Quadric Representation and Clifford Minimal Hypersurfaces

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Abstract

The Clifford minimal hypersurfaces in the $(n+1)$-dimensional unit hypersphere are the only compact, minimal and non-totally geodesic spherical hypersurfaces possessing the following property: Its quadric representation is mass-symmetric in some hypersphere and minimal in some concentric hyperquadric.

1 Introduction

It is known that the class of minimal hypersurfaces in the $(n+1)$-dimensional unit hypersphere S^{n+1} is extremely large. The aim of this paper is to show that within this class those hypersurfaces possessing certain eigenvalue-behaviour of the products of coordinate functions are rigid. By a well known result of T. Takahashi [6] the coordinate functions of minimal hypersurfaces in S^{n+1} are eigenfunctions of the Laplace operator with the same eigenvalue. Consequently, in order to study the eigenvalue-behaviour of the products of coordinate functions it is natural to immerse the unit sphere by its second standard immersion.

In [5] A. Ros developed this idea in order to study compact, minimal submanifolds in S^{n+1}. He studied this problem by organizing the products of coordinate functions as a new isometric immersion into the space $SM(n+2)$ of $(n+2)\times(n+2)$ symmetric matrices, the so called quadric representation (for some details, see the next section). By using this idea M. Barros and O.J. Garay in [1] obtained a new characterization of Clifford torus among all compact, minimal and non-totally geodesic

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surfaces in S^3. They proved that: The Clifford torus is the only compact minimal and non-totally geodesic surface in S^3, whose quadric representation is mass-symmetric in some hypersphere and minimal in some concentric hyperquadric. The proof of their result is based on the study of the nodal sets associated with the coordinate functions due to S.Y. Cheng [2].

In this paper we obtain a characterization of Clifford minimal hypersurfaces in S^{n+1}. More precisely we prove the next result.

Theorem. *Let $x : M^n \to S^{n+1}$ be a compact, minimal hypersurface in the unit hypersphere S^{n+1}. Let $\varphi : M^n \to SM(n+2)$ be its quadric representation with center of mass $\varphi_0 \in SM(n+2)$. Then φ is mass-symmetric in some hypersphere and minimal in some hyperquadric of $SM(n+2)$ centered at φ_0, if and only if, either M^n is totally geodesic in S^{n+1} or M^n is a Clifford minimal hypersurface $S^p(\sqrt{\frac{p}{n}}) \times S^{n-p}(\sqrt{\frac{n-p}{n}})$ with $1 \le p < n$ and φ_0 is a matrix similar to a diagonal matrix with $(p+1)$ eigenvalues equal to $\frac{p}{n(p+1)}$ and the remaining $(n-p+1)$ eigenvalues equal to $\frac{n-p}{n(n-p+1)}$.*

2 Preliminaries

Let $SM(n+2) = \{P \in gl(n+2, R)/P = P^t\}$ be the space of the symmetric matrices of order $n+2$, where P^t denotes the transpose. We define on $SM(n+2)$ the metric $g(P, Q) = \frac{1}{2} tr\, PQ$ for all P, Q in $SM(n+2)$. Let $x : S^{n+1} \to E^{n+2}$ be the unit hypersphere centered at the origin embedded in the standard way. Regarding the vectors of E^{n+2} as column matrices in E^{n+2}, the map $f : S^{n+1} \to SM(n+2)$ given by $f(x) = xx^t$ defines an isometric immersion of S^{n+1} into $SM(n+2)$ which is actually the second standard immersion of S^{n+1}.

