A Note on Tensor Products of Polar Spaces Over Finite Fields.

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Abstract

A symplectic or orthogonal space admitting a hyperbolic basis over a finite field is tensored with its Galois conjugates to obtain a symplectic or orthogonal space over a smaller field. A mapping between these spaces is defined which takes absolute points to absolute points. It is shown that caps go to caps. Combined with a result of Dye's one obtains a simple proof of a result due to Blokhuis and Moorehouse that ovoids do not exist on hyperbolic quadrics in dimension ten over a field of characteristic two.

Let $k = GF(q)$, q a prime power, and $K = GF(q^m)$ for some positive integer m. Let $V =< x_1, x_2 > \oplus < x_3, x_4 > \oplus \ldots \oplus < x_{2n-1}, x_{2n} >$ be a vector space over K. Let τ be the automorphism of K given by $\alpha^\tau = \alpha^q$ so that $< \tau > = T = Gal(K/k)$. For each $\sigma \in T$ let V^σ be a vector space with basis $x_1^\sigma, x_2^\sigma, \ldots, x_{2n}^\sigma$. Set $M = V \otimes V^\tau \otimes V^{\tau^2} \otimes \ldots \otimes V^{\tau^{m-1}}$. This is a space of dimension $(2n)^m$ over K. Let $\Im = \{1, 2, \ldots, 2n\}^m$ and for $I = (i_1, i_1, \ldots, i_m) \in \Im$, set $x_I = x_{i_1} \otimes x_{i_2}^\tau \otimes x_{i_3}^{\tau^2} \otimes \ldots \otimes x_{i_m}^{\tau^{m-1}}$. Then $B = \{x_I : I \in \Im\}$, is a basis for M.

We next define a semilinear action of τ on M as follows: For $I = (i_1, i_1, \ldots, i_m) \in \Im$, set $I^\tau = (i_{m-1}, i_0, i_1, \ldots, i_{m-2})$ and then for $a \in K, I \in \{1, 2, \ldots, 2n\}^m$ define $(ax_I)^\tau = a^\tau x_{I^\tau}$ and extend by additivity to all of M. Denote by M^T the set of all vectors of M fixed under this action. This is a vector space over k.

Proposition 1: As a vector space over k, $dim_k M^T = (2n)^m$.

Proof: Let $\Omega_1, \Omega_2, \ldots, \Omega_t$ be the orbits of T in B. Then M^T is the direct sum of the fixed points of τ in $< \Omega_i >_K$ for $i = 1, 2, \ldots, t$. Let $\Omega = \Omega_i$ for some $i, 1 \le i \le t$ and let $x = x_I$ be in Ω, assume that $< \tau^l >$ is the stablizer of x_I in T and set

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$L = K^{<\tau^l>}$. If $w \in < \Omega >_K^T$ then there is an $\alpha \in L$ such that $w = \alpha x + \alpha^\tau x^\tau + \ldots + \alpha^{\tau^{l-1}} x^{\tau^{l-1}}$. Since the stablizer of x in T is $< \tau^l >$ it follows that $card(\Omega) = m/l$. On the other hand, $dim_L(K) = l$ so that $dim_k(L) = m/l = card(\Omega) = dim_K(< \Omega >_K)$. We therefore have that $dim_k(M^T) = card(B) = dim_K(M)$.□

We now assume that V is equipped with an alternate or symmetric bilinear form γ such that the set of vectors $\{x_1, x_2, \ldots, x_{2n}\}$ is a hyperbolic basis for V with respect to γ. More precisely, we let γ: $V \times V \to K$ be a bilinear form which satisfies $\gamma(x_{2i-1}, x_{2i}) = 1$ for $i = 1, 2, \ldots, n$ and $\gamma(x_s, x_t) = 0$ for all other pairs x_s, x_t, with $s < t \in \{1, 2, \ldots, 2n\}$. Note that $\gamma(x_i, x_i) = 0$ for every i. Now for each $\sigma \in T$ define γ^σ to be a reflexive bilinear map of the same type as γ such that $\gamma^\sigma(x_i^\sigma, x_j^\sigma) = \gamma(x_i, x_j)$ for all $i, j \in \{1, 2, \ldots, 2n\}$. We may then define a bilinear form $\widehat{\gamma}$: $M \times M \to K$ as follows: let $I = (i_1, i_2, \ldots, i_m)$ and $J = (j_1, j_2, \ldots, j_m) \in \Im$, define $\widehat{\gamma}(x_I, x_J) = \prod_{l=1}^m \gamma^{\tau^{l-1}}(x_{i_l}^{\tau^{l-1}}, x_{j_l}^{\tau^{l-1})}$. Under this definition, for each $I \in \Im$ there is a unique $J \in \Im$ such that $\widehat{\gamma}(x_I, x_J) \neq 0$, namely the $J = (j_1, j_2, \ldots, j_m)$ with $j_l = i_l + 1$ if i_l is odd, and $j_l = i_l - 1$ if i_l is even. We denote this J by I'. Note that $\widehat{\gamma}(x_I, x_{I'}) = \pm 1$. Extend $\widehat{\gamma}$ to all of M by bilinearity. It then follows that for a suitable ordering of the x_I, B is a hyperbolic basis of M with respect to $\widehat{\gamma}$.

