Moufang Polygons, I. Root data

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Dedicated to J. A. Thas on his fiftieth birthday

Abstract

As a first step in the classification of all thick Moufang polygons, it is shown that every root ray datum of type $\tilde{A}_2, \tilde{B}_2$ or $\tilde{G}_2$ has a filtration by an ordinary root datum.

1 Introduction

Let n be an integer greater or equal 3. A *generalized n-gon*, or simply an *n-gon*, is a bipartite graph with diameter n and girth $2n$. (Here, graphs are undirected, with no loops or double edges; a graph is *bipartite* if its cycles have even length.) These properties imply that an n-gon is connected and that every vertex has order at least 2. In fact, we shall only consider *thick* n-gons, that is, assume that all vertices have order at least 3.

The generalized polygons are nothing else but the buildings of rank 2 and spherical types (cf. e.g. [9]). The buildings of (irreducible) spherical type and rank at least 3 are completely classified in *loc.cit.*; roughly speaking, they are the buildings associated to algebraic simple groups and classical groups of rank ≥ 3. In short, we shall say that they are "of algebraic origin". There is no such result in the rank 2 case; in fact, the existence of a "free construction" (cf. e.g. [13],4.4) indicates that generalized n-gons are too general objects to allow classification in any reasonable sense. Thus, in order to characterize geometrically the polygons "of algebraic origin", an extra-condition is necessary. The "Moufang condition", introduced in [9], p.274 (cf. also [11]), the statement of which will be recalled below, appears to be

Received by the editors in February 1994
AMS Mathematics Subject Classification: Primary 51E12, Secondary 20E42
Keywords: Moufang polygons, root systems.

the right one: this is the content of the conjecture stated in [11], 3.3. The present paper is one of a series, the final goal of which is the proof of that conjecture. Let us recall that

* the nonexistence of Moufang n-gons for $n \neq 3$, 4, 6, 8 was proved in [12] (a different, very nice proof of a more general result was given by R.Weiss in [15]);

* in [14], it was shown (among other things) that the only existing Moufang octagons are those associated with the Ree groups of type ${}^2F_4$;

* the enumeration of all Moufang hexagons is given in [11], 4.7, without proof;

* the preprint [10] deals with an exotic, very exceptional type of Moufang quadrangles (already described in [11], 4.5) which needs a special treatment;

* when $n = 3$, the cited conjecture amounts to the theorem of R. Moufang [7] characterizing the projective planes over alternative division rings as those in which the "little Desargues theorem" holds, together with the classification of the rings in question, due to R. Bruck and E. Kleinfeld ([2, 6]).

Finally, it should be mentioned that, in [4], J. Faulkner gives partial classification results for a class of generalized n-gons ($n = 4$ and 6) somewhat more restricted than that of Moufang n-gons.

Let Δ be a generalized n-gon and let $\Gamma = (s_0, s_1, \ldots, s_n)$ be an n-path (for us, this means that if $0 \leq i < n$, s_i and s_{i+1} are distinct vertices of Δ, connected by an edge, and $s_{i-1} \neq s_{i+1}$ if $i \neq 0, n$). It is a well-known consequence of theorem 4.1.1 of [9] that the group $U(\Gamma)$ of all automorphisms of Δ fixing Γ and all vertices adjacent to $s_1, \ldots, s_{n-1}$ operates freely on the set of all $2n$-cycles containing Γ. The n-gon Δ is said to be *Moufang* if $U(\Gamma)$ is transitive (hence simply transitive) on the set in question.

Let us now choose a $2n$-cycle $(s_z | z \in \mathbb{Z}, s_{z+2n} = s_z)$ in Δ and set, for all integers z, $U_z = U(s_z, \ldots, s_{z+n})$. If z, z' are two integers, we denote by $U_{[z,z']}$ the subgroup of AutΔ generated by all $U_{z''}$ for z'' satisfying $z \leq z'' \leq z'$. The commutator $xyx^{-1}y^{-1}$ of two elements x, y of a group is denoted by $[x, y]$ and if X, Y are subgroups of a group, we represent by $[X, Y]$ the *group generated by* all commutators $[x, y]$ for $x \in X$ and $y \in Y$. (This double use of the square brackets should not cause confusion.) It is well known, and easily shown, that if Δ is a Moufang n-gon, the groups U_z have the following properties, where we set $U_z^* = U_z \setminus \{1\}$:

(MP0) $U_z = U_{z+2n} \neq \{1\}$ for all z ;

(MP1) for $i < j < i + n$, one has $[U_i, U_j] \subset U_{[i+1,j-1]}$;

(MP2) for any integer i and any $u \in U_i^*$, there exists an element m in $U_{i+n} \cdot u \cdot U_{i+n}$ such that, for all integers j, the group ${}^mU_j$, conjugate of U_j by m, is equal to U_{2i+n-j} ;

(MP3) if U_+ denotes the group generated by U_1, U_2, $\ldots$, U_n, the product mapping $U_1 \times U_2 \times \ldots \times U_n \to U_+$ is injective (hence bijective, because of (MP1)).