The normal space of the immersion f at any point x of S^{n+1} is given by

$$T_x^{\perp} S^{n+1} = \{P \in SM(n+2)/Px = \lambda x, \text{for some real } \lambda\}.$$

In particular, we have $f(x) \in T_x^{\perp} S^{n+1}$. If $\bar{\sigma}$ denotes the second fundamental form of the immersion f, we have

$$\bar{\sigma}(X, Y) = XY^t + YX^t - 2\langle X, Y\rangle f(x) \tag{1}$$

for all X, Y in $T_x S^{n+1}$, where $\langle , \rangle$ is the standard inner product in E^{n+2}. It is well known [5] that $\bar{\sigma}$ is parallel and satisfies the following properties:

$$g(\bar{\sigma}(X, Y), \bar{\sigma}(V, W)) = 2\langle X, Y\rangle\langle V, W\rangle + \langle X, V\rangle\langle Y, W\rangle + \langle X, W\rangle\langle Y, V\rangle,$$

$$\bar{A}_{\bar{\sigma}(X,Y)} V = 2\langle X, Y\rangle V + \langle X, V\rangle Y + \langle Y, V\rangle X,$$

$$g(\bar{\sigma}(X, Y), f(x)) = -\langle X, Y\rangle,$$

$$g(\bar{\sigma}(X, Y), I) = 0,$$

where I is the identity matrix in $SM(n+2)$, $\bar{A}$ is the Weingarten map of f and X, Y, V, W are tangent vectors to S^{n+1}. Moreover S^{n+1} is immersed by the second standard immersion f as a minimal submanifold of a hypersphere of $SM(n+2)$ centered at $I/n+2$ and with radius $\sqrt{\frac{n+1}{2(n+2)}}$ (see [5]).

Let, now, $x : M^n \to S^{n+1}$ be an isometric immersion of a compact Riemannian manifold M^n into S^{n+1}. The isometric immersion $\varphi = f \circ x : M^n \to SM(n+2)$ is called the quadric representation of x since coordinates of φ depend on x in a quadric manner. The center of mass of φ is the symmetric matrix (α_{ij}) with $\alpha_{ij} = \frac{1}{vol(M^n)} \int_{M^n} x_i x_j dM^n$, where x_i $(i = 1, \dots, n+2)$ are the coordinate functions of x with respect to a constant coordinate system in E^{n+2} and dM^n denotes the volume element of M^n. We note that the quadric representation of an isometric immersion $x : M^n \to S^{n+1}$, and its center of mass depend on the chosen coordinate system of E^{n+2}.

Using (1) we find that the mean curvature vector field H of the quadric representation is given by

$$H = H_1 x^t + x H_1^t + \frac{2}{n} \sum_{i=1}^{n} E_i E_i^t - 2 x x^t, \tag{2}$$

where $\{E_1, \dots, E_n\}$ denotes an orthonormal frame of M^n and H_1 the mean curvature vector field of the immersion $x : M^n \to S^{n+1}$.

Example. Let $S^q(r)$ denote a q-dimensional sphere in E^{q+1} with radius r. Let n, p be positive integers such that $p < n$ and the Riemannian product $M_{p,n-p} = S^p(\sqrt{\frac{p}{n}}) \times S^{n-p}(\sqrt{\frac{n-p}{n}})$. We imbed $M_{p,n-p}$ into S^{n+1} as follows. Let (u, v) be a point of $M_{p,n-p}$ where u (resp. v) is a vector in E^{p+1} (resp. E^{n-p+1}) of length $\sqrt{\frac{p}{n}}$ (resp. $\sqrt{\frac{n-p}{n}}$). We can consider (u, v) as a unit vector in $E^{n+2} = E^{p+1} \times E^{n-p+1}$. Then $M_{p,n-p}$ is a minimal hypersurface in the unit hypersphere S^{n+1}, the so called Clifford minimal hypersurface. More precisely we have

$$x = \sqrt{\frac{p}{n}} \vec{\theta}_1 + \sqrt{\frac{n-p}{n}} \vec{\theta}_2,$$

where $\vec{\theta}_1$ is the position vector field of the unit hypersphere in E^{p+1} and $\vec{\theta}_2$ is the position vector field of the unit hypersphere in E^{n-p+1}. It is obvious that the Laplace operator Δ of $M_{p,n-p}$ is given by $\Delta = \frac{n}{p}\Delta_1 + \frac{n}{n-p}\Delta_2$, where Δ_i is the Laplace operator of $\vec{\theta}_i$. Consider now the quadric representation $\varphi : M_{p,n-p} \to SM(n+2)$. An easy calculation, by using the averaging principle, shows the following spectral behaviour

