Geometric hyperplanes of the half-spin geometries arise from embeddings

*E. E. Shult**

Dedicated to J. A. Thas on his fiftieth birthday

Abstract

Let the point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ be a half-spin geometry of type $D_{n,n}$. Then, for every embedding of Γ in the projective space $\mathbb{P}(V)$, where V is a vector space of dimension 2^{n-1}, it is true that every hyperplane of Γ arises from that embedding. It follows that any embedding of this dimension is universal. There are no embeddings of higher dimension. A corollary of this result and the fact that Veldkamp lines exist ([6]), is that the Veldkamp space of any half-spin geometry ($n \geq 4$) is a projective space.

1 Introduction

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a rank 2 incidence system, which we will call a *point-line* geometry. A *subspace* X is a subset of the set of points with the property that any line having at least two of its incident points in X, in fact has all its incident points in X. A proper subspace X is called a *geometric hyperplane* of Γ if and only if every line has at least one of its points in X.

Example. If $\mathbb{P} = \mathrm{PG}(n, F)$ is a projective space of (projective) dimension $n \geq 2$, truncated to its points and lines, then an ordinary projective hyperplane is a geometric hyperplane. (We shall often drop the adjective "geometric" and simply refer to geometric hyperplanes as "hyperplanes".)

*This work has been partially supported by a grant by the National Science Foundation.
Received by the editors February 1994
AMS Mathematics Subject Classification : Primary 51E24, Secondary 05B25
Keywords : Veldkamp spaces, embedding, half-spin geometry

An embedding $e : \Gamma \to \mathbb{P} = \mathbb{P}(V)$ of the point-line geometry Γ into the desarguesian projective space $\mathbb{P}(V)$ obtained from a vector space V is a pair of injective mappings

$$\begin{aligned} e_1 : \mathcal{P} &\to \{\text{1-subspaces of } V\} \\ e_2 : \mathcal{L} &\to \{\text{2-subspaces of } V\} \end{aligned}$$

such that

(i) any 1-subspace of a 2-space $e_2(L)$ is an image $e_1(p)$ for some point p incident with line L, and

(ii) the set $e_1(\mathcal{P})$ spans $\mathbb{P}(V)$.

Example. All the classical polar spaces have natural embeddings as the collections of all isotropic (or totally singular) 1-spaces and 2-spaces of a finite dimensional vector space V with respect to a non-degenerate (σ, ϵ)-hermitian (or pseudo quadratic) form.

We say that a geometric hyperplane H of Γ *arises from an embedding* $e : \Gamma \to \mathbb{P}$ if and only if there is an ordinary projective hyperplane $\mathbb{H}$ of $\mathbb{P}$ such that

$$H = e^{-1}\left(e(\mathcal{P}) \cap \mathbb{H}\right). \tag{1}$$

(It is easy to see that any subset H defined by the right hand side of (1) must be a geometric hyperplane of Γ.)

Some effort has been spent showing that hyperplanes of various geometries arise from an embedding $e : \Gamma \to \mathbb{P}$ ([3, 7]). The principal motivation has been to clarify the possible conclusion geometries which would arise in characterizing geometries whose planes are affine. For example in [2], there is a characterization involving relatively simple axioms, whose conclusion reads that Γ is a classical polar space with a geometric hyperplane removed. One can easily forsee up the road characterizations of geometries based on affine planes whose conclusion reads "Γ is a certain Lie incidence geometry with a hyperplane removed." If we know all such hyperplanes arise from an embedding $e : \Gamma \to \mathbb{P}(V)$, a study of the module V often can elucidate what these hyperplanes are.

There is a second motivation: knowing that all hyperplanes of Γ arise from an embedding $e : \Gamma \to \mathbb{P}$ provides quite a bit of information – facts about e as well as internal information about Γ. Indeed, one may conclude:

1) *The embedding e is universal.*

2) *If subspaces of codimension* 2 *in $\mathbb{P}$ are spanned by the image points, then Veldkamp lines exist.*

3) *If Veldkamp lines exist, then the entire Veldkamp space (see below) is a projective space* (see [8]).