Conversely, if $2n$ subgroups U_z ($z \in \mathbb{Z}$, $U_z = U_{z+2n}$) of some group satisfy the conditions (MP0) to (MP3), then there exist a system $(\Delta; (s_z | z \in \mathbb{Z}, s_z = s_{z+2n}))$, unique up to unique isomorphism, consisting of a Moufang n-gon and a $2n$-cycle in it, and a homomorphism of the group G° generated by all U_z in AutΔ mapping each U_z bijectively onto the corresponding group $U(s_z, \dots, s_{z+n})$, with the above notation. The homomorphism is not necessarily injective, but its kernel is central in G° and intersects all U_z trivially. The proof of those facts, using [9], 3.2.6, and [14], 2.5, 2.7, 2.8, is straightforward; the Moufang property essentially reflects condition (MP1).

The set of conditions (MP0) to (MP3) is the rank 2 special case of the system of axioms (2.1) to (2.4) of [8], used again at various occasions since then. As motivation to what follows, let us describe its main features. We consider a real vector space V endowed with a Euclidean metric. A subset of V of the form $\mathbb{R}_+ \cdot v$, with $v \in V \setminus \{0\}$ is called a *ray* (in [8], instead of rays, I was considering half-spaces, which is of course equivalent). To each ray α we associate the *reflection* r_α with respect to the hyperplane orthogonal to α and containing 0. We define a *root ray system* ("système de racine" in the terminology of [8]) as a finite set of rays generating V linearly and stable under the reflections associated to its elements. Notice that root ray systems are in 1-to-1 correspondence with finite reflection groups of V having no fixed point except 0. Let $\tilde{\Phi}$ be such a system. By *root ray datum* of type $\tilde{\Phi}$ in a group G, we shall understand a system of subgroups U_α indexed by the elements α of $\tilde{\Phi}$ and satisfying the axioms (2.1) to (2.4) of [8]. (In [8], such a system was called "donnée radicielle".) Rephrased with our present terminology and notation, they take the form of four conditions (RRD0) to (RRD3), generalizing respectively (MP0) to (MP3), the latter corresponding to the case where dim$V = 2$ and $\tilde{\Phi}$ is the root ray system associated to a (dihedral) reflection group of order $2n$. For our present purpose, it will be useful to state explicitly the axiom

(RRD1) for $\alpha, \beta \in \Phi$, the group $[U_\alpha, U_\beta]$ is contained in the group generated by all U_γ with $\gamma \subset \alpha + \beta$.

The motivation for the introduction of that notion in [8] was of course the application to isotropic simple algebraic groups, but in case of algebraic groups, one deals with an *a priori* much more restricted type of structure, namely a *root datum*; by this, we mean a "donnée radicielle" in the sense of [3], 6.1, except that here, we fix our attention only on the subgroups U_a. More precisely, if $\Phi \subset V$ is a *root system* (cf. e.g. [1], p.142), a system $(U_a)_{a \in \Phi}$ of subgroups of a group G, indexed by Φ, will be called a *root datum of type* Φ if it satisfies the following conditions:

(RD0) The groups U_a are all different from $\{1\}$ and, if $2a \in \Phi$, then $U_{2a} \neq U_a$.

(RD1) For $a, b \in \Phi$ such that $b \notin -\mathbb{R}_+ a$, the group $[U_a, U_b]$ is contained in the group generated by all U_c with $c = pa + qb \in \Phi$, $p, q \in \mathbb{N}$, $p > 0$, $q > 0$.

(RD2) For $a \in \Phi$ and $u \in U_a \setminus \{1\}$, there exists $m \in U_{-a} \cdot u \cdot U_{-a}$ which conjugates U_b onto $U_{r_a(b)}$ for all $b \in \Phi$, where r_a represents the reflection of V with respect to the hyperplane orthogonal to a.

(RD3) For any choice of a basis Ψ of Φ and any element a of Ψ, the group U_+ generated by the groups U_b corresponding to positive roots b (roots which are linear combinations of elements of Ψ with positive coefficients) does not contain U_{-a}.

Note that, in view of (RD2), if (RD3) is true for some basis Ψ of Φ, it is true for all of them.

The relation between these axioms and those of [3], 6.1 is as follows: the subgroups U_a in a system satisfying the axioms of [3] clearly satisfy the axioms (RD0) to (RD3); conversely, it is easily seen that if $(U_a)_{a\in\Phi}$ is a root datum in a group G, if T is the intersection of the normalizers of the U_a in G and if, for all $a \in \Phi$, M_a denotes the product of T by any element m as described in axiom (RD2), the system $(T, (U_a, M_a)_{a\in\Phi})$ is a "donnée radicielle" in the sense of [3], 6.1.1.

Except for (RRD1), the axioms of root ray data, which we did not reproduce here, are similar to the corresponding axioms of root data. On the other hand, as one can see, there is a major difference between (RRD1) and (RD1); for instance, the latter clearly implies that all U_a are nilpotent of class at most 2 (and that the group U_+ considered above is nilpotent), whereas the axioms of root ray data, in particular (RRD1), impose no obvious restriction on the structure of the groups U_α.