$$\Delta(\varphi_{ij} - \alpha_{ij}) = \lambda_{ij}(\varphi_{ij} - \alpha_{ij}), \tag{3}$$

where the center of mass $\varphi_0 = (\alpha_{ij})$ of φ is given by

$$\alpha_{ij} = \begin{cases} \dfrac{p}{n(p+1)}, & i = j \le p+1 \\ 0, & i \ne j \\ \dfrac{n-p}{n(n-p+1)}, & i = j \ge p+2 \end{cases}$$

and the eigenvalues are given by

$$\lambda_{ij} = \begin{cases} \dfrac{2n(p+1)}{p}, & i,j \leq p+1 \\ 2n, & i \geq p+2 \text{ and } j \leq p+1 \text{ or } i \leq p+1 \text{ and } j \geq p+2 \\ \dfrac{2n(n+1-p)}{n-p}, & i,j \geq p+2. \end{cases}$$

Moreover $M_{p,n-p}$ is immersed via φ into some hypersphere of $SM(n+2)$ centered at (α_{ij}). Furthermore, using (3) we deduce that φ is minimal in the hyperquadric of $SM(n+2)$ given by

$$\sum_{i,j=1}^{n+2} \lambda_{ij}(\varphi_{ij} - \alpha_{ij})^2 = n.$$

The next lemma was proved in [1].

Lemma 2.1. *Let $x : M^n \to E^m$ be an isometric immersion of a compact Riemannian manifold M^n in the Euclidean space E^m. Let $b = (b_1, b_2, \ldots, b_m)$ denote the center of mass of M^n. Then*

$$\Delta(x_i - b_i) = \lambda_i(x_i - b_i), \qquad i = 1, \ldots, m$$

if and only if:
(i) x is mass-symmetric in some hypersphere S_b^{m-1} centered at b and
(ii) x is minimal in the hyperquadric, concentric with S_b^{m-1} given by

$$\sum_{i=1}^{m} \alpha_i(x_i - b_i)^2 = k.$$

Moreover, in this case $\lambda_i = \frac{n\alpha_i}{k}$, $1 \leq i \leq m$.

Remark 2.2. Comparing this Lemma with Theorem 2.2 of [4], we see that the results are basically the same. However, this result is an extension of a result due to T. Takahashi [6].

3 Proof of Theorem

At first, it is convenient to prove some lemmas.

Lemma 3.1. *Let $x : M^n \to S^{n+1}$ be an isometric immersion of the compact manifold M^n in S^{n+1}. Assume that $A = (\alpha_{ij})$ is the center of mass of its quadric representation $\varphi : M^n \to SM(n+2)$. Let P be an $(n+2) \times (n+2)$ orthogonal matrix and $\psi : M^n \to SM(n+2)$ the quadric representation of the isometric immersion $y = Px : M^n \to S^{n+1}$. If $B = (b_{ij})$ is the center of mass of the quadric representation ψ, then $B = PAP^t$.*

Proof. By integration, since $yy^t = Pxx^tP^t$, we obtain the desired result. Thus the centers of mass A, B are similar matrices. □

Remark 3.2. The center of mass A of the quadric representation φ of an isometric immersion $x : M^n \to S^{n+1}$ is a positive-definite matrix unless $x(M^n)$ is totally geodesic in S^{n+1}. In fact, since A is a symmetric matrix there exists an orthogonal matrix P such that $PAP^t = diag[\lambda_1, \ldots, \lambda_{n+2}]$, where λ_i are the eigenvalues of A. The matrix $diag[\lambda_1, \ldots, \lambda_{n+2}]$ is the center of mass of the quadric representation of the isometric immersion $y = Px$. Since, $\lambda_i Vol(M^n) = \int_{M^n} y_i^2 dM^n$ we conclude, by using Lemma 3.1, that the matrix A is a positive semi-definite symmetric matrix. If $\lambda_i = 0$, for some $i = 1, \ldots, n+2$, then the coordinate function y_i is zero and thus $y(M^n)$ is totally geodesic in S^{n+1}. Therefore $x(M^n)$ is totally geodesic in S^{n+1} since P is a linear rigid motion in E^{n+2}.

