense subset of $\Omega \cap E_m$. For every $n \in \mathbb{N}$ and $k \in \mathbb{N}$, there is a point $a_{m,n,k}$ of $E_m \setminus \Omega$ such that

$$\|y_{m,n} - a_{m,n,k}\|_{E_m} \leq d_{E_m}(y_{m,n}, E_m \setminus \Omega) + \tfrac{1}{k}.$$

There is then a point $b_{m,n,k}$ of $\partial_{E_m}(\Omega \cap E_m)$ which belongs to the segment joining $y_{m,n}$ to $a_{m,n,k}$ and such that

$$\{y_{m,n} + t(b_{m,n,k} - y_{m,n}) : t \in [0, 1[\} \subset \Omega.$$

Then $\{x_{m,j} : j \in \mathbb{N}\}$ is just an ordering of the set $\{b_{m,n,k} : n \in \mathbb{N}, k \in \mathbb{N}\}$.

b) Proof of the property.

Of course, $(\partial\Omega_0) \cap \Omega_1$ is not void and contained in $\partial\Omega$. Let x be a point of $(\partial\Omega_0) \cap \Omega_1$. Belonging to Ω_1, there is a semi-ball V centered at x and contained in Ω_1. Let us then consider $z \in V \cap \Omega_0$ and an integer m such that x and z belong to E_m. Since $V \cap E_m$ is a convex hence connected subset of E_m such that $(V \cap E_m) \cap (\Omega_0 \cap E_m) \neq \emptyset$ and $(V \cap E_m) \setminus (\Omega_0 \cap E_m) \neq \emptyset$, one gets $(V \cap E_m) \cap \partial_{E_m}(\Omega_0 \cap E_m) \neq \emptyset$ and therefore $(\Omega_1 \cap E_m) \cap \partial_{E_m}(\Omega_0 \cap E_m) \neq \emptyset$. Let now y be a point of $(\Omega_1 \cap E_m) \cap \partial_{E_m}(\Omega_0 \cap E_m)$. There are then a ball $b_m(y, r)$ contained in $\Omega_1 \cap E_m$ and a point $y_{m,n}$ such that $\|y_{m,n} - y\|_{E_m} < \frac{r}{2}$. Since $y \notin \Omega$, one has $d_{E_m}(y_{m,n}, E_m \setminus \Omega) < \frac{r}{2}$, therefore $\|y_{m,n} - a_{m,n,k}\|_{E_m} < \frac{r}{2}$ for k large enough. In conclusion, one verifies directly that the corresponding $b_{m,n,k}$ belongs to $\Omega_1 \cap \partial_{E_m}(\Omega_0 \cap E_m)$. As a matter of fact, on one hand, the segment joining $y_{m,n}$ to $a_{m,n,k}$ is contained in $b_m(y, r)$ and $b_{m,n,k}$ belongs to Ω_1. On the other hand, since $S = \{y_{m,n} + t(b_{m,n,k} - y_{m,n}) : t \in [0, 1[\} \subset \Omega \cap \Omega_1$ contains the point $y_{m,n}$ of Ω_0, one gets $S \subset \Omega_0 \cap E_m$. Consequently, $b_{m,n,k}$ belongs to $\Omega_1 \cap \partial_{E_m}(\Omega_0 \cap E_m)$. ■

3 Proof of the proposition 1.4

To prove that every non void, open and convex subset Ω of E is a domain of analytic existence, we can of course assume to have $0 \in \Omega \neq E$. By the use of the lemma 2.1, we get for every $m \in \mathbb{N}$ a particular subset $\{x_{m,j} : j \in \mathbb{N}\}$ of the boundary of Ω. By a new enumeration, we let $\{x_j : j \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$ denote respectively the sets of the $x_{m,j}$'s and of the $y_{m,n}$'s. Let us now introduce a special sequence of functions φ_j.