Let Φ be a root system, let $\tilde{\Phi}$ denote the root ray system consisting of all rays of V containing at least one element of Φ and let $(U_a | a \in \Phi)$ be a root datum of type Φ. For $\alpha \in \tilde{\Phi}$, let U_α denote the union of all U_a with a contained in α, or, equivalently, the biggest one of them (there are at most two, totally ordered by inclusion!). Then, taking into account [3], 6.1.6, one can show that the groups U_α form a root ray datum of type $\tilde{\Phi}$, and we shall say that this datum is *filtered* by the root datum (U_a). Question: does every root ray datum have such a filtration? At first sight, this certainly seems most unlikely, considering what we have just said about the axioms (RRD1) and (RD1). Yet, the answer is "almost affirmative". Let us denote by $\tilde{A}_1$ (resp. $\tilde{I}_2(n)$) the type of the root ray system of dimension 1 (resp. the system of dimension 2 associated with the dihedral group of order $2n$). Then:

any root ray datum whose type has no direct factor of type $\tilde{A}_1$ or $\tilde{I}_2(8)$ has a filtration by an ordinary root datum.

The exception $I_2(8)$ is not a serious one as it can be disposed of via a slight enlargement of the notion of root system and root datum (cf. [14] and section 5 below). To prove the above assertion, it suffices to consider the rank 2 case: if the type of the root ray datum under consideration has no direct factor of type $\tilde{B}_n = \tilde{C}_n$ ($n \geq 3$) or $\tilde{F}_4$, this is clear; for $\tilde{B}_n$ and $\tilde{F}_4$, one can use the classification of buildings of those types, given in [9], but a classification free proof is also possible, though not completely straightforward, using Proposition 2 below.

As for the rank 2 situation, that is, the case of a system (U_z) satisfying the axioms (MP0) to (MP3), we know by [12] (or [15]) that it can exist only if $n = 3$, 4, 6 or 8. For $n = 8$, all such systems have been determined in [14] and, anyway, that value has been excluded from the above statement. There remains therefore to prove that, for $n = 3$, 4 or 6, all systems (U_z) satisfying the conditions (MP) are filtered by a root datum. That is the purpose of the present paper.

2 General lemmas (unspecified n)

In the whole paper, n denotes an integer ≥ 3 and $(U_z|z \in \mathbb{Z}, U_{z+2n} = U_z)$ is a system of subgroups of a group G, satisfying the conditions (MP0) to (MP3) above. We first recall some elementary facts from [14], 2.3. Given $u \in U_i$, the element m of condition (MP3) is unique; we denote it by $\mu(u)$. Also, the elements u' and u'' of U_{i+n} such that $m = u'uu''$ are unique. We set $u'' = \nu(u)$; this defines a bijection ν of the disjoint union of all U_i onto itself mapping U_i onto $U_{i+n} = U_{i-n}$, one has $\mu\nu = \mu$, hence $m = \nu^{-1}(u) \cdot u \cdot \nu(u) = u \cdot \nu(u) \cdot \nu^2(u)$.

We observe that the system of axioms (MP0) to (MP3) is preserved when any given integer is added to all indices; indeed, it is invariant by the substitutions $j \mapsto n + 2 - j$ (by (MP2)) and $j \mapsto n + 1 - j$ (obviously) on indices, hence also by $j \mapsto j + 1$. This allows us to adopt the following convention: in the statements of most lemmas, the indices occurring involve an indeterminate i and we shall feel free to set $i = 0$ in the proof without further justification. Whenever, for $u \in U_i$, we set $\mu(u) = u'uu''$, we shall mean that $u' = \nu^{-1}(u)$ and $u'' = \nu(u)$.

Lemma 2.1 *For $u \in U_i^*$ and $x \in U_{n+i-1}$, the $(i+1)$-component x_1 of $[u, x]$, defined by $[u, x] \in x_1 \cdot U_{[i+2,n+i-2]}$, is the conjugate of x by $\mu(u)$. In particular, the map $x \mapsto x_1$ is an isomorphism of U_{n+i-1} onto U_{i+1}.*

Proof. We set $m = \mu(u) = u'uu''$. Then ${}^{u'^{-1}m}x = {}^{uu''}x = {}^{u}x = [u, x] \cdot x$. Equating the $(i+1)$-components of the two extreme members of that relation, we get ${}^{m}x = x_1$, as desired. □

Lemma 2.2 *For $u \in U_i^*$ and $x \in U_{n+i-2}$, the $(i + 1)$-component of $[u, x]$ is the conjugate of $[\nu(u), x]^{-1}$ by $\mu(u)$; if x commutes with $\nu(u)$, the $(i + 2)$-component of $[u, x]$ is the conjugate of x by $\mu(u)$.*

Proof. We set again $m = \mu(u) = u'uu''$, hence $u'' = \nu(u)$, and let x_1 denote the $(i + 1)$-component of $[u, x]$. We have

$$\begin{aligned} [u'^{-1}, {}^{m}x] \cdot {}^{m}x &= {}^{u'^{-1}m}x = {}^{uu''}x = {}^{u}([u'', x] \cdot x) \\ &= {}^{u}[u'', x] \cdot ([u, x] \cdot x) \end{aligned} \tag{1}$$