Some of the terms here require explanation. The assertion that "Veldkamp lines exist" is simply the property

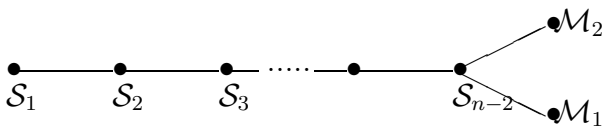
(V) *For any three pairwise distinct hyperplanes, A, B and C; $A \cap B \subseteq C$ implies $A \cap B = A \cap C$.*

If (V) holds for Γ, one can construct a linear space $(\mathcal{H}, \mathcal{V})$ (called the *Veldkamp space*) where $\mathcal{H}$ is the collection of all hyperplanes of Γ, and $\mathcal{V}$ is the set of intersections of two distinct hyperplanes (the *Veldkamp lines*) with containment defining incidence. Then (V) just says that any Veldkamp line is uniquely determined by any two of its points.

We now describe the half-spin geometries. Let $Q : V \to F$ be a non-degenerate quadratic form on a finite dimensional vector space V over a field F. Suppose $\dim V = 2n$, and V has a totally singular space of dimension n. Then all maximal totally singular subspaces have dimension n and are partitioned into two classes $\mathcal{M}_1$ and $\mathcal{M}_2$ subject to these rules:

> Two maximal totally singular spaces M_1 and M_2 of V belong to the same class if and only if $M_1 \cap M_2$ has even codimension in M_1 (or equivalently, M_2).

Let S_j be the collection of all j-dimensional singular subspaces of V. Then these varieties form a geometry Δ called a *building* of type D_n, which is a diagram geometry with diagram



The *half-spin geometry of type* D_n, is the point-line geometry $(\mathcal{M}_1, S_{n-2}) = (\mathcal{P}, \mathcal{L})$. Thus the *points* are the subspaces of V in $\mathcal{M}_1$, and the *lines* are the totally singular subspaces of V of dimension $n-2$.

We can now state the main result.

Theorem 1.1 *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a half-spin geometry of type D_n and suppose $e : \Gamma \to \mathbb{P}(V)$ is an embedding with $\dim V \geq 2^{n-1}$. Then $\dim V = 2^{n-1}$ and every hyperplane of Γ arises from the embedding e.*

It follows, as remarked before, that the embedding e is universal.

Such embeddings do exist. (They are doubtless unique up to isomorphism but that is not proved here). The standard half-spin module (obtained from a minimal ideal in the Clifford algebra) is an example.

The basic idea for this proof sprang from an insightful comment of J. A. Thas in the context of Grassmann spaces. The idea is to show that there are two disjoint subgeometries, say A and B, belonging to the same parameterized family of geometries as Γ, such that for the embedding $e : \Gamma \to \mathbb{P}(V)$ hypothesized, one has

$$V = \langle e(A) \rangle \oplus \langle e(B) \rangle.$$

Then if H is a geometric hyperplane of Γ, induction arguments can be used to show that $H \cap A$ and $H \cap B$ are hyperplanes of A and B (that is, neither A nor B is contained in H) and that

$$\langle e(H)\rangle = \langle e(H \cap A)\rangle \oplus \langle e(H \cap B)\rangle \oplus e(x)$$

where

$$x \in H - e^{-1}\left(\mathbb{P}\left(\langle e(H \cap A), e(H \cap B)\rangle\right)\right).$$

The arguments of this paper, showing that the same space $\langle e(H)\rangle$ obtains for every choice of x, depend critically on the following result:

Lemma 1.2 (Shult [6]) *The half-spin geometries of type $D_{n,n}$ possess Veldkamp lines.*

(This is true for a wider class of strong parapolar spaces such as $E_{6,1}$ and $E_{7,1}$. In fact, half-spin geometries possess Veldkamp planes. But we do not require these results here.)

Thas' idea has been formally developed and is exploited in forthcoming joint work on the hyperplanes of the dual polar spaces ([9]).

Section 2 developes the necessary properties of the half-spin geometries needed to carry out the proof, which appears in section 3.

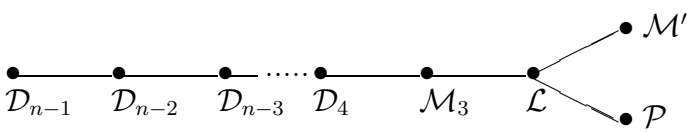
The author gratefully acknowledges the warm hospitality and support of the Universiteit Gent and the support of the National Fund of Scientific Research of Belgium, during this work.

