Generalized divisor problem

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The authors study the generalized divisor problem by means of the residue theorem of Cauchy and the properties of the Riemann Zeta function.

By an elementary argument similar to that used by Dirichlet in the divisor problem, Landau [7] proved that if α and β are fixed positive numbers and if $\alpha \neq \beta$, then

$$\sum_{m^\alpha n^\beta \leq x} 1 = \zeta(\beta/\alpha)x^{1/\alpha} + \zeta(\alpha/\beta)x^{1/\beta} + O(x^{1/(\alpha+\beta)}).$$

In 1952 H.E. Richert by means of the theory of Exponents Pairs (developed by J.G. van der Korput and E. Phillips) improved the above O-term (see [8] or [4] pag. 221). In 1969 E. Krätzel studied the three-dimensional problem. Besides, M.Vogts (1981) and A. Ivić (1981) got some interesting results which generalize the work of P.G. Schmidt of 1968. In 1987 A.Ivić obtained Ω-results for $\int_1^T \Delta_k^2(a_1,\ldots,a_k,x)dx$ where $\Delta_k(a_1,\ldots,a_k,x)$ is the error-term of the summatory function of $d(a_1,\ldots,a_k,n)$. Moreover, he proved that his results hold for the function

$$\zeta^{q_1}(b_1 s)\zeta^{q_2}(b_2 s)\zeta^{q_3}(b_3 s)\ldots, \quad 1 \leq b_1 < b_2 < b_3 \ldots$$

q_j, b_j being some positive integers. In 1983 E.Krätzel studied the many-dimensional problem

$$\sum_{m_1^{b_1}\ldots m_n^{b_n} \leq x} m_1^{a_1}\ldots m_n^{a_n},$$

using the properties of the Riemann Zeta-function. M. Vogts (1985) also studied that problem but using elementary methods. Here, we will analyse some sums of the above kind by means of non-elementary methods.

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Theorem. *Let $a_j > -1$, $b_j > 0$, $(j = 1, \ldots, n)$, $(n \geq 2)$ be real numbers . Suppose that the fractions $\frac{1+a_j}{b_j}$ belong to the gap*

$$(c - \frac{1}{2b_1}, c + \min\{0, \frac{1}{2b_j} - \frac{1}{2b_1}\}], j = 1, \ldots, n$$

and

$$c = \frac{1+a_1}{b_1} = \max_{1 \leq j \leq n} \frac{1+a_j}{b_j}.$$

If

$$\sum_{j=1}^{n} \frac{1}{m(b_j(c - \frac{1}{2b_1}) - a_j)} \leq 1 \tag{1}$$

where

$$m(\sigma) = \sup\{m : \int_1^T |\zeta(\sigma + it)|^m \ll T^{1+\epsilon}, \forall \epsilon > 0\}$$

then we have

$$\sum_{m_1^{b_1} \ldots m_n^{b_n} \leq x} m_1^{a_1} \ldots m_n^{a_n} = \sum \operatorname*{Res}_{s=\frac{1+a_k}{b_k}} \left(\prod_{j=1}^{n} \zeta(b_j s - a_j) \right) \frac{x^s}{s} + O(x^{c - \frac{1}{2b_1} + \epsilon}).$$

When the fractions $\frac{1+a_j}{b_j}$ are different, the point $s = \frac{1+a_j}{b_j}$ is a simple pole of $\zeta(b_j s - a_j)$ and we have

$$\sum \operatorname*{Res}_{s=\frac{1+a_k}{b_k}} \left(\prod_{j=1}^{n} \zeta(b_j s - a_j) \right) \frac{x^s}{s} = \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j \neq k}}^{n} \zeta(b_j \frac{1+a_k}{b_k} - a_j) x^{\frac{1+a_k}{b_k}}.$$

When

$$\frac{1+a_r}{b_r} = \frac{1+a_{r+1}}{b_{r+1}} = \ldots = \frac{1+a_{r+q-1}}{b_{r+q-1}}$$

$(1 \leq r < r+q-1 \leq n)$, the sum

$$\sum_{k=r}^{r+q-1} \prod_{\substack{j=1 \\ j \neq k}} \zeta(b_j \frac{1+a_k}{b_k} - a_j) x^{\frac{1+a_k}{b_k}}$$

is replaced by $x^{\frac{1+a_r}{b_r}} P_{r,q}(\log x)$ where $P_{r,q}(\log x)$ is a polynomial in $\log x$ of degree $q-1$.