Lemma 3.3. *Let $x : M^n \to S^{n+1}$ be a minimal isometric immersion in the unit hypersphere S^{n+1} $(n \geq 2)$. Assume that the coordinate functions x_i $(i = 1, \ldots, n+2)$ of M^n satisfy the analytic equation $\sum_{i=1}^{n+2} \mu_i x_i^2 = c$, where $c, \mu_1, \ldots, \mu_{n+2}$ are some positive constants; then at most three of μ_i are distinct. Moreover, if exactly three of them are distinct, say μ_1, μ_2 and μ_3, then we have*

$$(\mu_1 - c)(\mu_2 - c)(\mu_3 - c) = 0 \tag{4}$$

and

$$(k_1 - 1)\mu_1 + (k_2 - 1)\mu_2 + (k_3 - 1)\mu_3 = c(n-1), \tag{5}$$

where k_i is the multiplicity of μ_i.

Proof. We set $f_1(x_1, \ldots, x_{n+2}) = \sum_{i=1}^{n+2} x_i^2 - 1$ and $f_2(x_1, \ldots, x_{n+2}) = \sum_{i=1}^{n+2} \mu_i x_i^2 - c$. The gradient vector fields $\bar{\nabla} f_1$, $\bar{\nabla} f_2$, where $\bar{\nabla}$ stands for the gradient operator in E^{n+2}, must either be linearly dependent everywhere on M^n, or linearly independent on some open subset U of M^n. In the first case all μ_i $(i = 1, \ldots, n+2)$ are equal.

Henceforth we shall assume that $\bar{\nabla} f_1$ and $\bar{\nabla} f_2$ are linearly independent on U. Then the unit vector fields

$$\xi_1 = \frac{\bar{\nabla} f_1}{|\bar{\nabla} f_1|} = (x_1, \ldots, x_{n+2}), \xi_2 = \frac{\bar{\nabla} f_2 - \langle \bar{\nabla} f_2, \xi_1 \rangle \xi_1}{|\bar{\nabla} f_2 - \langle \bar{\nabla} f_2, \xi_1 \rangle \xi_1|}$$

generate the normal space of U in E^{n+2}. Moreover, ξ_2 is the unit normal vector field of U in S^{n+1}. Since M^n is minimal in S^{n+1}, we have that *trace* $A_{\xi_2} = 0$, where A_{ξ_2} denotes the Weingarten map of M^n in E^{n+2} with respect to ξ_2. By a straightforward computation we see that *trace* $A_{\xi_2} = 0$ is equivalent to the following

$$\sum_{i=1}^{n+2} \alpha_i x_i^2 = -c^3 - dc^2, \tag{6}$$

where $d = \sum_{i=1}^{n+2} \mu_i - c(n+1)$ and $\alpha_i = \mu_i^2(\mu_i - 2c - d)$. Obviously, the μ_i's are not all equal since the $\bar{\nabla} f_1$, $\bar{\nabla} f_2$ are linearly independent on U. We shall prove that at most three of them are different. Without loss of generality, we suppose that $\mu_1 \neq \mu_2$. Then, solving the system $\sum_{i=1}^{n+2} x_i^2 = 1$ and $\sum_{i=1}^{n+2} \mu_i x_i^2 = c$ with respect to x_1^2, x_2^2 and substituting into (6) we get that the quadric polynomial

$$\sum_{j=3}^{n+2} \Big(\alpha_1(\mu_j - \mu_2) + \alpha_2(\mu_1 - \mu_j) + \alpha_j(\mu_2 - \mu_1)\Big) x_j^2$$

$$+\alpha_1(\mu_2 - c) + \alpha_2(c - \mu_1) - (\mu_2 - \mu_1)(-c^3 - dc^2),$$

vanishes identically on an open subset of E^n. However, from the coefficient of x_i^2, $i \geq 3$ and taking account of the expression for α_i we obtain

$$\Big(\sum_{j=1}^{n+2} \mu_j - \mu_1 - \mu_2 - \mu_i - c(n-1)\Big)(\mu_1 - \mu_2)(\mu_i - \mu_2)(\mu_i - \mu_1) = 0, \quad i \geq 3. \tag{7}$$

The last relation implies that at most three of μ_i's are distinct, say μ_1, μ_2 and μ_3. In that case, relation (7) implies that

$$\sum_{i=1}^{n+2} \mu_i - \mu_1 - \mu_2 - \mu_3 = c(n-1), \tag{8}$$

from which follows relation (5).