Now suppose that γ is an alternate form so that $\gamma(u, v) = -\gamma(v, u)$ for every $u, v \in V$. Then if m is even the form $\widehat{\gamma}$ is symmetric, while if m is odd, then $\widehat{\gamma}$ is alternate. In the former case, we can define a quadratic form $\widehat{Q}$ on M so that $\widehat{Q}(x_I) = 0$, $\widehat{\gamma}(x_I, x_J) = \widehat{Q}(x_I + x_J) - \widehat{Q}(x_I) - \widehat{Q}(x_J)$. When γ is symmetric, $\widehat{\gamma}$ is again symmetric and if for each $\sigma \in T$, Q^σ is the quadratic form from V^σ to K such that $Q^\sigma(\sum_{i=1}^{2n} \alpha_i x_i^\sigma) = \sum_{j=1}^n \alpha_{2j-1}\alpha_{2j}$ so that $Q^\sigma(x_i^\sigma) = 0$, and $\gamma^\sigma(x_i, x_j) = Q^\sigma(x_i + x_j) - Q^\sigma(x_i) - Q^\sigma(x_j)$, then in a similar fashion we can define a quadratic form $\widehat{Q} : M \to K$.

Lemma: I. Let $u, v \in M^T$, then $\widehat{\gamma}(u, v) \in k$. II. Assume one of the following: (a) γ is symmetric and V is equipped with a quadratic form; or (b) γ is alternate and m is even. Let $\widehat{Q} : M \to K$ be the quadratic form defined as above. Then for any $v \in M, \widehat{Q}(v) \in k$.

Proof: I. M^T is the direct sum of the spaces $< \Omega >_K^T$ taken over the orbits Ω of T in B. For an orbit Ω of T in B let $\Omega' = \{x_{I'} | x_I \in \Omega\}$. Now for any orbit Δ of T in B other than Ω, Ω' the spaces $< \Delta >_K$ and $< \Omega, \Omega' >_K$ are orthogonal with respect to $\widehat{\gamma}$. By the additivity of $\widehat{\gamma}$ it suffices to consider the case that $u \in < \Omega >_K^T$, $v \in < \Omega' >_K^T$. Let $x = x_I$ be in Ω and assume that the stablizer of x_I is $< \tau^l >$ and set $L = K^{<\tau^l>}$ the fixed field of τ^l in K. Then also $< \tau^l >$ is the stabilizer of $x' = x_{I'}$ in T. Note that $\widehat{\gamma}(x_I, x_{I'}) = \widehat{\gamma}(x_{I^{\tau^s}}, x_{(I')^{\tau^s}})$, for $0 \leq s \leq l-1$. Now a typical element of $< \Omega >_K^T$ is $u = \alpha x + \alpha^\tau x^\tau + \ldots + \alpha^{\tau^{l-1}} x^{\tau^{l-1}}$ where α is an element of L and similarly, if v is an element of $< \Omega' >_K^T$ then there is a $\beta \in L$ such that $w' = \beta x' + \beta^\tau(x') + \ldots + \beta^{\tau^{l-1}}(x')^{\tau^{l-1}}$. Then $\widehat{\gamma}(u, v) = \alpha\beta + \alpha^\tau\beta^\tau + \ldots + \alpha^{\tau^{l-1}}\beta^{\tau^{l-1}} = Tr_{L/k}(\alpha\beta)$ which is an element of k.

II. From the above it suffices to assume that $v \in < \Omega >_K^T + < \Omega' >_K^T$ and show that $\widehat{Q}(v) \in k$. There are two cases to consider: (i) $\Omega \neq \Omega'$; and (ii) $\Omega = \Omega'$.

