Classification of Riemannian 3-manifolds with distinct constant principal Ricci curvatures *

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Abstract

We prove that the local isometry classes of Riemannian 3-manifolds with distinct constant Ricci eigenvalues are parametrized by three arbitrary functions of two variables. This improves essentially the earlier result by A.Spiro and F.Tricerri from [9].

1 Introduction

The problem of how many Riemannian metrics exist on the open domains of $\boldsymbol{R}^3$ with prescribed constant Ricci eigenvalues $\rho_1 = \rho_2 \neq \rho_3$ was completely solved in the series of papers [3],[2] and [7]. The main existence theorem says that the local isometry classes of these metrics are always parametrized by *two arbitrary functions of one variable*. Some nontrivial explicit examples are presented in [3], as well.

The case of distinct constant Ricci eigenvalues is more interesting. Here the first nontrivial examples have been presented by K.Yamato [10], and some others in [4]. Finally, in [5], nontrivial explicit examples have been constructed for *every choice* of the Ricci eigenvalues $\rho_1 > \rho_2 > \rho_3$. (All examples in [10] are *complete* Riemannian manifolds but the range of the admissible triplets of Ricci eigenvalues is restricted by certain algebraic inequalities. Outside this range it seems that the corresponding metrics must always be incomplete.) In [6] an *explicit classification* was done under some additional geometric conditions.

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The problem of *how many* local isometry classes of solutions exist has remained open until recently. In [9] the authors have shown that, in the real analytic case, the local isometry classes depend on an *infinite number of parameters*. Their method is based on the theory of formally integrable analytic differential systems.

In the present paper we give a definitive solution of the existence problem for the real analytic case. Only elementary techniques are used, i.e. the calculus of exterior differential forms and the Cauchy-Kowalewski Theorem. In fact, the basic partial differential equations for the problem have been derived already in [5] but the manipulation with the integrability conditions appeared to be cumbersome. The present authors now succeeded to settle these equations using computer assistance (and "Maple V", (C) Waterloo Maple Software) for the necessary routine manipulations.

For more information about the background and related problems (and for more references) see Introduction in [9] and especially [1].

2 The basic system of PDE for the problem

In this section we recall the basic preparatory results from [5] (omitting routine computational details) and we draw some simple consequences of them.

We assume here that (M, g) is a Riemannian 3-manifold of class C^∞ with distinct constant Ricci eigenvalues ρ_1, ρ_2, ρ_3. Choose an open domain $U \subset M$ and a smooth orthonormal moving frame $\{E_1, E_2, E_3\}$ consisting of the corresponding Ricci eigenvectors at each point of U. Denoting by R_{ijkl} and R_{ij} the corresponding components of the curvature tensor and the Ricci tensor respectively, we obtain

$$\begin{aligned} &R_{ii} = \rho_i \ (i = 1, 2, 3), \qquad R_{ij} = 0 \text{ for } i \neq j, && (1)\\ &R_{1212} = \lambda_3,\ R_{1313} = \lambda_2,\ R_{2323} = \lambda_1, \text{ where } \lambda_i \text{ are constants}, && (2)\\ &R_{ijkl} = 0 \text{ if at least three indices are distinct.} \end{aligned}$$

Moreover, the numbers λ_i are connected with the numbers ρ_i as follows:

$$\lambda_i - \lambda_j = -(\rho_i - \rho_j),\ i, j = 1, 2, 3. \tag{3}$$

In a neighborhood U_m of any point $m \in U$ one can construct a local coordinate system (w, x, y) such that

$$E_3 = \frac{\partial}{\partial y} \qquad \text{on } U_m. \tag{4}$$

Consider the orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ which is dual to $\{E_1, E_2, E_3\}$. Then the coordinate expression of the coframe $\{\omega^1, \omega^2, \omega^3\}$ in U_m must be of the form

$$\begin{aligned} \omega^1 &= Adw + Bdx \\ \omega^2 &= Cdw + Ddx \\ \omega^3 &= dy + Gdw + Hdx \end{aligned} \tag{5}$$

where A, B, C, D, G, H are unknown functions to be determined.