Now, from the constant term of the polynomial and taking account of the expressions for α_i and d we obtain

$$(\mu_1 - c)(\mu_2 - c)(\mu_1 - \mu_2)\Big(\mu_1 + \mu_2 + cn - \sum_{j=1}^{n+2} \mu_j\Big) = 0. \tag{9}$$

The relation above implies (4) because of (8). □

Lemma 3.4. *Let $x : M^n \to S^{n+1}$ be a compact minimal hypersurface of S^{n+1} whose quadric representation φ is mass-symmetric in some hypersphere of $SM(n+2)$ centered at $A = (\alpha_{ij})$. Then A has at most two distinct eigenvalues.*

Proof. Suppose that φ is mass-symmetric in a hypersphere $\tilde{S}$ centered at A. It is also well known that $\varphi(M^n)$ is contained in a hypersphere S centered at $I/n+2$ (see section 2). Hereafter we assume that $A \neq I/n+2$. This implies that $\varphi(M^n)$ is contained in the intersection $\tilde{S} \cap S$. Since $\varphi(M^n)$ is mass-symmetric in $\tilde{S}$ we conclude that $x(M^n)$ is contained in the hyperquadric N of E^{n+2} given by

$$\sum_{i,j=1}^{n+2} \alpha_{ij} x_i x_j = c,$$

where $c = \sum_{i,j=1}^{n+2} \alpha_{ij}^2$.

Let P be an orthogonal matrix such that $PAP^t = diag[\mu_1, \ldots, \mu_{n+2}]$, where μ_i $(i = 1, \ldots, n+2)$ are the eigenvalues of A.
The coordinate functions y_i $(i = 1, \ldots, n+2)$ of the isometric immersion $y = Px$ satisfy the equations

$$\sum_{i=1}^{n+2} y_i^2 = 1 \tag{10}$$

$$\sum_{i=1}^{n+2} \mu_i y_i^2 = c. \tag{11}$$

Using Lemma 3.3 we conclude that at most three of the eigenvalues μ_i's are distinct. Actually we shall prove that at most two of them are distinct. We argue now indirectly. Suppose that μ_1, μ_2, μ_3 are distinct with corresponding multiplicities k_1, k_2, k_3. Using Lemma 3.3 we obtain

$$(\mu_1 - c)(\mu_2 - c)(\mu_3 - c) = 0 \tag{12}$$

and

$$(k_1 - 1)\mu_1 + (k_2 - 1)\mu_2 + (k_3 - 1)\mu_3 = c(n-1). \tag{13}$$

Without loss of generality, because of (12), we may assume $\mu_2 = c$. It will be convenient, in what follows, to put

$$\sum_{i=1}^{k_1} y_i^2 = R_1, \qquad \sum_{i=k_1+1}^{k_1+k_2} y_i^2 = R_2 \quad \text{and} \quad \sum_{i=k_1+k_2+1}^{n+2} y_i^2 = R_3.$$

Then from (10) and (11) we find that

$$(\mu_1 - c)R_1 + (\mu_3 - c)R_3 = 0, \tag{14}$$

and thus

$$(\mu_1 - c)(\mu_3 - c) < 0. \tag{15}$$

Moreover, since the matrix $diag[\mu_1, \ldots, \mu_{n+2}]$ is the center of mass for the quadric representation of $y = Px$, from (10) we have $\sum_{i=1}^{n+2} \mu_i = 1$, and thus