2 Basic facts about the half-spin geometries

We first fix a totally singular k-subspace $U \in S_k$ and consider $D(U) : \mathcal{M}_1 \cap \mathrm{Res}(U)$, the set of maximal singular subspaces of V in $\mathcal{M}_1$ which contain U. Now the collection of all singular subspaces of V which contain U is closed under intersection; thus if two of them meet at an $(n-2)$-space B, then every member of $\mathcal{M}_1$, containing B contains U. This means that as a subset of $\mathcal{P}$, $D(U)$ is a subspace. We denote the collection of subspaces of the form $D(U)$, where U ranges over S_k by the symbol $\mathcal{D}_{n-k}$. This notation is intended to be suggestive of the fact that the singular subspaces of V containing U are precisely the singular subspaces of $U^{\perp}/U$ with respect to the induced quadratic form.

Finally, an easy induction proof shows that if M_1 and M_2 are singular subspaces in $\mathcal{M}_1$, then any *geodesic* in Γ consists of spaces of $\mathcal{M}_1$ containing $M_1 \cap M_2$ and has length equal to the codimension of $M_1 \cap M_2$ in M_1. We record this as

Theorem 2.1 (i) *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a half-spin geometry of type D_n. Then Γ is a diagram geometry with diagram*



where each subspace $X \in \mathcal{D}_k$ is a convex subspace of Γ which itself is a half-spin geometry of type D_k, $k = 4, 5, \ldots, n-1$.

(ii) *Γ has diameter $[n/2]$.*

(iii) *If two points x and y are at distance $d(x, y) = d \geq 2$ in the collinearity graph on points, then the convex closure $\langle x, y\rangle$ (that is, the smallest convex subspace containing x and y) is a member of $\mathcal{D}_{2d}$.*

Theorem 2.2 (i) *The elements of $\mathcal{M}_2$ in Δ correspond to a class of maximal singular subspaces (that is, subspaces which are maximal with respect to being linear spaces) which are projective spaces of type* $\mathrm{PG}(n-1, F)$. *(This class is denoted $\mathcal{M}'$ in the diagram of proposition* 2.1.*)*

(ii) *The class S_{n-3} of Δ corresponds to a class of singular subspaces of Γ (denoted $\mathcal{M}_3$ in proposition* 2.1 *of type* $\mathrm{PG}(3, F)$.

(iii) *Every singular subspace of Γ lies in a member of $\mathcal{M}_3$ or a member of $\mathcal{M}'$.*

(iv) *Each subspace $X \in \mathcal{D}_k$, $4 \leq k \leq n-1$, also possesses two classes of maximal singular subspaces, $\mathcal{M}_3(X)$ consisting of* $\mathrm{PG}(3, F)$*'s and $\mathcal{M}'(X)$ consisting of* $\mathrm{PG}(k-1, F)$. *We have*

(a) $\mathcal{M}_3(X) \subseteq \mathcal{M}_3$
and

(b) *each subspace $A \in \mathcal{M}'(X)$ lies in a unique member $\hat{A}$ of $\mathcal{M}'$.*

(v) *Two members of $\mathcal{M}'$ intersect at either a line or the empty set.*

(vi) *If $(D, M) \in \mathcal{D}_{n-1} \times \mathcal{M}'$, then $D \cap M$ is either a hyperplane of M (in $\mathcal{M}'(D)$) or a single point.*

(vii) *If $(p, M) \in \mathcal{P} \times \mathcal{M}'$, then $p^\perp \cap M = \emptyset$ or is a plane, or else $p \in M$.*

(viii) *If $(S, D) \in \mathcal{D}_4 \times \mathcal{D}_{n-1}$, either $S \subseteq D$, $S \cap D$ is a member of $\mathcal{M}_3$ or $S \cap D$ is empty.*

Theorem 2.3 (Subspaces belonging to $\mathcal{D}_{n-1}$) (i) *Two distinct members of $\mathcal{D}_{n-1}$ either intersect at a subspace of $\mathcal{D}_{n-2}$, or are disjoint (in which case we say they are* opposite*).*

(ii) *If $(p, D) \in \mathcal{P} \times \mathcal{D}_{n-1}$ and p is not incident with D, then $A_p = p^\perp \cap D \in \mathcal{M}'(D)$ and $\langle p, p^\perp \cap D\rangle$ is $\hat{A}_p$, the unique element of $\mathcal{M}'$ containing A_p (see* proposition 2.2(iv)(b)*).*