Now, we shall compute the components ω^i_j of the connection form. These are determined by the standard formulas

$$d\omega^i + \sum \omega^i_j \wedge \omega^j = 0,\ \omega^i_j + \omega^j_i = 0,\ i,j = 1,2,3\ . \tag{6}$$

We put

$$\omega^i_j = \sum_k a^i_{jk}\omega^k\ . \tag{7}$$

The components Ω^i_j of the curvature form are determined by the standard formula

$$\Omega^i_j = d\omega^i_j + \sum \omega^i_k \wedge \omega^k_j\ . \tag{8}$$

From (2) we obtain at once

$$\begin{aligned} d\omega^1_2 + \omega^1_3 \wedge \omega^3_2 &= \lambda_3\omega^1 \wedge \omega^2\ ,\\ d\omega^1_3 + \omega^1_2 \wedge \omega^2_3 &= \lambda_2\omega^1 \wedge \omega^3\ ,\\ d\omega^2_3 + \omega^2_1 \wedge \omega^1_3 &= \lambda_1\omega^2 \wedge \omega^3\ . \end{aligned} \tag{9}$$

Differentiating (9) and substituting (9) and (6) in the new equations, we obtain

$$\begin{aligned} (\lambda_1-\lambda_3)\omega^2\wedge\omega^3\wedge\omega^1_3 + (\lambda_3-\lambda_2)\omega^1\wedge\omega^3\wedge\omega^2_3 &= 0\ ,\\ (\lambda_3-\lambda_2)\omega^1\wedge\omega^2\wedge\omega^2_3 + (\lambda_2-\lambda_1)\omega^2\wedge\omega^3\wedge\omega^1_2 &= 0\ ,\\ (\lambda_2-\lambda_1)\omega^1\wedge\omega^3\wedge\omega^1_2 + (\lambda_1-\lambda_3)\omega^1\wedge\omega^2\wedge\omega^1_3 &= 0\ . \end{aligned} \tag{10}$$

Using the notation (7) we obtain, more explicitly,

$$\begin{aligned} (\lambda_1-\lambda_3)a^1_{31} + (\lambda_3-\lambda_2)(-a^2_{32}) &= 0\ ,\\ (\lambda_3-\lambda_2)a^2_{33} + (\lambda_2-\lambda_1)a^1_{21} &= 0\ ,\\ (\lambda_2-\lambda_1)(-a^1_{22}) + (\lambda_1-\lambda_3)a^1_{33} &= 0\ . \end{aligned} \tag{11}$$

Putting

$$\alpha = \frac{\lambda_1-\lambda_3}{\lambda_3-\lambda_2} = \frac{\rho_1-\rho_3}{\rho_3-\rho_2} \tag{12}$$

(where obviously $\alpha \neq 0, -1$), we get (11) in the unified form

$$a^2_{32} = \alpha a^1_{31},\ a^2_{33} = (\alpha+1)a^1_{21},\ a^1_{33} = -\Big(\frac{\alpha+1}{\alpha}\Big)a^1_{22}\ . \tag{13}$$

Now, we shall calculate the coefficients a^i_{jk} using only (5) and (6). First we introduce new functions $\mathcal{D}, \mathcal{E}, \mathcal{F}$ (where $\mathcal{D} \neq 0$) by

$$\mathcal{D} = AD - BC,\ \mathcal{E} = AH - BG,\ \mathcal{F} = CH - DG\ . \tag{14}$$