In the case of (i) if $v = w + w'$ with $w \in < \Omega >_K^T$ and $w' \in < \Omega' >_K^{\cdot}$ then $\widehat{Q}(v) = \widehat{Q}(w + w') = \widehat{\gamma}(w, w') \in k$ by I. Thus, we may assume (ii). Then for each $x \in \Omega$ also

$x' \in \Omega$ and therefore l is even. Let $l_0 = l/2$. Then $x' = x^{\tau^{l_0}}$. Now let $w \in <\Omega>_K^T$. As remarked in I there is an $\alpha \in L$ such that $w = \alpha x + \alpha^\tau + \ldots + \alpha^{\tau^{l-1}} x^{\tau^{l-1}}$. Then $\widehat{Q}(w) = \alpha\alpha^{\tau^{l_0}} + \alpha^\tau \alpha^{\tau^{l_0+1}} + \ldots + \alpha^{\tau^{l_0-1}} \alpha^{\tau^{2l_0-1}}$. But this is clearly fixed by τ, whence is an element of k.□

In light of the lemma we can assume that the bilinear form $\gamma^T = \widehat{\gamma}|M^T \times M^T$ and the quadratic form $Q^T = \widehat{Q}|M^T$ are defined over k. Now for a vector $v = \sum_{i=1}^{2n} \alpha_i x_i \in V$, and $\sigma \in T$ define $v^\sigma = \sum_{i=1}^{2n} \alpha_i^\sigma x_i^\sigma$ an element of V^σ. This is a semilinear map from V to V^σ. For $v \in V$ set $v^T = v \otimes v^\tau \otimes \ldots \otimes v^{\tau^{m-1}}$. This is a vector in M^T. Our main results now follow:

Proposition 2: Let the hypothesis be as in the second part of the previous lemma. Then $Q^T(v^T) = N_{K/k}(Q(v))$.

Proof: Let $v = \sum_{i=1}^{2n} \alpha_i x_i$ so that $v^T =$

$$(\sum_{i=1}^{2n} \alpha_i x_i) \otimes (\sum_{i=1}^{2n} \alpha_i^\tau x_i^\tau) \otimes \ldots \otimes (\sum_{i=1}^{2n} \alpha_i^{\tau^{m-1}} x_i^{\tau^{m-1}})$$

$$= \sum \alpha_{i_1} \alpha_{i_2}^\tau \ldots \alpha_{i_m}^{\tau^{m-1}}$$

where the sum is taken over all $I = (i_1, i_2, \ldots, i_m) \in \Im$. It then follows that

$$Q^T(v^T) = \sum \alpha_{i_1} \alpha_{j_1} \alpha_{i_2}^\tau \alpha_{j_2}^\tau \ldots \alpha_{i_m}^{\tau^{m-1}} \alpha_{j_m}^{\tau^{m_1}}$$

where $J = (j_1, j_2, \ldots, j_m) = I'$ and the sum is taken over the pairs $\{I, I'\}$ from $\Im$. This is equal to

$$\sum (\alpha_{i_1} \alpha_{j_1})(\alpha_{i_2} \alpha_{j_2})^\tau \ldots (\alpha_{i_m} \alpha_{j_m})^{\tau^{m-1}}$$

$=$

$$\prod_{l=0}^{m} (\alpha_1 \alpha_2 + \alpha_3 \alpha_4 \ldots + \alpha_{2n-1} \alpha_{2n})^{\tau^l} = N_{K/k}(Q(v)).\square$$

In out next proposition we establish a similar formula for $\gamma^\tau(v^T, w^T)$.

Proposition 3: For $v, w \in V$, $\gamma^T(v^T, w^T) = N_{K/k}(\gamma(v, w))$.