(iii) *If x and y are two distinct points of $\mathcal{P} - D$ for a subspace $D \in \mathcal{D}_{n-1}$, then the two subspaces $A_x := x^\perp \cap D$ and $A_y := y^\perp \cap D$ of $\mathcal{M}'(D)$ intersect at a line, if and only if x is collinear with a point $y' \in \langle y, A_y\rangle - A_y := \hat{A}_y - D$.*

(iv) *Let D_1 and D_2 be opposite subspaces in $\mathcal{D}_{n-1}$ (i.e. $D_j \in \mathcal{D}_{n-1}$, $j = 1, 2$, and $D_1 \cap D_2 = \emptyset$). Then there exists an isomorphism (which we call a* duality mapping*)*

$$(D_1, \mathcal{L}(D_1)) \to (\mathcal{M}'(D_2), \mathcal{L}(D_2))$$

which takes $x \in D_1$ to $A_x := x^\perp \cap D_2 \in \mathcal{M}'(D_2)$.

(v) *For each* $D \in \mathcal{D}_{n-1}$*, there is a duality automorphism*

$$\sigma : (D, \mathcal{L}(D)) \to (\mathcal{M}'(D), \mathcal{L}(D)) .$$

(vi) *Let* X *and* Y *be disjoint subspaces belonging to one of the varieties of the diagram in proposition* 2.1*, but not belonging to* $\mathcal{M}'$*. Then there exists a subspace* $D \in \mathcal{D}_{n-1}$ *such that* $X \subseteq D$ *but* $D \cap Y = \emptyset$*.*

(vii) *Let* D_1 *and* D_2 *be opposite members of* $\mathcal{D}_{n-1}$*. Let* $Z \in \mathcal{D}_k$*,* $4 \leq k \leq n$ *(where* $\mathcal{D}_n$ *is interpreted to be the subspace* $\{\mathcal{P}\}$*) and suppose* Z *contains a point* x *not in* $D_1 \cup D_2$*. Then there exists a subspace* $Y \in \mathcal{D}_{k-1}$ *containing* x *such that* $D_1 \cap Y = \emptyset = D_2 \cap Y$*.*

(viii) *Let* D_1 *and* D_2 *be opposite members of* $\mathcal{D}_{n-1}$*. Then each point* x *of* $\mathcal{P} - (D_1 \cup D_2)$ *lies on a unique line* L_x *(called a transversal line) which intersects each* D_i *at a point.*

(ix) *Let* D_1 *and* D_2 *be opposite members of* $\mathcal{D}_{n-1}$ *and suppose* $Y \in \mathcal{D}_k$*,* $4 \leq k \leq n-1$ *with* $Y \cap D_1 = \emptyset = Y \cap D_2$*. Then there exist embeddings*

$$\pi_i : Y \to D_i$$

which are isomorphisms of Y *into a subspace of* D_i *with* $\{\pi_i(y)\} = L_y \cap D_i$*, for each* $y \in Y$*,* $i = 1, 2$*. (Here* L_y *is the unique transversal line on* y *(*see part (viii) of this proposition*).)*

All of the above, are elementary consequences of a polar space interpretation of Γ.

In proposition 2.2, the subspaces (D, M) in $\mathcal{D}_{n-1} \times \mathcal{M}'$ are interpreted as a pair $(d, m) \in S_1 \times \mathcal{M}_2$ of Δ, that is, a pair consisting of a point d and a maximal totally singular subspace m belonging to the class $\mathcal{M}_1$. Of course either d is incident with m or there is a unique space $\langle d, d^\perp \cap m \rangle_\Delta$ in $\mathcal{M}_1 (= \mathcal{P})$ incident with both d and m. Reinterpreting these alternatives back into the point-line language of Γ yields the results.

Similarly proposition 2.1(i) is equivalent to the assertion that in the polar space Δ, two polar points (elements of S_1) either lie in no maximal singular subspace of $\mathcal{M}_1$ or are incident with a common polar line (element of S_2).

In the same vein, proposition 2.3(vi), when interpreted in terms of Δ, just says that if X and Y are singular subspaces of V not in $\mathcal{M}_2$ and not both incident with a member of $\mathcal{M}_1$, then there is a polar point in X not incident with a member of $\mathcal{M}_1$ in common with Y. The restrictions on X and Y mean that their incidence with a member of $\mathcal{M}_1$ is that of containment, and that $X \not\subseteq Y^\perp$. So the desired polar point is any point of $X - Y^\perp$.