We also define a bracket of two functions f, g by

$$[f,g] = f'_y g - f g'_y\ . \tag{15}$$

Then we obtain, by a routine calculation

$$a^1_{21} = \frac{1}{\mathcal{D}}(GB'_y - HA'_y + A'_x - B'_w),\ a^1_{31} = \frac{1}{\mathcal{D}}(DA'_y - CB'_y), \tag{16}$$

$$a^1_{22} = \frac{1}{\mathcal{D}}(GD'_y - HC'_y + C'_x - D'_w),\ a^2_{32} = \frac{1}{\mathcal{D}}(AD'_y - BC'_y), \tag{17}$$

$$a^1_{33} = \frac{1}{\mathcal{D}}(DG'_y - CH'_y),\ a^2_{33} = \frac{1}{\mathcal{D}}(AH'_y - BG'_y), \tag{18}$$

$$\begin{aligned} a^1_{23} &= \frac{1}{2\mathcal{D}}\{[C,D] + [A,B] - [G,H] + (G'_x - H'_w)\}\ , \\ a^2_{31} &= \frac{1}{2\mathcal{D}}\{[C,D] - [A,B] + [G,H] - (G'_x - H'_w)\}\ , \\ a^1_{32} &= \frac{1}{2\mathcal{D}}\{[C,D] - [A,B] - [G,H] + (G'_x - H'_w)\}\ . \end{aligned} \tag{19}$$

(In [5], there is a sign misprint in the last formula.)
Due to (13), we have only six basic coefficient functions, namely

$$a^1_{31},\ a^1_{21},\ a^1_{22},\ a^1_{23},\ a^2_{31},\ a^1_{32}\ .$$

For the abbreviation, we put

$$p = a^1_{23},\ q = a^2_{31},\ r = a^1_{32},\ s = a^1_{22},\ t = a^1_{21},\ u = a^1_{31}\ . \tag{20}$$

Now, taking into account the formulas (13), we can re-write (16)-(19) as a system of partial differential equations

$$\begin{aligned} A'_y &= Au + C(r-p)\ , \\ B'_y &= Bu + D(r-p)\ , \\ C'_y &= A(p+q) + \alpha Cu\ , \\ D'_y &= B(p+q) + \alpha Du\ , \\ G'_y &= (\alpha+1)Ct - \frac{\alpha+1}{\alpha}As\ , \\ H'_y &= (\alpha+1)Dv - \frac{\alpha+1}{\alpha}Bs\ . \end{aligned} \tag{21}$$

$$\begin{aligned} A'_x - B'_w &= \mathcal{D}t + \mathcal{E}u + \mathcal{F}(r-p)\ , \\ C'_x - D'_w &= \mathcal{D}s + \mathcal{E}(p+q) + \alpha\mathcal{F}u\ , \\ G'_x - H'_w &= \mathcal{D}(r-q) - \frac{\alpha+1}{\alpha}\mathcal{E}s + (\alpha+1)\mathcal{F}t\ . \end{aligned} \tag{22}$$

Next, we express explicitly the conditions (9) for the curvature components. After lengthy but routine calculations we obtain the following system of partial differential equations (which is again re-arranged in two parts and in which the formulas (13) are used):

$$\begin{aligned}
&Aq'_y + \alpha Cu'_y + (\alpha+1)Gt'_y - (\alpha+1)t'_w - AV_1 - C(W_1-\lambda_1) = 0 \ ,\\
&Bq'_y + \alpha Du'_y + (\alpha+1)Ht'_y - (\alpha+1)t'_x - BV_1 - D(W_1-\lambda_1) = 0 \ ,\\
&Au'_y + Cr'_y - \frac{\alpha+1}{\alpha}Gs'_y + \frac{\alpha+1}{\alpha}s'_w - A(V_2-\lambda_2) - CW_2 = 0 \ ,\\
&Bu'_y + Dr'_y - \frac{\alpha+1}{\alpha}Hs'_y + \frac{\alpha+1}{\alpha}s'_x - B(V_2-\lambda_2) - DW_2 = 0 \ ,\\
&At'_y + Cs'_y + Gp'_y - p'_w - AV_3 - CW_3 = 0 \ ,\\
&Bt'_y + Ds'_y + Hp'_y - p'_x - BV_3 - CW_3 = 0 \ ;
\end{aligned} \tag{23}$$