The first member belongs to $U_{[i+2,n+i-1]}$. Since the $(i + 1)$-component is a multiplicative function in $U_{[i+1,n+i-1]}$, the product of the $(i + 1)$-components of the two factors of the last member of (1), namely ${}^{m}[u'', x]$ (by lemma 2.1) and x_1 must be trivial; this is the first assertion of the lemma. The second assertion immediately follows from (1) by equating the $(i + 2)$-components of its first and last member. □

Lemma 2.3 *Suppose n is even: $n = 2n'$. Then, if an element x of $U_{i+n'}$ commutes with an element u of U_i^*, it also commutes with $\mu(u)$.*

Proof. Suppose $i = 0$, without loss of generality. Setting $u' = \nu^{-2}(u), u'' = \nu^{-1}(u)$ and $m = u'u''u = \mu(u)$, we have ${}^{u'^{-1}m}x = {}^{u''u}x = {}^{u''}x$. The first member belongs to $U_{[1,n'-1]} \cdot {}^{m}x$ and the last one to $x \cdot U_{[n'+1,n-1]}$, while both x and ${}^{m}x$ belong to $U_{n'}$. The assertion ensues. □

Lemma 2.4 *For all i, one has $U_i \cap U_{[i+1,i+n]} = U_i \cap U_{[i-n,i-1]} = \{1\}$.*

Proof. It suffices to show that $U_0 \cap U_{[1,n]}$ is reduced to 1. Suppose this is not the case and let $u \neq 1$ belong to that intersection. We have $\mu(u) \in U_n \cdot u \cdot U_n \subset U_{[1,n]}$. Since the group $U_{[2,n]}$ is normal in $U_{[1,n]}$, it follows that $U_1 = {}^{\mu(u)}U_{n-1} \subset U_{[2,n]}$, in contradiction with (MP3). □

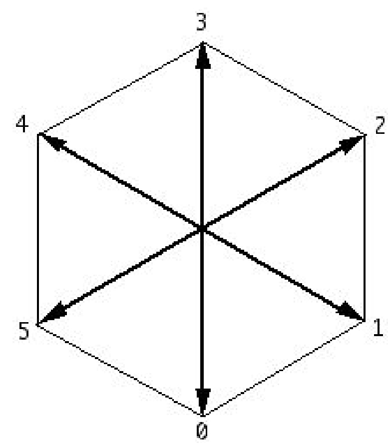
3 The case $n = 3$

Lemma 3.1 *If $n = 3$ and $u \in U_{i-1}$, the map $x \mapsto [u, x]$ of U_{i+1} in U_i is bijective, one has $[U_{i-1}, U_{i+1}] = U_i$ and the group U_i is abelian.*

Proof. The first assertion is a special case of lemma 2.1 and the second one follows from the first. Finally, since U_i commutes with both U_{i-1} and U_{i+1}, it commutes with their commutator U_i; it is therefore commutative. □

The above lemma, together with conditions (MP0) to (MP3) and lemma 2.4 readily imply:

Proposition 3.2 *Labelled as shown below, the groups U_i form a root datum of type A_2.*



Root system of type A_2

4 The case $n = 4$

In this section, we suppose n =4 and, for all i, we set $V_i = [U_{i-1}, U_{i+1}]$.

Lemma 4.1 *The group V_i is central in U_i.*

Proof. This is clear since U_{i-1} and U_{i+1} commute with U_i. □

Lemma 4.2 *The commutator of U_{i-1} and V_{i+1} is central in $U_{[i-1,i+2]}$.*

Proof. Indeed, it is central in $U_{[i,i+2]}$ (because this group is normalized by U_{i-1} and centralized by V_{i+1}), and it centralizes U_{i-1} because it is contained in U_i. □

Lemma 4.3 *One has* $[U_i, U_i] = [U_{i-1}, V_{i+1}] \subset Z(U_i)$.

Proof. We suppose $i = 0$, without loss of generality, and choose arbitrary elements $u \in U_{-1}$, $x \in U_0$ and $y \in U_2$. Let v denote the commutator $[x, y]$, which is an element of U_1, and let $x' \in U_0$ and $v' \in U_1$ be given by $[u, y] = x'v'$. Since v' commutes with x and y, hence also with v, and since v commutes with x', we have

$$\begin{aligned} {}^u v &= {}^u[x, y] = [x, x'v'y] = xx'v' \cdot yx^{-1}v'^{-1}x'^{-1} \\ &= xx'v' \cdot x^{-1}vy \cdot y^{-1}v'^{-1}x'^{-1} = [x, x'] \cdot v, \end{aligned}$$

that is

$$[u, v] \;=\; [x, y]. \tag{2}$$

Since the map $v \mapsto [u, v]$ is a homomorphism of U_2 in U_1, the elements $[u, v]$, as above, generate $[U_{-1}, V_1]$, therefore (2) implies that $[U_{-1}, V_1] \subset [U_0, U_0]$. The opposite inclusion also follows from (2) and from the fact that, in view of lemma 2.1, x' can be any element of U_0, independently of the choice of x. The last inclusion of the statement is just lemma 4.1. □

Observe that we now already know that all U_i are nilpotent, of class at most 2.