Proposition 2.3(vii) is similar. D_1 and D_2 correspond to two non-collinear polar points d_1 and d_2 of Δ, and Z is regarded as a totally singular subspace of V of dimension $n - k$ (possibly zero) contained in the maximal singular space X in $\mathcal{M}_1$. Then there is a 1-space s in X not contained in $d_1^\perp \cup d_2^\perp$ (for X cannot be the union of just two of its hyperplanes $d_i^\perp \cap X, i = 1, 2$). We then form the totally singular

$(k+1)$-space $Y = \langle s, Z\rangle$. Then $Z \leq Y \leq X$ and $\langle Y, d_i\rangle$ is not totally singular, $i = 1, 2$. Thus, as a subspace of $\mathcal{D}_{k-1}$, Y satisfies the desired conclusions in Γ.

The remaining parts of the proposition can be proved directly in Γ. Let D_1 and D_2 be two opposite members of $\mathcal{D}_{n-1}$. Suppose x is a point of $\mathcal{P}$ outside $D_1 \cup D_2$. Then by proposition 2.3(ii), $x^\perp \cap D_1 \in \mathcal{M}'(D_1)$ and is a hyperplane of $M_1 = \langle x, x^\perp \cap D_1\rangle \in \mathcal{M}'$. Then as $D_1 \cap D_2 = \emptyset$, the intersection $D_2 \cap M_1$ is at most a point. By proposition 2.2(vi), $D_2 \cap M$ *does* consist of a single point x_2. Then the line L_x on x and x_2 must intersect the hyperplane $x^\perp \cap D_1$ of M_1 at a point x_1. Thus L_x is a transversal line. But setting $M_2 = \langle x, x^\perp \cap D_2\rangle \in \mathcal{M}'$, we see that $M_1 \cap M_2$ contains any transversal line, and by proposition 2.2(v), $M_1 \cap M_2$ is a unique line, and hence the unique transversal line. Thus proposition 2.3(viii) holds.

Now assume Y is a subspace in $\mathcal{D}_k$, $4 \leq k \leq n-1$, with $Y \cap D_1 = \emptyset = Y \cap D_2$. For each point $y \in Y$, let L_y be the unique transversal line on the point y. We define the maps $\pi_i : Y \to D_i$ by setting $L_y \cap D_i = \{\pi_i(y)\}$, $i = 1, 2$.

Suppose $\pi_1(x) = \pi_1(y)$ for x and y in Y. Set $p = \pi_1(x)$, $A_p = p^\perp \cap D_2 \in \mathcal{M}'(D_2)$ and $\langle p, p^\perp \cap D_2\rangle = \hat{A}_p \in \mathcal{M}'$. Then x and y are points of $Y \cap \hat{A}_p$. But Y is a subspace disjoint from D_2 and hence disjoint from the hyperplane A_p of $\hat{A}_p$. Thus $Y \cap \hat{A}_p$ is a single point so $x = y$. Thus π_1 is injective. Similarly π_2 is injective.

Now suppose x and y are distinct collinear points of Y. Then by part (iii), the two subspaces $A_x = x^\perp \cap D_1$ and $A_y = y^\perp \cap D_1$ of $\mathcal{M}'(D_1)$ meet at a line. But $A_x = x_2^\perp \cap D_1$ and $A_y = y_2^\perp \cap D_1$ where $x_i = L_x \cap D_i$, $y_i = L_y \cap D_i$, $i = 1, 2$ and L_x and L_y are the unique transversal lines on x and y, respectively. Then as (x_2, A_x) and (y_2, A_y) are corresponding pairs under the duality mapping $(D_2, \mathcal{L}(D_2)) \to (\mathcal{M}'(D_1), \mathcal{L}(D_1))$ of part (iii), we deduce that $x_2 = \pi_2(x)$ is collinear with $y_2 = \pi_2(y)$. Thus $\pi_2 : Y \to D_2$ preserves collinearity. A similar argument yields the same result for π_1.