Proof. The function $\prod_{j=1}^{n} \zeta(b_j s - a_j)$ can be expresed as a Dirichlet series absolutely convergent for $\sigma > c$

$$\sum_{N=1}^{\infty} \left(\sum_{m_1^{b_1} \ldots m_n^{b_n} = N} m_1^{a_1} \ldots m_n^{a_n} \right) N^{-s}.$$

By Lemma 3.12 [10] if x is the half of an odd integer we have

$$\sum_{m_1^{b_1}\dots m_n^{b_n}\le x} m_1^{a_1}\dots m_n^{a_n} = \frac{1}{2\pi i}\int_{c+\epsilon-iT}^{c+\epsilon+iT}\prod_{j=1}^{n}\zeta(b_js-a_j)\frac{x^s}{s}ds + O(\frac{x^{c+\epsilon}}{T}).$$

We will consider the rectangle R_T with vertices $c-\frac{1}{2b_1}\pm iT, c+\epsilon\pm iT$. Therefore,

$$\frac{1}{2\pi i}\int_{c+\epsilon-iT}^{c+\epsilon+iT}\prod_{j=1}^{n}\zeta(b_js-a_j)\frac{x^s}{s}ds =$$

$$=\sum \operatorname*{Res}_{s=\frac{1+a_k}{b_k}}\left(\prod_{j=1}^{n}\zeta(b_js-a_j)\right)\frac{x^s}{s}+\frac{1}{2\pi i}(I_1+I_2+I_3)$$

where I_1, I_2, I_3 are the integrals along the other three sides of the rectangle

$$I_1+I_3 \ll \int_{c-\frac{1}{2b_1}}^{c+\epsilon}\prod_{j=1}^{n}|\zeta(b_j\sigma-a_j+ib_jT)|x^\sigma T^{-1}d\sigma.$$

Let

$$\mu(\sigma)=\inf\{\xi, \zeta(\sigma+it)=O(|t|^\xi), \text{as} \quad t\to\infty\}$$

then

$$I_1+I_3 \ll \int_{c-\frac{1}{2b_1}}^{c+\epsilon} T^{\sum_{j=1}^{n}\mu(b_j\sigma-a_j)+\epsilon}T^{-1}x^\sigma d\sigma \ll$$

$$T^{-1+\sum_{j=1}^{n}\mu(b_j(c+\epsilon)-a_j)+\epsilon}x^{c+\epsilon}+T^{-1+\sum_{j=1}^{n}\mu(b_j(c-\frac{1}{2b_1})-a_j)+\epsilon}x^{c-\frac{1}{2b_1}}.$$

From hypothesis (1) we can deduce

$$-1+\sum_{j=1}^{n}\mu(b_j(c-\frac{1}{2b_1})-a_j)\le 0$$

therefore

$$I_1+I_3\ll x^{c-\frac{1}{2b_1}}T^\epsilon+\frac{x^{c+\epsilon}}{T}.$$

For the integral I_2 we have

$$I_2 \ll x^{c-\frac{1}{2b_1}}+x^{c-\frac{1}{2b_1}}\int_1^T\prod_{j=1}^{n}|\zeta(b_j(c-\frac{1}{2b_1})-a_j+b_jt)|\frac{dt}{t}\ll$$

$$\ll x^{c-\frac{1}{2b_1}}+x^{c-\frac{1}{2b_1}}\frac{1}{T}\int_1^T\prod_{j=1}^{n}|\zeta(b_j(c-\frac{1}{2b_1})-a_j+b_jt)|dt.$$