Lemma 4.4 *If* U_i *contains a nontrivial central element of* $U_{[i,i+2]}$*, then* $[U_{i+1}, U_{i+1}] = [U_i, V_{i+2}] = \{1\}$.

Proof. Let u be a nontrivial element of U_0, central in $U_{[0,2]}$. Since U_3 normalizes $U_{[0,2]}$, U_1 centralizes the commutator of u and U_3, hence also the 1-component of that commutator, which is the whole of U_1 by lemma 2.1. Therefore, U_1 is commutative, and there just remains to use lemma 4.3. □

Lemma 4.5 *One of the two groups* U_i *and* U_{i+1} *is commutative.*

Proof. By lemma 4.2 and lemma 4.3, the commutator group of U_0 is central in $U_{[-1,2]}$. If it is not trivial, then lemma 4.4 implies that U_1 is commutative. □

Lemma 4.6 *For any* $u \in U_i^*$, $\nu(u)$ *is conjugate to* u *by an element of* $\mu(U_{i+1}) \cdot \mu(U_{i-1})$.

Proof. Indeed, by [12],I,lemma 9, there exists a system of elements $u_z \in U_z^*$, $z \in \mathbb{Z}$, such that $u = u_i$, that $\nu(u) = u_{i+4}$ and that $\mu(u_z)$ conjugates u_{z+1} onto u_{z+3} for all z. Then, $\mu(u_{i+1}) \cdot \mu(u_{i-1})$ conjugates u onto $\nu(u)$. □

Let N° denote the group generated by all $\mu(U_i)$. It normalizes the system (U_i) and permutes its elements U_i according to the dihedral group D_{2n} of order $2n$. Let T° denote the group of all elements of N° normalizing each U_i ; thus $N^\circ/T^\circ \cong D_{2n}$. For any integer i, let Y_i denote the intersection of U_i with the center of $U_{[i-2,i+2]}$. We observe that, in view of lemma 4.1 and lemma 4.3,

$$\text{if } U_{i-1} \text{ is commutative, then } V_i \subset Y_i. \tag{3}$$

Indeed, by lemma 4.3, V_i then centralizes U_{i-2} and, symmetrically, it centralizes U_{i+2} (since U_{i+1} is then also commutative); furthermore, it centralizes U_i by lemma 4.1, and U_{i-1}, U_{i+1} by (MP1). Clearly,

$$T^\circ \text{ normalizes } V_i \text{ and } Y_i \text{ for all } i. \tag{4}$$

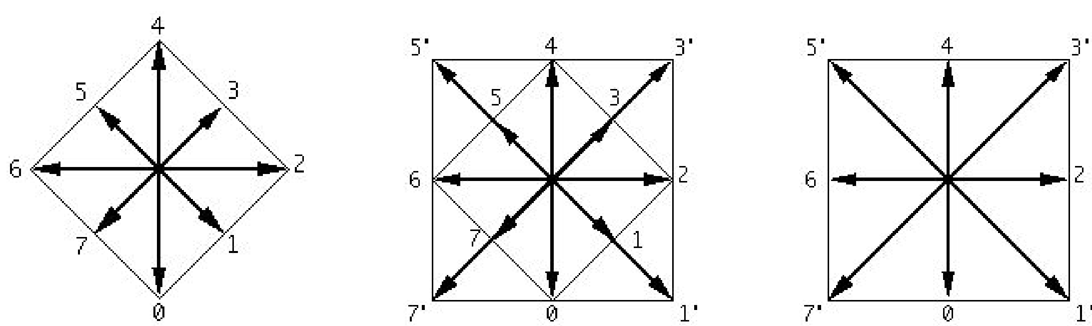
Proposition 4.7 *We assume (without loss of generality by lemma* 4.3*), that U_0 is commutative, and choose arbitrarily a subgroup $U_{1'}$ of Y_1 containing V_1 (cf.* (3)*) and normalized by T° (cf.* (4)*). For all odd $z \in \mathbb{Z}$, let $U_{z'}$ denote the conjugate of $U_{1'}$ by any element of N° conjugating U_1 onto U_z. Then, according as*

$$U_{1'} = \{1\}, \ \text{or } \{1\} \neq U_{1'} \neq U_1, \ \text{or } U_{1'} = U_1$$

the system of subgroups

$$(U_z)_{z\in\mathbb{Z}}, \ \text{resp. } (U_z, U_{(2z+1)'})_{z\in\mathbb{Z}}, \ \text{resp. } (U_{2z}, U_{(2z+1)'})_{z\in\mathbb{Z}}$$

form a root datum of type B_2, resp. BC_2, resp. C_2 for the labelling of those subgroups shown below.



Root systems of types B_2, BC_2 and C_2

N.B. Since there is no difference between root systems of type B_2 and root systems of type C_2, there is no compelling reason for using the name B_2 in the first case of the proposition and the name C_2 in the third one: it would be perfectly correct, mathematically, to use for instance C_2 in both cases. However, it is useful to have different names for the two cases, and our choice is suggested by the following consideration: in higher ranks, the root system one gets when removing the longest (resp. shortest) roots from a system of type BC_n is a system of type B_n (resp. C_n); here, this corresponds to the vanishing of the groups $U_{(2z+1)'}$ (resp. $U_{2z+1}/U_{(2z+1)'}$).