Proof: Let $v = \sum_{i=1}^{2n} \alpha_i x_i$ and $w = \sum_{i=1}^{2n} \beta_i x_i$. Then

$$v^T = (\sum_{i=1}^{2n} \alpha_i x_i) \otimes (\sum_{i=1}^{2n} \alpha_i^\tau x_i^\tau) \otimes \ldots \otimes (\sum_{i=1}^{2n} \alpha_i^{\tau^{m-1}} x_i^{\tau^{m-1}})$$

and

$$w^T = (\sum_{i=1}^{2n} \beta_i x_i) \otimes (\sum_{i=1}^{2n} \beta_i^\tau x_i^\tau) \otimes \ldots \otimes (\sum_{i=1}^{2n} \beta_i^{\tau^{m-1}} x_i^{\tau^{m-1}}).$$

Then $\gamma^T(v^T, w^T) = \sum (\alpha_{i_1} \beta_{j_1})(\alpha_{i_2} \beta_{j_2})^\tau \ldots (\alpha_{i_m} \beta_{j_m})^{\tau^{m-1}}$ where, as in the previous proposition $J = (j_1, j_1, \ldots, j_m) = I'$ and the sum is taken over all pairs $\{I, I'\}$. This is equal to

$$\prod_{l=0}^{m-1} (\alpha_1 \beta_1 + \alpha_2 \beta_2 + \ldots + \alpha_{2n} \beta_{2n})^{\tau^{l-1}}$$

which is, indeed, equal to $N_{K/k}(\gamma(v,w))$ as claimed. □

Corollary: If $v, w \in V$ and $\gamma(v,w) \neq 0$, then $\gamma^T(v^T, w^T) \neq 0$.

Definition: Let V be equipped with an alternate form γ. A set of points O of $PG(V)$ (one spaces of V) is a **cap** if for all distinct $U, W \in O, \gamma(U,W) \neq 0$, that is, U, W are non-orthogonal. If V is an orthogonal space with a quadratic form Q and associated symmetric form γ then a **cap** is a set O of singular points (one spaces U of V such that $Q(U) = 0$) which are pairwise non-orthogonal with respect to γ. The bound on the cardinality of a cap in a hyperbolic orthogonal space V (i.e. an orthogonal space which has a hyperbolic basis) is $q^{n-1}+1$ (cf [K,T]). A cap in a hyperbolic orthogonal space which realizes this bound is called an **ovoid**. When $n = 3$ (dimension of $V = 6$), via the Klein correspondence, an ovoid is nothing more than an affine translation plane (see [MS]) of dimensional at most two over its kernal. Ovoids are much rarer when $n = 4$ but a number of families have been constructed (see [CKW, K, M1, M2]). It is conjectured that ovoids do not exist for $n \geq 5$. This has been proved in the case the field K has characteristic 2, 3, or 5 [BM]. From what we have shown, together with a result from [D] we can obtain a simple proof of the non-existence of ovoids on hyperbolic quadrics in $PG(2n-1, 2^m)$ for $n \geq 5$.

Theorem[BM]: Let $n \geq 5, q = 2$. Then (V, Q) does not contain an ovoid.

Proof: It suffices to prove that (V, Q) does not contain an ovoid when $n = 5$ (cf [T]). Let C be an ovoid in V. Let $D = \{< v^T > \mid < v > \in C\}$. Note D is well-defined, for if $< v > \in C$ and $\alpha \in K$ then $(\alpha v)^T = N_{GF(2^m)/GF(2)}(\alpha) v^T = v^T$. By Proposition 2, D consists of singular points, and by Proposition 3, D is a cap of M^T. By Theorem 1 (ii) of [D], $card(D) \leq dim_{GF(2)}(M^T) + 1 = (10)^m + 1$, since M^T is a hyperbolic space. On the other hand, $card(D) = card(C) = (2^m)^4 + 1 = 16^m + 1$ which is greater than $(10)^m + 1$, a contradiction. □

We can also make use of the results in [D] to prove an ovoid O in a hyperbolic space V of eight dimensions over $GF(2^m)$ must span the entire space:

Theorem[BM,T]: Let (V, Q) be an orthogonal space with hyperbolic basis $x_1, \ldots, x_8$ defined over the field $K = GF(2^m)$. Let O be an ovoid of (V, Q), then $< O >_K = V$.

Proof: Let $W = < O >_K$. The cap $O^T = \{< v^T > \mid < v > \in O\}$ in M^T has cardinality $(2^m)^3 + 1 = 8^m + 1 = dim_{GF(2)}(M^T)$. Since (M^T, Q^T) is a hyperbolic space over $GF(2)$ it follows from Theorem 1 (iv) [D] that $< O^T >_{GF(2)}$ spans M^T and therefore $< O^T >_{GF(2^m)}$ spans M. However, if W were a proper subspace of V then $< O^T >_K$ would be contained in the subspace $W \otimes W^\tau \otimes^{\tau^2} \otimes \ldots \otimes W^{\tau^{m-1}}$ which is a proper subspace of M.□

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