$$\begin{aligned}
Aq'_x - Bq'_w + \alpha Cu'_x - \alpha Du'_w + (\alpha+1)Gt'_x - (\alpha+1)Ht'_w&\\
-\mathcal{D}U_1 - \mathcal{E}V_1 - \mathcal{F}(W_1-\lambda_1) = 0 \ ,&\\
Au'_x - Bu'_w + Cr'_x - Dr'_w - \frac{\alpha+1}{\alpha}Gs'_x + \frac{\alpha+1}{\alpha}Hs'_w&\\
-\mathcal{D}U_2 - \mathcal{E}(V_2-\lambda_2) - \mathcal{F}W_2 = 0 \ ,&\\
At'_x - Bt'_w + Cs'_x - Ds'_w + Gp'_x - Hp'_w - \mathcal{D}(U_3-\lambda_3) - \mathcal{E}V_3 - \mathcal{F}W_3 = 0 \ .&
\end{aligned} \tag{24}$$

Here U_i, V_i, W_i $(i = 1, 2, 3)$ are auxiliary functions defined by

$$\begin{aligned}
U_1 &= \alpha tq - (\alpha-1)su - (\alpha+2)tr \ ,\\
V_1 &= \frac{(\alpha+1)(\alpha+2)}{\alpha}ts - (\alpha+1)uq - (\alpha-1)up \ ,\\
W_1 &= \frac{\alpha+1}{\alpha}s^2 - (\alpha+1)^2t^2 - \alpha^2u^2 + pq - rq + rp \ ,\\
U_2 &= \frac{1}{\alpha}sr + (\alpha-1)tu - \frac{2\alpha+1}{\alpha}sq \ ,\\
V_2 &= (\alpha+1)t^2 - u^2 - \frac{(\alpha+1)^2}{\alpha^2}s^2 - pq - pr - qr \ ,\\
W_2 &= (1-\alpha)pu - (\alpha+1)ru + \frac{(2\alpha+1)(\alpha+1)}{\alpha}st \ ,\\
U_3 &= -t^2 - s^2 - \alpha u^2 + pq + qr - pr \ ,\\
V_3 &= \frac{1}{\alpha}sp - (\alpha+2)tu - \frac{2\alpha+1}{\alpha}sq \ ,\\
W_3 &= -\alpha tp - (\alpha+2)tr - (2\alpha+1)su \ .
\end{aligned} \tag{25}$$

Now we want to solve the system (23) explicitly with respect to the $y-$derivatives of the functions $p, q, \ldots, t$. For this purpose and also for the aim of the next section, we shall prove

Lemma 1. *In some neighborhood $V_m \subset U_m$ there is a system of local coordinates $(\overline{w}, \overline{x}, \overline{y})$ such that*

a) the orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ preserves the form (5) ,

b) $\overline{w}(m) = \overline{x}(m) = \overline{y}(m) = 0$,

c) $\overline{\mathcal{D}}\ \overline{\mathcal{E}}\ \overline{\mathcal{F}} \neq 0$ in V_m .

Proof. We define the new local coordinates in U_m by the formulas $\overline{w} = w - w(m)$, $\overline{x} = x - x(m)$, $\overline{y} = y - \Phi(w,x)$, where Φ is a smooth function such that $y(m) - \Phi(w(m), x(m)) = 0$. Then (5) preserves its form in the new coordinates and we get

$$\overline{A} = A,\ \overline{B} = B,\ \overline{C} = C,\ \overline{D} = D,\ \overline{G} = G + \Phi'_w,\ \overline{H} = H + \Phi'_x\ .$$

Hence $\overline{\mathcal{E}} = \mathcal{E} + A\Phi'_x - B\Phi'_w$, $\overline{\mathcal{F}} = \mathcal{F} + C\Phi'_x - D\Phi'_w$. We see that, unless Φ satisfies a specific partial differential equation, $\overline{\mathcal{E}\mathcal{F}} \neq 0$ in a neighborhood V_m and $\overline{\mathcal{D}} = \mathcal{D} \neq 0$ also holds. This completes the proof. ∎