Calling

$$m_j=m(b_j(c-\frac{1}{2b_1})-a_j)$$

by the Hölder inequality and hypothesis (1) we get

$$I_2 \ll x^{c-\frac{1}{2b_1}} T^{\epsilon}.$$

Then

$$\sum_{m_1^{b_1}\dots m_n^{b_n}\le x} m_1^{a_1}\dots m_n^{a_n} = \sum \operatorname*{Res}_{s=\frac{1+a_k}{b_k}} \left(\prod_{j=1}^{n} \zeta(b_j s - a_j)\right) \frac{x^s}{s} +$$

$$+O(\frac{x^{c+\epsilon}}{T}) + O(x^{c-\frac{1}{2b_1}} T^{\epsilon}).$$

Taking $T = x^d$ for some sufficiently large $d > 0$ we obtain (2) and theorem is proved. ∎

Case A. Let $b_1 = \dots = b_n = 1$, $a_j > -1$, $0 \le a_1 - a_j < 1/2$ for $j = 1, \dots, n$. If

$$\sum_{j=1}^{n} \frac{1}{m((1/2) + a_1 - a_j)} \le 1$$

then

$$\sum_{m_1\dots m_n\le x} m_1^{a_1}\dots m_n^{a_n} = \sum \operatorname*{Res}_{s=1+a_j} \left(\prod_{j=1}^{n} \zeta(s - a_j)\right) \frac{x^s}{s} + O(x^{a_1+(1/2)+\epsilon})$$

for every $\epsilon > 0$.

For example, let $a_1 > -1$ and $a_j = a_1 - \frac{j-1}{2n}$, $(j = 2, \dots, n)$. be real numbers. For $4 \le n \le 10$, from Theorem 8.4. [2], the hypothesis (1) is verified and the above formula holds.

Case B. Let $a_1 = \dots = a_n = a > -1$ y $0 < b_1 \le \dots \le b_n$. If

$$\frac{1}{b_1}(\frac{1}{2} + a) < \frac{1+a}{b_j} \le \frac{1}{b_1}(\frac{1}{2} + a) + \frac{1}{2b_j}, j = 1, \dots, n$$

and, besides

$$\sum_{j=1}^{n} \frac{1}{m((b_j/b_1)(\frac{1}{2} + a) - a)} \le 1$$

then

$$\sum_{m_1^{b_1}\dots m_n^{b_n}\le x} (m_1\dots m_n)^a = \sum \operatorname*{Res}_{s=\frac{1+a}{b_j}} \left(\prod_{j=1}^{n} \zeta(b_j s - a)\right) \frac{x^s}{s}$$

$$+O(x^{\frac{1}{b_1}(a+(1/2)+\epsilon)}).$$

In particular if $a = 1$ and $0 < b_1 \le b_j < \frac{4b_1}{3}$ and if the condition about the moments is satisfied, the order of the O-term is $x^{\frac{3}{2b_1}+\epsilon}$ for every $\epsilon > 0$.

References

[1] A. Ivić, On the number of finite non isomorphic abelian groups in short intervals. *Math. Nachr.* 101(1981), 257-271.

[2] A. Ivić, The Riemann zeta function. *John Wiley and sons.* New York 1985.

[3] A. Ivić, The general divisor problem. *Journal of number theory,* 27 (1987), 73-91.

[4] E. Krätzel, Lattice Points. *Kluwer Academic Publishers.* Dordrech Boston London. 1988.

[5] E. Krätzel, Teilerproblem in drei dimensionen. *Math. Nachr.* 42 (1969), 275-288

[6] E. Krätzel, Divisor problems and powerful numbers. *Math. Nachr.* 114 (1983) 97-104.

[7] E. Landau, Au sujet d′une certaine expression asymptotique. *L'intermédiaire des math.* Vol. 20 (1913), p.155.

[8] H.E. Richert, "Uber die anzahl abelscher gruppen gegebener ordnung I. *Math. Zeit.* 56 (1952), 21-32.

[9] P.G.Schmidt. Zur Anzahl abelscher gruppen gegebener ordnung I. *J.Reine Angew. Math.* 229 (1968), 34-42.

[10] E.C.Titchmarsh, The theory of the Riemann zeta function. *Clarendon Press ,* Oxford (1951).

[11] M. Vogts, Teilerproblem in drei dimensionen. *Math. Nachr.* 101(1981), 243-256.

[12] M. Vogts, Many dimensional generalized divisor problems. *Math. Nachr.* 124 (1985) 103-121.

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