Proof. (of proposition 4.7).

The axiom (RD0) is obviously satisfied in all three cases.

In order to verify (RD1), one must consider separately the various possible configurations of the pair of roots (a, b). The three cases B_2, BC_2 and C_2 may be handled simultaneously but, to fix ideas, the reader may just think about BC_2 since it "contains" the two other cases. Up to reflections, there are nine inclusions to be proved:

$$[U_{2i}, U_{2i}] \quad = \quad \{1\} \tag{5}$$

$$[U_{2i+1}, U_{2i+1}] \subset U_{(2i+1)'} \tag{6}$$
$$[U_{2i+1}, U_{(2i+1)'}] = \{1\} \tag{7}$$
$$[U_i, U_{i+1}] = \{1\} \tag{8}$$
$$[U_{2i}, U_{2i+2}] \subset U_{(2i+1)'} \tag{9}$$
$$[U_{2i-1}, U_{2i+1}] \subset U_{2i} \tag{10}$$
$$[U_{2i-1}, U_{(2i+1)'}] = \{1\} \tag{11}$$
$$[U_i, U_{i+3}] \subset U_{i+1}U_{i+2} \tag{12}$$
$$[U_{2i}, U_{(2i+3)'}] \subset U_{(2i+1)'}U_{2i+2}. \tag{13}$$

The relations (8), (10) and (12) are special cases of (MP1). As for (5), (6), (7), (9), (11), (13), they are immediate consequences of, respectively,

the commutativity of U_0 (hypothesis of the proposition),

lemma 4.3 and the relation $[U_{2z}, U_{2z+2}] = V_{2z+1} \subset U_{(2z+1)'}$,

the inclusion $V_{2z+1} \subset U_{(2z+1)'} \subset Y_{2z+1} \subset Z(U_{2z+1})$,

the inclusion $V_{2z+1} \subset U_{(2z+1)'}$,

this same inclusion and lemma 4.3,

(MP1) and lemma 2.1.

Axiom (RD2) readily follows from (MP2) in view of the invariance of the system of groups $(U_{(2z+1)'})$ by N° and the fact that if u is a nontrivial element of $U_{(2z+1)'}$, then $\nu(u)$ and $\nu^{-1}(u)$ belong to $U_{(2z+5)'}$, by lemma 4.6.

Finally, lemma 2.4 clearly implies the validity of axiom (RD3). □

Remark. In principle, the above proposition gives all filtrations of any given root ray datum of type $\tilde{B}_2$ by root data. Let us say that two such filtrations are *similar*, or are related by a *similitude*, if they consist of the same groups, the labellings differing only by a similitude (isometry up to a constant factor) between the root systems. It turns out that, "in most cases", a root ray datum of type $\tilde{B}_2$ has a unique filtration up to similitude. For instance, it is so whenever, with the notation of the proposition, V_i contains Y_i for all i, which is often the case.

The main result of [10] is that the root ray data described there, and also in [11], are the only data of type $\tilde{B}_2$ having both a filtration of type B_2 and a filtration of type C_2 (with the notation of proposition 4.7 and for a fixed choice of U_0). With the description of [11], the root ray data in question depend on a field K_1 of characteristic 2, a subfield k_1 of K_1 containing K_1^2, a subspace K of the k_1-vector space K_1 and a subspace k of the K_1^2-vector space k_1. Straightforward application of proposition 4.7 shows that when the k_1-vector spaces K and K_1/K and the K_1^2-vector spaces k and k_1/k all have dimension at least 2, the situation one is in is the extreme opposite of the uniqueness case described above: here, for *any* root system Φ supported by the root ray system labelling the given root ray datum, the latter is filtered by at least one root datum of type Φ. Up to homothetic transformations, there are four different such Φ (among which, two of type BC_2).

5 The case $n = 6$

In this section, we assume $n = 6$. For all integers i, we denote by V_i the intersection of U_i with the centralizer of $U_{i-2} \cup U_{i+2}$ and we set $V_i^* = V_i \setminus \{1\}$. As in section 3, N° represents the group generated by all $\mu(U_i)$ and T° the intersection of the normalizers of all U_i in N°. We observe that N° preserves the system of subgroups (V_i) by conjugation.

Lemma 5.1 *We have*

$$
\begin{aligned}
[V_i, U_{i\pm4}] &\subset U_{i\pm1} \cdot V_{i\pm2}, && (14)\\
[\nu^{-1}(V_{i+6}^*), U_{i\pm4}] &\subset U_{i\pm2} \cdot U_{i\pm3}, && (15)\\
[V_i, V_{i\pm4}] &= V_{i\pm2}. && (16)
\end{aligned}
$$

Proof. For the proof, we take $i = 0$, without loss of generality. The inclusion (14) follows from the first assertion of lemma 2.2 since V_0 commutes with $U_{\pm2}$, hence with $\nu(U_{\mp4})$. The same assertion of lemma 2.2 implies (15) since $\nu(\nu^{-1}(V_6^*)) = V_6^*$ commutes with $U_{\pm4}$. Finally, by two applications of (14), the first member of (16) is contained in $U_{\pm1}V_{\pm2}$ and in $V_{\pm2}U_{\pm3}$, hence in $V_{\pm2}$, and the opposite inclusion follows from the last assertion of lemma 2.2. □