Thus, in the following we can always suppose that our local coordinates satisfy the conditions of Lemma 1. Now routine calculations using the Cramer's rule and the first equation of (24) show easily that the system (23) is equivalent to the explicit system

$$\begin{aligned}
p'_y &= \frac{1}{\mathcal{F}}(Cp'_x - Dp'_w) + \frac{\mathcal{D}}{\mathcal{E}\mathcal{F}}(At'_x - Bt'_w) \\
&\quad + \frac{\mathcal{D}^2}{2(\alpha+1)\mathcal{E}\mathcal{F}}\Big\{(W_1 - \lambda_1) - \alpha(V_2 - \lambda_2) - (\alpha+1)(U_3 - \lambda_3)\Big\} - \frac{\mathcal{D}}{\mathcal{F}}V_3\ , \\
q'_y &= \frac{\alpha+1}{\mathcal{E}}(Gt'_w - Ht'_x) \\
&\quad + \frac{\mathcal{F}}{2\mathcal{E}}\Big\{(W_1 - \lambda_1) - \alpha(V_2 - \lambda_2) - (\alpha+1)(U_3 - \lambda_3)\Big\} + V_1\ , \\
r'_y &= \frac{\alpha+1}{\alpha\mathcal{F}}(Hs'_x - Gs'_w) \\
&\quad - \frac{\mathcal{E}}{2\alpha\mathcal{F}}\Big\{(W_1 - \lambda_1) - \alpha(V_2 - \lambda_2) + (\alpha+1)(U_3 - \lambda_3)\Big\} + W_2\ , \\
s'_y &= \frac{1}{\mathcal{F}}(Cs'_x - Ds'_w) - \frac{\mathcal{D}}{2(\alpha+1)\mathcal{F}}\Big\{(W_1 - \lambda_1) - \alpha(V_2 - \lambda_2) + (\alpha+1)(U_3 - \lambda_3)\Big\}\ , \\
t'_y &= \frac{1}{\mathcal{E}}(At'_x - Bt'_w) + \frac{\mathcal{D}}{2(\alpha+1)\mathcal{E}}\Big\{(W_1 - \lambda_1) - \alpha(V_2 - \lambda_2) - (\alpha+1)(U_3 - \lambda_3)\Big\}\ , \\
u'_y &= \frac{1}{2\alpha}\Big\{(W_1 - \lambda_1) + \alpha(V_2 - \lambda_2) + (\alpha+1)(U_3 - \lambda_3)\Big\}\ .
\end{aligned} \tag{26}$$

3 The main theorem

Now we want to apply the Cauchy-Kowalewski Theorem to the previous system of partial differential equations. To this aim we need some preparatory lemma. We shall limit ourselves to a special case, and we prefer an informal way of presentation. An expert can see easily how to formulate this result in the full generality and rigour, using the language of rings of real analytic functions (whose variables are chosen from a countable set of formal symbols) and their ideals.

Lemma 2. *Let*

$$f_\alpha\Big(z^\gamma, \frac{\partial z^\gamma}{\partial x_i}\Big) = 0 \qquad (\alpha = 1,\dots,a;\ i = 1,\dots,n) \tag{A}$$

$$h_\beta\Big(z^\gamma, \frac{\partial z^\gamma}{\partial x_j}\Big) = 0 \qquad (\beta = 1,\dots,b;\ j = 1,\dots,n-1) \tag{B}$$

be a system of $a+b$ partial differential equations of 1st order for c unknown functions $z^1, \ldots, z^c$ of n variables $x_1, \ldots, x_n$, where f_α and h_β are real analytic functions of the corresponding variables. Assume that each differential equation

$$\frac{\partial}{\partial x_n} h_\beta(z^\gamma, \frac{\partial z^\gamma}{\partial x_j}) = 0 \qquad (\beta = 1, \ldots, b)$$

is an algebraic consequence of the equations $f_\alpha = 0$, their first partial derivatives and the equations $h_\beta = 0$.