1) Construction of the functions φ_j

If Ω is an open and convex subset of E containing 0 then $\omega = (\Omega) \cap (-\Omega)$ is an open and absolutely convex subset of E and its Minkowski gauge p_ω is a continuous semi-norm on E. Let us fix j, $n \in \mathbb{N}$. As $x_j \in \partial\Omega$ and $y_n \in \Omega$, one gets $p_\omega(x_j - y_n) > 0$. In fact, the map

$$\varphi : E \to \mathbb{R} \qquad e \mapsto \inf\{r > 0 : e \in r\Omega\}$$

is such that

$$\varphi(re) = r\varphi(e) \text{ for every } e \in E \text{ and } r \in [0, +\infty[,$$
$$\varphi(e_1 + e_2) \leq \varphi(e_1) + \varphi(e_2) \text{ for every } e_1, e_2 \in E$$

and one has $\Omega = \{e \in E : \varphi(e) < 1\}$. Therefore one gets

$$\begin{aligned} p_\omega(x_j - y_n) &= \inf\{r > 0 : x_j - y_n \in r\omega\} \\ &\geq \inf\{r > 0 : x_j - y_n \in r\Omega\} = \varphi(x_j - y_n) \\ &\geq \varphi(x_j) - \varphi(y_n) > 0. \end{aligned}$$

There is then $\omega_{j,n} \in E'$ such that

$$\langle x_j - y_n \, , \, \omega_{j,n} \rangle = p_\omega(x_j - y_n)$$

and

$$|\langle e \, , \, \omega_{j,n} \rangle| \leq p_\omega(e) \quad \text{for every } e \in E.$$

Now, for every $j \in \mathbb{N}$, we introduce the function

$$\varphi_j : E \to \mathbb{R} \qquad x \mapsto \frac{1}{p_\omega(x_j)} \sum_{k=1}^{+\infty} \frac{(\langle x_j - x \, , \, \omega_{j,k} \rangle)^2}{k!} \, .$$

For every $j \in \mathbb{N}$, φ_j is an analytic function on E for the semi-norm p_ω and takes its values in $[0, +\infty[$. Moreover, one has $\varphi_j^{(k)} = 0$ for every $k \geq 3$.

2) Properties of the functions $1/(\varphi_j + \varepsilon)$

For every $j \in \mathbb{N}$ and $\varepsilon > 0$, $1/(\varphi_j + \varepsilon)$ is an analytic function on E for the semi-norm p_ω.

a) The sequence $(k_n)_{n \in \mathbb{N}}$

Following the method used in [4], for every $n \in \mathbb{N}$, we set

$$b_n = \{x \in E : p_\omega(x - y_n) \leq \tfrac{1}{2} d_{p_\omega}(y_n, E \setminus \Omega)\}$$

(let us notice that $d_{p_\omega}(y_n, E \setminus \Omega)$ is strictly positive for every $n \in \mathbb{N}$)

$$\begin{aligned} B_n &= \cup_{j=1}^n b_j, \\ f_n &= \sup_{j \in \mathbb{N}} \sup_{x \in B_n} \|\varphi_j^{(1)}(x)\|_{p_\omega}, \\ g &= \sup_{j \in \mathbb{N}} \sup_{x \in E} \|\varphi_j^{(2)}(x)\|_{p_\omega}, \\ h_n &= \inf_{j \in \mathbb{N}} \inf_{x \in B_n} \varphi_j(x). \end{aligned}$$

One can prove very easily that for every $n \in \mathbb{N}$, $f_n \leq 2\mathrm{e}(1 + \sup_{x \in B_n} p_\omega(x))$ and $g \leq 2\mathrm{e}$. Moreover, for every $n \in \mathbb{N}$, h_n is a strictly positive real number. Indeed, we have

$$h_n \geq \inf_{j \in \mathbb{N}} \inf_{l=1,\dots,n} \inf_{x \in b_l} \frac{1}{p_\omega(x_j)} \frac{(\langle x_j - x \, , \, \omega_{j,l} \rangle)^2}{l!}$$

and to establish that $h_n > 0$, we have just to note that for every $j \in \mathbb{N}$ and $l \in \{1, \dots, n\}$ such that $p_\omega(x_j) < 2(p_\omega(y_l) + d_{p_\omega}(y_l, E \setminus \Omega)) = 2A$, one has

$$\inf_{x \in b_l} \frac{(\langle x_j - x \, , \, \omega_{j,l} \rangle)^2}{l! p_\omega(x_j)} \geq \frac{d_{p_\omega}^2(y_l, E \setminus \Omega)}{8A \; l!}$$

and for every $j \in \mathbb{N}$ and $l \in \{1, \dots, n\}$ such that $p_\omega(x_j) \geq 2A$, one has

$$\inf_{x \in b_l} \frac{(\langle x_j - x \, , \, \omega_{j,l} \rangle)^2}{l! p_\omega(x_j)} \geq \inf_{\substack{r \geq 2A \\ r \geq 1}} \frac{(r - A)^2}{l! r} \geq \inf_{\substack{r \geq 2A \\ r \geq 1}} \frac{(r - A)^2}{l! r^2} \geq \frac{1}{4 \; l!}.$$