Lemma 5.2 *If $u \in U_i^*$ and $u' \in U_{i+4}^*$ are such that $[u, u']$ belongs to U_{i+2}, then $\mu(u)$ conjugates u' onto $[u, u']$, $\mu(u')^{-1}$ conjugates $[u, u']$ onto u^{-1} and $\mu(u')^{-1}\mu(u)$ conjugates u' onto u^{-1}.*

Proof. By the first assertion of lemma 2.2, $\nu(u)$ commutes with u', and the second assertion of lemma 2.2 then implies that $[u, u']$ is the conjugate of u' by $\mu(u)$. Since $[u', u] = [u, u']^{-1}$, the same argument shows that $\mu(u')^{-1}$ conjugates $[u, u']$ onto u^{-1}. Now the last part of the lemma ensues. □

Lemma 5.3 *If $V_i \neq \{1\}$, then $\nu^{-1}(V_i^*) \subset V_{i+6}$, the group V_i coincides with U_i and the conjugation by T° has a single orbit in V_i^*.*

Proof. We take $i = 0$. Let v be an element of V_0^*, let y be an element of V_4, and set $m = \mu(v) = v'vv''$ with $v', v'' \in U_6$. Since y commutes with U_6 and U_2, and since ${}^m y = [v, y]$ by lemma 2.2, we have

$$
{}^m y = {}^{v'v}y = {}^{v'}([v, y] \cdot y) = [v', [v, y]] \cdot [v, y] \cdot y,
$$

hence $[v', [v, y]] = y^{-1}$. By lemma 5.2, this implies that v' is conjugate to y by an element of N°, hence our first assertion. Now, (14) and (15) imply that $[v', U_2] \subset V_4$; again by lemma 5.2, it follows that $U_2 \subset V_2$, which proves the second assertion. Finally, the equality (16) and lemma 5.2 imply that any element of V_0^* can be conjugated into any element of V_4^* by an element of $\mu(V_0)\mu(V_4)$; since the quotient of two elements of this last set belongs to T°, the last assertion of the lemma follows. □

Lemma 5.4 *The commutator group of U_i is contained in V_i.*

Proof. As before, we take $i = 0$. Let x, y be two arbitrary elements of U_0, and let $u \in U_2$. Since x, y and u centralize U_1, so do $[u, x]$ and $[u, y]$. Those two commutators also centralize U_0 and U_2, therefore, they are central in $U_{[0,2]}$, and we have

$$^u[x, y] = [^u x, ^u y] = [[u, x] \cdot x \cdot [u, y] \cdot y] = [x, y].$$

Thus, $[x, y]$ centralizes U_2. By symmetry, it also centralizes U_{-2} and belongs therefore to V_0, hence the claim. □

Lemma 5.5 *The groups V_i are commutative.*

Proof. This follows from the equality (16), since V_i commutes with V_{i-2} and V_{i+2}. □

Lemma 5.6 *The groups V_i and U_{i+3} centralize each other.*

Proof. It will be sufficient to show that $[V_0, U_3] = \{1\}$. Suppose the contrary and let u be an element of U_5 which does not centralize V_2. By (16), we have $V_2 = [V_0, V_4]$, hence, using lemma 2.1 and lemma 5.5,

$$^u V_2 = [^u V_0, ^u V_4] \subset [U_{[0,3]} \cdot V_4, V_4] \subset U_{[0,3]}.$$

Consequently,

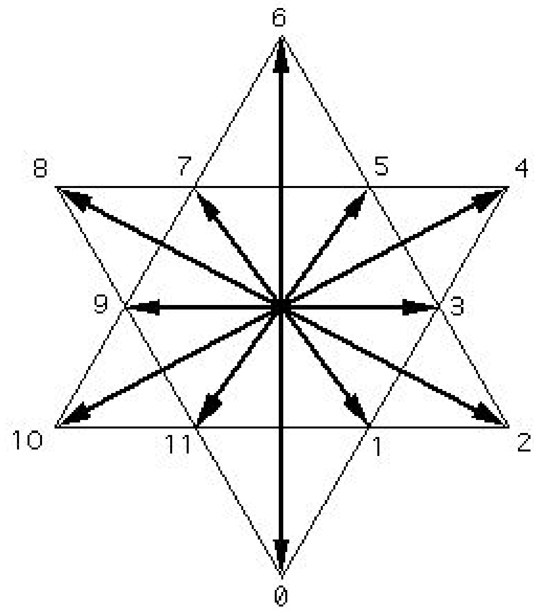
$$[u, V_2] \subset U_{[0,3]} \cap U_{[3,4]} \quad = \quad U_3. \tag{17}$$

Since V_2 centralizes $U_{[0,4]}$ (by lemma 5.5 and the definition of V_i), so does $[u, V_2]$. If x is any element of U_0, lemma 2.3 and (17) now imply that $[u, V_2]$ also commutes with $\mu(x)$, hence with $^{\mu(x)}U_{[0,4]} = U_{[2,6]}$. It follows that $[u, V_2] \subset V_3$, hence, by lemma 5.3, that $V_3 = U_3$ and that U_3 centralizes U_0, since $[u, V_2]$ does, a contradiction. □