Let $\{z^\gamma = P^\gamma(x_1, \ldots, x_n),\ \gamma = 1, \ldots, c\}$ be a real analytic solution of the subsystem (A) defined on an open neighborhood $\Omega \subset \boldsymbol{R}^n[x_1, \ldots, x_n]$ of the origin. Suppose that the functions $\overline{P}^\gamma(x_1, \ldots, x_{n-1}) = P^\gamma(x_1, \ldots, x_{n-1}, 0)$ satisfy the subsystem (B) in a neighborhood of the origin in $\boldsymbol{R}^{n-1}[x_1, \ldots, x_{n-1}]$. Then the functions $P^\gamma(x_1, \ldots, x_n)$ satisfy the subsystem (B) in a neighborhood $\Omega' \subset \Omega$ of the origin in $\boldsymbol{R}^n$.

Proof. By the induction we see that each equation

$$\frac{\partial^k}{(\partial x_n)^k} h_\beta(z^\gamma, \frac{\partial z^\gamma}{\partial x_j}) = 0 \qquad (\beta = 1, \ldots, b) \tag{27}$$

is an algebraic consequence of the equations $f_\alpha = 0$, their partial derivatives up to order k, and of the equations $h_\beta = 0$. This means that all equations (27) will be satisfied if we substitute first $z^\gamma = P^\gamma(x_1, \ldots, x_n)$ and then $x_n = 0$. In other words, we have

$$\left.\frac{\partial^k}{(\partial x_n)^k}\right|_{x_n=0} h_\beta(P^\gamma, \frac{\partial P^\gamma}{\partial x_j}) = 0,\ \ \gamma = 1, \ldots, b, \tag{28}$$

for all integers $k \geq 0$. Because the functions $h_\beta(P^\gamma, \dfrac{\partial P^\gamma}{\partial x_j})$ are real analytic, we get $h_\beta(P^\gamma, \dfrac{\partial P^\gamma}{\partial x_j}) = 0$ for all β in a neighborhood of the origin in $\boldsymbol{R}^n$. ■

From now on, we assume our Riemannian 3-manifold (M, g) to be real analytic. We see easily that all calculations and constructions from the previous section are still valid inside the category C^ω. We consider a neighborhood $V_m \subset M$ with a local coordinate system (w, x, y) satisfying the conditions of Lemma 1. We are going to prove

Theorem 1. *The general solution of the system of partial differential equations (21)-(24) depends on six arbitrary functions of two variables and six arbitrary functions of one variable.*

Proof. The functions z^γ from Lemma 2 will be chosen as our functions $A, \ldots, H$, $p, \ldots, u$ and the independent variables will be $x_1 = w,\ x_2 = x,\ x_3 = y$. We see that the subsystem (21)+(23) is of the type (A) and the subsystem (22)+(24) is of the type (B). (Here $a = 12, b = 6$.) Now, let us consider (22)+(24) as a system of PDE for the functions of *two* variables w, x, which will be denoted as $A_0, B_0, \ldots, t_0, u_0$, respectively. Choose $B_0, D_0, H_0, p_0, r_0, s_0$ as arbitrary analytic functions of w, x. Then (22) and (24) give a system of six PDE for the six unknown functions A_0, C_0, G_0, q_0, t_0

and u_0, which can be expressed explicitly with respect to the corresponding x-derivatives and we can use the Cauchy-Kowalewski Theorem. (For the correctness it sufices to choose the initial condition $A_0(w,0) \neq 0$.) The general solution then involves, in addition, six arbitrary functions of one variable w.

Next, consider the partial differential equations (21) and (26) (which are equivalent to (21) and (23)). Then the Cauchy-Kowalewski Theorem implies that there is a unique solution $(A, B, \dots, t, u)$ in a neigborhood of the origin in $\boldsymbol{R}^3$ such that

$$A(w,x,0) = A_0(w,x),\ B(w,x,0) = B_0(w,x),\ \dots,\ u(w,x,0) = u_0(w,x)$$

in a neighborhood of the origin in $\boldsymbol{R}^2[w,x]$. The initial conditions are given, of course, as solutions of the system (22)+(24).