Finally, we set $k_n = \sup\{\frac{f_n}{h_n}, \frac{g}{h_n}, 1\}$ for every $n \in \mathbb{N}$. The sequence $(k_n)_{n \in \mathbb{N}}$ is of course increasing.

b) The functions $\psi_{j,\varepsilon,m}$

For every $j \in \mathbb{N}$, $\varepsilon > 0$ and $m \in \mathbb{N}$, let us set

$$\psi_{j,\varepsilon,m} = \frac{1}{(\varphi_j + \varepsilon)^m}.$$

One gets

$$\psi^{(1)}_{j,\varepsilon,m}(x) = -m\,\psi_{j,\varepsilon,m+1}(x) \otimes \varphi^{(1)}_j(x)$$

and

$$\psi^{(p+1)}_{j,\varepsilon,m}(x) = -m\,\mathrm{sym}(\psi^{(p)}_{j,\varepsilon,m+1}(x) \otimes \varphi^{(1)}_j(x) + p\,\psi^{(p-1)}_{j,\varepsilon,m+1}(x) \otimes \varphi^{(2)}_j(x))$$

for every $p \in \mathbb{N}$.

One can also prove by induction on p that for every $j \in \mathbb{N}$, $\varepsilon > 0$, $m \in \mathbb{N}$, $p \in \mathbb{N}$ and $x \in B_n$,

$$\|\psi^{(p)}_{j,\varepsilon,m}(x)\|_{p_\omega} \leq \frac{m(m+1)...(m+p-1)}{h_n^m}(2k_n)^p.$$

In particular, for $m = 1$, that statement implies that

$$\left\| \left(\frac{1}{\varphi_j + \varepsilon}\right)^{(p)} (x) \right\|_{p_\omega} \leq \frac{p!}{h_n}(2k_n)^p$$

for every $j \in \mathbb{N}$, $\varepsilon > 0$, $p \in \mathbb{N}_0$ and $x \in B_n$.

c) By use of a similar argument, one gets the uniform boundedness of $(\frac{1}{\varphi_j+\varepsilon})^{(p)}$ on $\{x \in E : p_\omega(x) \leq n\}$ for every $j \in \mathbb{N}$, $\varepsilon > 0$, $p \in \mathbb{N}_0$ and $n \in \mathbb{N}$. In fact, by setting

$$\begin{aligned} f'_n &= \sup_{j\in\mathbb{N}} \sup_{p_\omega(x)\leq n} \|\varphi^{(1)}_j(x)\|_{p_\omega}, \\ g &= \sup_{j\in\mathbb{N}} \sup_{x\in E} \|\varphi^{(2)}_j(x)\|_{p_\omega}, \\ k'_n &= \sup\{f'_n, g, \varepsilon\}, \end{aligned}$$

one gets

$$\|\psi^{(p)}_{j,\varepsilon,m}(x)\|_{p_\omega} \leq \frac{m(m+1)...(m+p-1)}{\varepsilon^m}\left(\frac{2k'_n}{\varepsilon}\right)^p$$

for every $j \in \mathbb{N}$, $\varepsilon > 0$, $m \in \mathbb{N}$, $p \in \mathbb{N}$.

Therefore, by taking the value $m = 1$, one gets

$$\left\| \left(\frac{1}{\varphi_j + \varepsilon}\right)^{(p)} (x) \right\|_{p_\omega} \leq \frac{p!}{\varepsilon}\left(\frac{2k'_n}{g\varepsilon}\right)^p$$

for every $j \in \mathbb{N}$, $\varepsilon > 0$, $p \in \mathbb{N}_0$ and x such that $p_\omega(x) \leq n$.

d) Value of the functions $1/(\varphi_j + \varepsilon)$ and of its differentials at x_j