Lemma 5.7 *Not all V_i are trivial.*

Proof. Suppose the contrary. By lemma 5.4, U_0 is commutative, hence central in $U_{[-1,1]}$. Therefore, the commutator $[U_0, U_2]$ is also central in $U_{[-1,1]}$. Symmetrically, it is central in $U_{[1,3]}$. As a result, it is contained in V_1, hence reduced to $\{1\}$. Similarly, $[U_0, U_{-2}] = \{1\}$. But then, $U_0 = V_0$, a contradiction. □

Proposition 5.8 *We assume (without loss of generality by lemma 5.7) that $V_1 \neq \{1\}$. Then, the groups U_i, labelled as in figure 5, form a root datum of type G_2.*



Root system of type G_2

Proof. By lemma 5.3, $U_1 = V_1$, hence $U_{2i+1} = V_{2i+1}$ for all i.

The axiom (RD0) is clearly satisfied and (RD2) is nothing else but (MP2).

For all i, U_i is commutative: this follows from lemma 5.3 and lemma 5.4. In order to prove axiom (RD1), one may therefore assume the roots a and b are not proportional. Passing in review the various possible configurations of the pair (a, b), one sees that, in all cases, the inclusion to be proved is an immediate consequence of one of the following statements: (MP1), lemma 5.6, the definition of V_i and the relation (16) in lemma 5.1.

Finally, the validity of (RD3) readily follows from lemma 2.4. □

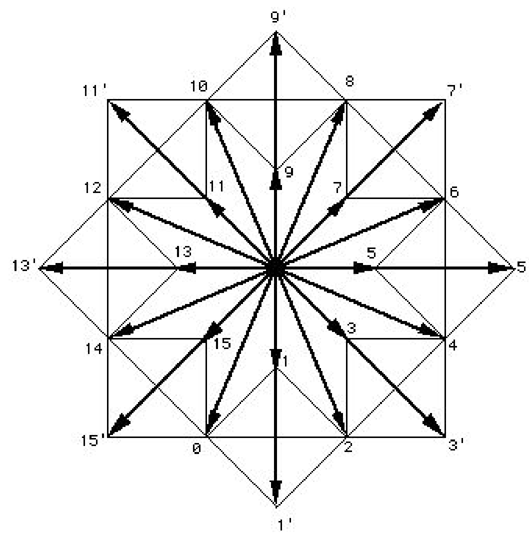
Remark. The above proposition shows that if V_i is not equal to U_i for all i — in other words, if U_i does not always commute with U_{i+2} —, then the root ray datum $(U_z)_{z\in\mathbb{Z}}$ has a unique filtration by a root datum of type G_2, up to similitude. If, on the other hand, U_i commutes with U_{i+2} for all i, then, there are exactly two such filtrations (the long roots for one of them corresponding to the short roots for the other). The classification of Moufang hexagons stated in [11], 4.7, shows that this happens only for the split groups of type G_2 in characteristic 3 and their "mixed" variations, described in [9], 10.3.2 (cf. also [11], 4.7, Remark, case (ii)).

6 The case $n = 8$

To be complete, let us briefly recall what happens if $n = 8$. The general reference for this case is [14]. Here,[1] 8 of the 16 groups U_i are of exponent 2 (hence abelian). We assume, without loss of generality, that this happens when i is even. For all i, the elements of order 1 or 2 in U_i form a central subgroup which we denote by $U_{i'}$, when i is odd; the quotient $U_i/U_{i'}$ is also a group of exponent 2. Let us label the

[1] I thank H. Van Maldeghem for pointing out an error in an earlier version of the manuscript.

24 groups U_z and $U_{(2z+1)'}$ by the 24 vectors a_z and $a_{(2z+1)'}$ shown on figure 6 below (where, as before, only the indices z and $(2z+1)'$ are written):



Root system of type $^2F_4$

We call this set of vectors a *root system of type* $^2F_4$, its elements being the *roots*. The results of [14], especially 1.4 and 1.7.1, show that the system of groups $U_i = U_{a_i}$, $U_{(2i+1)'} = U_{a_{(2i+1)'}}$ is a *root datum of type* $^2F_4$ in the following modified sense:

in (RD0), 2 must be replaced by $1+\sqrt{2}$;

in (RD1), $\mathbb{N}$ must be replaced by $\mathbb{N}+\mathbb{N}\sqrt{2}$;

in (RD2), the root a must be taken nondivisible, i.e. of the form a_i (whereas b can be any root) ;

finally, (RD3) is unchanged provided one defines bases of the root system in an appropriate way (e.g. as all images of the pair (a_1, a_8) by elements of the Weyl group).

N.B. The root system of type $^2F_4$ was introduced in [14],Figure 1, p. 574. The above representation, due to J.-Y. Hée ([5], Figure 3, p. 129), has, among others, the advantage of making all necessary numerical information (e.g. the coordinates of the roots with respect to a basis) graphically apparent.

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