As the final step, we have to prove that this solution satifies also the equations (22)+(24). Here one can show by lengthy but routine calculations that these equations differentiated with respect to the variable y are algebraic consequences of the equations (21)+(23) (or (21)+(26)), their derivatives with respect to w and x, and of the equations (22)+(24). (Here the calculations by hand were realized for the subsystem (22) and a computer with the sofware package "Maple V, rel.3.0", (C) Waterloo Maple Software, was used for more cumbersome subsystem (24)). Hence we can apply Lemma 2 and the proof of the Theorem 1 is completed. ∎

We shall now formulate our main result.

Theorem 2. *The isometry classes of germs of three-dimensional (real analytic) Riemannian metrics with prescribed distinct constant Ricci eigenvalues are parametrized by triplets of germs of arbitrary (real analytic) functions of two variables.*

Proof. Let $(M,g), (\overline{M},\overline{g})$ be two real analytic Riemannian 3-manifolds with the same constant Ricci eigenvalues $\rho_1 > \rho_2 > \rho_3$. Let $F : U \to \overline{U}$ be an isometry between two open domains of M and $\overline{M}$, respectively. We construct the "Ricci adapted" orthonormal coframes $\{\omega^i\}, \{\overline{\omega}^i\}$ and the local coordinate systems $(w,x,y), (\overline{w},\overline{x},\overline{y})$ in the neighborhoods $U_m \subset U$ and $\overline{U}_{F(m)} = F(U_m) \subset \overline{U}$, respectively, such that g and $\overline{g}$ are both of the form (5). We obviously have

$$F^*(\overline{\omega}^i) = \epsilon_i \omega^i, \qquad \epsilon_i \in \{-1,1\}, \qquad i = 1,2,3\ . \tag{29}$$

Hence we see easily that the corresponding parametrization of F in local coordinates must be of the form

$$\overline{w} = \Phi_1(w,x), \qquad \overline{x} = \Phi_2(w,x), \qquad \overline{y} = \epsilon y + \Phi_3(w,x) \tag{30}$$

where $\epsilon = \pm 1$ and $\Phi_i(w,x)$ are arbitrary (real analytic) functions of two variables. Conversely, every local transformation F of the form (30) determines a local isometry which preserves the formulas (5) via (29). The result now follows from Theorem 1. ∎

Let us notice that we neglect here six arbitrary functions of one variable. This is completely justified by the following example: consider a system of partial differential equations of the form

$$\frac{\partial^2}{(\partial x)^2} A_j(w,x) = B_j(w,x), \qquad j = 1,2,3\ , \tag{31}$$

for six unknown functions. A person "A" can say that the general solution of (31) depends on three arbitrary functions A_j of two variables. On the other hand, a person "B" can say, with the same legitimacy, that the general solution depends on three arbitrary functions B_j of two variables *and* six additional functions of one variable w.

Remark. A Riemannian manifold (M, g) is said to be *curvature homogeneous* if, for any pair of points p and q of M, there is a linear isometry $F : T_pM \to T_qM$ between the corresponding tangent spaces such that $F^*R_q = R_p$ (where R denotes the curvature tensor of type $(0, 4)$). I.M.Singer in 1960 (see [8]) asked the question whether there exist curvature homogeneous spaces which are not locally homogeneous. The first example was constructed by K.Sekigawa in 1973 (cf.[5],[6] and [1] for more details, futher development and references). In dimension three, a Riemannian manifold is curvature homogeneous if and only if it has constant Ricci eigenvalues. Theorem 2 then implies that *the local isometry classes of curvature homogeneous Riemannian spaces of dimension 3 are parametrized by 3 arbitrary functions of two variables.* It was not known, until now, that there exists so many curvature homogeneous spaces. The class of all locally homogeneous spaces is really negligible in this context.

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