$$k_1\mu_1 + k_2\mu_2 + k_3\mu_3 = 1. \tag{16}$$

On the other hand, combining (13) with $k_1 + k_2 + k_3 = n + 2$ we find

$$(k_1 - 1)(\mu_1 - c) + (k_3 - 1)(\mu_3 - c) = 0. \tag{17}$$

We distinguish three cases

Case i. $k_1 = 1$ or $k_3 = 1$. If just one is equal to 1, say k_1, then from (17) we conclude that $\mu_3 = c$, a contradiction. Assume that $k_1 = 1$ and $k_3 = 1$. In that case (14) implies that $y(M^n)$ lies on $(\mu_1 - c)y_1^2 + (\mu_3 - c)y_{n+2}^2 = 0$, a pair of hyperplanes, which means that $y(M^n)$ is totally geodesic. In this case one can, easily, verify that the center of mass of a totally geodesic hypersurface has exactly two distinct eigenvalues, a contradiction.

Case ii. $k_1 > 1$, $k_2 > 1$, $k_3 > 1$. In this case, from (14) and (17) we have $(k_1 - 1)R_3 - (k_3 - 1)R_1 = 0$ and because of $R_1 + R_3 = 1 - R_2$, we find

$$R_1 = \frac{k_1 - 1}{k_1 + k_3 - 2} R_2, \qquad R_3 = \frac{k_3 - 1}{k_1 - k_3 - 2} R_2. \tag{18}$$

Now, a parametrization of $y(M^n)$ given by

$$y = \alpha \cos\varphi \vec{\theta}_1 + \sin\varphi \vec{\theta}_2 + b\cos\varphi \vec{\theta}_3, \qquad \varphi \in (0, \pi/2)$$

where $\alpha = \sqrt{\frac{k_1 - 1}{k_1 + k_3 - 2}}$, $b = \sqrt{\frac{k_3 - 1}{k_1 + k_3 - 2}}$ and $\vec{\theta}_i$ denotes the position vector field of the unit hypersphere in E^{k_i}. Denote by $B = (b_{ij})$ the center of mass of the quadric representation of $y = Px$. Using the above parametrization we find

$$\begin{aligned} b_{ij} &= 0, && i \neq j, \\ b_{ii} &= \frac{\alpha^2}{k_1} \cdot \frac{n + 1 - k_2}{n + 1}, && i = 1, \dots, k_1, \\ b_{ii} &= \frac{1}{n+1}, && i = k_1 + 1, \dots, k_1 + k_2, \\ b_{ii} &= \frac{b^2}{k_3} \cdot \frac{n + 1 - k_2}{n + 1}, && i = k_1 + k_2 + 1, \dots, n + 2. \end{aligned}$$

Consequently, we have

$$\mu_1 = \frac{\alpha^2}{k_1} \cdot \frac{n + 1 - k_2}{n + 1}, \tag{19}$$

$$\mu_2 = c = \frac{1}{n+1}, \tag{20}$$

$$\mu_3 = \frac{b^2}{k_3} \cdot \frac{n + 1 - k_2}{n + 1}. \tag{21}$$

Using (13), (16) and taking account of the expressions for μ_1, μ_2, μ_3 we find $k_1 = 1$ or $k_3 = 1$, which is a contradiction.

Case iii. $k_1 > 1$, $k_3 > 1$, $k_2 = 1$. In a similar way a parametrization of $y(M^n)$ is given by

$$y = \alpha \cos\varphi \vec{\theta}_1 + \sin\varphi e_{k_1+1} + b\cos\varphi\vec{\theta}_3, \qquad \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

where e_{k_1+1} denotes the (k_1+1)-th vector of the usual basis of E^{n+2}, $\alpha = \sqrt{\frac{k_1-1}{k_1+k_3-2}}$, $b = \sqrt{\frac{k_3-1}{k_1+k_3-2}}$ and $\vec{\theta}_i$ is the position vector field of the unit sphere in E^{k_i}. The center of mass $B = (b_{ij})$ of the quadric representation of $y = Px$ is given by

$$\begin{aligned} b_{ij} &= 0, && i \neq j, \\ b_{ii} &= \frac{\alpha^2}{k_1} \cdot \frac{n}{n+1}, && i = 1, \dots, k_1, \\ b_{ii} &= \frac{1}{n+1}, && i = k_1 + 1, \\ b_{ii} &= \frac{b^2}{k_3} \cdot \frac{n}{n+1}, && i = k_1 + 2, \dots, n+2. \end{aligned}$$

Proceeding as in the second case we obtain $k_1 = 1$ or $k_3 = 1$, a contradiction. □

Proof of Theorem. As we already have mentioned, the quadric representations of totally geodesic hypersurfaces and Clifford minimal hypersurfaces in S^{n+1} are mass-symmetric in some hypersphere of $SM(n+2)$ and minimal in a concentric hyperquadric.