One proves by induction on p that for every $p \in \mathbb{N}$,

$$\psi^{(2p-1)}_{j,\varepsilon,m}(x_j) = 0$$

$$\psi^{(2p)}_{j,\varepsilon,m}(x_j) = (-1)^p m(m+1)...(m+p-1)\frac{1}{\varepsilon^{m+p}}(2p-1)(2p-3)...1\bigotimes_{k=1}^{p}\varphi_j^{(2)}(x_j).$$

For $m = 1$, one gets

$$\left(\frac{1}{\varphi_j+\varepsilon}\right)^{(2p-1)}(x_j) = 0$$

and

$$\left(\frac{1}{\varphi_j+\varepsilon}\right)^{(2p)}(x_j) = \frac{(-1)^p}{\varepsilon}\frac{(2p)!}{2^p\varepsilon^p}\bigotimes_{j=1}^{p}\varphi_j^{(2)}(x_j)$$

for every $p \in \mathbb{N}$.

e) Let us finally prove that for every $j \in \mathbb{N}$, there is an integer m_j such that for every $\varepsilon > 0$, $p \in \mathbb{N}$ and $m \geq m_j$, one has

$$\left\|\left(\left.\frac{1}{\varphi_j+\varepsilon}\right|_{E_m}\right)^{(2p)}(x_j)\right\|_{L(^{2p}E_m,\mathbb{R})} = \frac{1}{\varepsilon}\frac{(2p)!}{2^p\varepsilon^p}(A_{j,m})^p$$

with $A_{j,m} = \|(\varphi_j|_{E_m})^{(2)}(x_j)\|_{L(^2E_m,\mathbb{R})} > 0$ and $x_j \in \partial_{E_m}(\Omega \cap E_m)$.

Since the function $1/(\varphi_j+\varepsilon)$ is C_∞ on E for the semi-norm p_ω, one gets

$$\left.\frac{1}{\varphi_j+\varepsilon}\right|_{E_m} \in C_\infty(E_m,\mathbb{R}),\ \ \forall m \in \mathbb{N},$$

and

$$\begin{aligned}&\left(\left.\frac{1}{\varphi_j+\varepsilon}\right|_{E_m}\right)^{k}(x_0)(e_1,...,e_k)\\ &= \left(\frac{1}{\varphi_j+\varepsilon}\right)^{(k)}(x_0)(e_1,...,e_k)\ \ \forall k \in \mathbb{N}_0, x_0, e_1,...,e_k \in E_m.\end{aligned}$$

Then the statement here above follows immediately if one proves that for every $j \in \mathbb{N}$, there is an integer m_j such that $x_j \in E_{m_j}$ and $A_{j,m_j} > 0$.

If it is not the case, for every m such that $x_j \in E_m$, one has $\varphi_j^{(2)}|_{E_m}(x_j) = 0$ therefore $\varphi_j^{(2)}(x_j) = 0$ and consequently $\varphi_j^{(2)} = 0$. That leads to $\varphi_j^{(1)} = 0$ and thus φ_j has to be constant which is contradictory.

Then for every $m \geq m_j$, one of course has $x_j \in E_m$ and $A_{j,m} \geq A_{j,m_j} > 0$. Finally, since Ω is convex, $x_j \in E_m$ and $x_j \in \partial_E\Omega$ imply $x_j \in \partial_{E_m}(\Omega \cap E_m)$ (one proves that last fact by using the equality $\partial_E\Omega = \{e \in E : \varphi(e) = 1\}$).

3) The space $AC_{\infty,p_\omega}(\Omega, E)$

Let $C_{\infty,p_\omega}(E)$ be the space of the functions which are C_∞ on E for the semi-norm p_ω and $A_{p_\omega}(\Omega)$ the space of the functions which are analytic on Ω for the semi-norm

p_ω. A function f belongs to $AC_{\infty,p_\omega}(\Omega,E)$ if $f\in C_{\infty,p_\omega}(E)\cap A_{p_\omega}(\Omega)$ and is such that

$$p_n(f)=\sum_{j=0}^{n}\|f^{(j)}(x)\|_{p_\omega,\{x\in E:p_\omega(x)\le n\}}+\sum_{j=1}^{n}\sup_{l\in\mathbb{N}_0}\frac{\|f^{(l)}(x)\|_{p_\omega,B_j}}{l!(2k_j)^l}<+\infty$$

for every $n\in\mathbb{N}$.