Now, suppose $x : M^n \to S^{n+1}$ is a non-totally geodesic compact minimal hypersurface in S^{n+1}. Then from (2) and Lemma 2.1 we have

$$\Omega = (\Omega_{ij}) = \Delta(\varphi - \varphi_0) = -nH = (\lambda_{ij}(\varphi_{ij} - \alpha_{ij})). \tag{22}$$

The matrix Ω can be certainly regarded as a field of endomorphisms on E^{n+2} along M^n. Denoting by ξ the unit normal vector field of M^n in S^{n+1}, then (2) implies

$$\Omega(\xi) = 0 \quad \text{and} \quad \Omega(x) = 2nx. \tag{23}$$

According to Lemma 3.4 the matrix $A = (\alpha_{ij})$ has at most two distinct eigenvalues. We distinguish two cases.

Case i. A has two distinct eigenvalues. Equations (10) and (11) imply that M^n is a Riemannian product of two hyperspheres and from the minimality of M^n we conclude that M^n is a Clifford minimal hypersurface.

Case ii. A has one eigenvalue λ of multiplicity $n+2$, that is $A = \lambda I$. Since M^n lies on the sphere S^{n+1} we have $\sum_{i=1}^{n+2} \alpha_{ii} = 1$, and so

$$\alpha_{ij} = \frac{\delta_{ij}}{n+2}. \tag{24}$$

The second equation of (23) by virtue of (24) implies that $M^n \subseteq f_i^{-1}(0)$ for all $i = 1, \ldots, n+2$, where

$$f_i(x_1, \ldots, x_{n+2}) = \sum_{j=1}^{n+2} \lambda_{ij} x_j^2 - \frac{\lambda_{ii}}{n+2} - 2n.$$

Moreover, because of $\bar{\nabla} f_i = \langle \bar{\nabla} f_i, x \rangle x + \langle \bar{\nabla} f_i, \xi \rangle \xi$, the equations (23) we deduce that

$$\Omega(\bar{\nabla} f_i) = 2n \langle \bar{\nabla} f_i, x \rangle x. \tag{25}$$

We note that if $\lambda_{ij} = \lambda$ for all $i, j = 1, \ldots, n+2$ then M^n should be totally geodesic in S^{n+1} ([3], Theorem 3). Thus we may assume that $\lambda_{i1}, \ldots, \lambda_{in+2}$, for some $i = 1, \ldots, n+2$, are not all equal. Using (25) we find

$$\sum_{j=1}^{n+2} \lambda_{ij}^2 x_j^2 = \frac{\lambda_{ii}^2}{n+2} + \frac{2n\lambda_{ii}}{n+2} + 4n^2. \tag{26}$$

Since M^n is minimal in S^{n+1} and lies in the hyperquadric $f_i(x) = 0$ we conclude, by Lemma 3.2, that at most three of $\lambda_{i1}, \ldots, \lambda_{in+2}$ are distinct. In this case, equations

$$\sum_{j=1}^{n+2} \lambda_{ij}^2 x_j^2 = \frac{\lambda_{ii}^2}{n+2} + \frac{2n\lambda_{ii}}{n+2} + 4n^2, \qquad \sum_{j=1}^{n+2} x_j^2 = 1, \qquad \sum_{j=1}^{n+2} \lambda_{ij} x_j^2 = \frac{\lambda_{ii}}{n+2} + 2n$$

and the minimality of M^n, imply that only two of $\lambda_{i1}, \ldots, \lambda_{in+2}$ must be distinct. So, M^n is a Clifford minimal hypersurface and the Theorem is proved. □

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