One verifies directly that, for $P=\{p_n:n\in\mathbb{N}\}$, $(AC_{\infty,p_\omega}(\Omega,E),P)$ is a Fréchet space hence a Baire space.

In addition, $\frac{1}{\varphi_j+\varepsilon}\in AC_{\infty,p_\omega}(\Omega,E)$ for every $j\in\mathbb{N}$ and $\varepsilon>0$.

4) The closed subsets $A_{j,q,r,k}$

For every $j,q,r,k\in\mathbb{N}$, we set

$$\begin{aligned}&A_{j,q,r,k}\\ &=\ \left\{f\in AC_{\infty,p_\omega}(\Omega,E):\sup_{l\in\mathbb{N}}\frac{\|(f|_{E_{m_j}})^{(l)}(x)\|_{m_j,\{x\in\Omega\cap E_{m_j}:\|x-x_j\|_{m_j}<\frac{1}{q}\}}}{l!k^l}\le r\right\}.\end{aligned}$$

These sets $A_{j,q,r,k}$ are countably many, closed in $AC_{\infty,p_\omega}(\Omega,E)$ and have empty interior in $AC_{\infty,p_\omega}(\Omega,E)$ (because $A_{j,q,r,k}$ is absolutely convex and may not contain any multiple of $\frac{1}{\varphi_j+\varepsilon}$ for ε small enough).

5) The function f

Since $AC_{\infty,p_\omega}(\Omega,E)$ is a Fréchet space thus a Baire space and since the sets $A_{j,q,r,k}$ are countably many, closed and have empty interior, the Baire category theorem provides a function $f\in AC_{\infty,p_\omega}(\Omega,E)$ which does not belong to any of the $A_{j,q,r,k}$'s. That function f is C_∞ on E for the semi-norm p_ω. Let us prove that f has Ω as domain of analytic existence. Let Ω_1 be a domain of $E=\mathrm{ind}E_m$ such that $\Omega_1\not\subset\Omega\not\subset E\setminus\Omega_1$ and let Ω_0 be a connected component of $\Omega\cap\Omega_1$. By use of the lemma 2.1, $\Omega_1\cap(\partial\Omega_0)$ contains one of the x_j's and we know that there is $m\in\mathbb{N}$ such that $x_j\in\partial_{E_m}(\Omega_0\cap E_m)\cap\Omega_1$. First of all, since $x_j\in E_{m_j}\cap\Omega_1$, then for q large enough, $\{x\in\Omega\cap E_{m_j}:\|x-x_j\|_{m_j}<\frac{1}{q}\}$ is contained in $\Omega\cap\Omega_1\cap E_{m_j}$ thus in Ω_0. Let us now prove by contradiction that $f|_{\Omega_0}$ has no analytic extension onto Ω_1. Suppose that g is an analytic extension of $f|_{\Omega_0}$ onto Ω_1. The restriction of g to the open subset $\Omega_1\cap E_{m_j}$ of E_{m_j} is of course analytic on $\Omega_1\cap E_{m_j}$. Then, there is a ball $b_{m_j}(x_j,r)$ contained in $\Omega_1\cap E_{m_j}$ and a constant $C>0$ such that

$$\sup_{l\in\mathbb{N}_0}\frac{\|(g|_{\Omega_1\cap E_{m_j}})^{(l)}(x)\|_{m_j,b_{m_j}(x_j,r)}}{l!C^l}<+\infty.$$

However $f\notin A_{j,q,r,k}$ for any q,r,$k\in\mathbb{N}$ and this finally leads to a contradiction because one gets at the same time

$$\sup_{l\in\mathbb{N}_0}\frac{\|(f|_{E_{m_j}})^{(l)}(x)\|_{m_j,b_{m_j}(x_j,r)\cap\Omega_0}}{l!C^l}<+\infty$$

and

$$\sup_{l\in\mathbb{N}_0}\frac{\|(f|_{E_{m_j}})^{(l)}(x)\|_{m_j,\{x\in\Omega\cap E_{m_j}:\|x-x_j\|_{m_j}<\frac{1}{q}\}}}{l!k^l}=+\infty$$

for every $k,q\in\mathbb{N}$. Hence the conclusion.

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Institut de Mathématique
Université de Liège
Bâtiment B37, Sart-Tilman
B-4000 LIEGE
BELGIUM