Finally, we must show that if $\pi_i(x)$ is collinear with $\pi_i(y)$, for x and y in Y, then x is collinear with y, $i = 1, 2$. Without loss we can take $i = 1$. It is an easy argument that Y lies in a subspace D_3 in $\mathcal{D}_{n-1}$ opposite both D_1 and D_2. (This entails only the reinterpretation of (D_1, D_2, Y) as two 1-spaces and a totally singular $(n-k)$-space (d_1, d_2, Y) of V, and choosing 1-space d_3 in Y outside its two hyperplanes $d_i^\perp \cap Y$, $i = 1, 2$.) Then the transversal lines L_x and L_y relative to D_1 and D_2, are also transversal lines on $\pi_1(x)$ and $\pi_1(y)$ relative to D_3 and D_1. Such transversal lines on points of D_1 define injections $\psi_2 : D_1 \to D_2$ and $\psi_3 : D_1 \to D_3$ which preserve collinearity. But the dual descriptions of the lines L_x and L_y show that

$$\psi_3(\pi_1(x)) = x \quad \text{and} \quad \psi_3(\pi_1(y)) = y$$

and so x is collinear with y as ψ_3 preserves collinearity. Thus the π_i are as stated and proposition 2.3(ix) holds.

We say that a half spin geometry of type D_n has *even type* if n is even – otherwise it is *of odd type*. There is a considerable difference in the internal structure of the geometries of odd type versus those of even type.

If x is a point of $\mathcal{P}$, and k is a positive integer, we let $\Delta_k(x)$ denote the set of points y such that $d(x, y) \leq k$.

Theorem 2.4 *Assume Γ is a half-spin geometry of even type D_n where $n = 2m \geq 4$.*

(i) *Let p be any point, and let M be a maximal singular subspace of Γ in the class $\mathcal{M}'$. Then either*

(a) *$\Delta_{m-2}(p) \cap M$ has codimension 0 or 3 in M and $M \subseteq \Delta_{m-1}(p)$, or*

(b) *$\Delta_{m-1}(p) \cap M$ is a hyperplane of M.*

(ii) *$\Delta_{m-1}(p)$ is a geometric hyperplane of Γ.*

Proof. (i) easily follows from the polar space interpretation that there is a $D \in \mathcal{D}_{n-1}$ containing p and meeting M at a hyperplane. The latter lies in $\Delta_{m-1}(p)$ as $\mathrm{diam}(D) = m - 1$. (ii) is an immediate consequence of (i). □

Theorem 2.5 *Assume Γ is a half-spin geometry of odd type D_n, where $n = 2m+1 \geq 5$. Assume p is a point and $M \in \mathcal{M}'$. Then one of the following holds*

(a) *$\Delta_{m-2}(p) \cap M$ has codimension ≤ 4 in M; $M \subseteq \Delta_{m-1}(p)$.*

(b) *$\Delta_{m-1}(p) \cap M$ has codimension 2 in M, or*

(c) *$\Delta_{m-1}(p) \cap M = \emptyset$.*

If Γ is of odd type with $\mathrm{diam}(\Gamma) = m$, and M is an element of $\mathcal{M}' = \mathcal{M}'(\Gamma)$, we say that a point p is *near* M if conditions (a) or (b) of proposition 2.5 hold – i.e. if $\Delta_{m-1}(p) \cap M \neq \emptyset$. We denote the set of points of Γ which are near M by the symbol $N(M)$. We need to make this concept relative to subspaces D in $\mathcal{D}_k$ where k is even ($4 \leq k \leq n-1$). In this case, $\mathrm{diam}(D) = k/2$, and for each $M \in \mathcal{M}'(D)$, the *points of D near M* are those of the set

$$N_D(M) := \left\{ p \in D \mid \Delta_{(k/2)-1}(p) \cap M \neq \emptyset \right\}.$$

Since, by proposition 2.3(v), there is a duality automorphism σ of Γ, there is an isomorphism of the collinearity graphs on $\mathcal{P}$ and on $\mathcal{M}'$. We therefore obtain a distance metric $d' : \mathcal{M}' \times \mathcal{M}' \to \mathbb{Z}$.

Theorem 2.6 *Let Γ be a half-spin geometry of odd type and diameter m. Let L be any line and let M be an element of $\mathcal{M}'$. We let $\mathcal{M}' \cap \mathrm{Res}(L)$ be the elements of $\mathcal{M}'$ which contain line L. Then one of the following holds:*

(a) *$L \cap N(M)$ contains a single point. Also, $d'(M, M') = m$ for each $M' \in \mathcal{M}' \cap \mathrm{Res}(L)$.*

(b) *$L \subseteq N(M)$, $d'(M, M') = m - 1$ for a unique element M' in $\mathcal{M}' \cap \mathrm{Res}(L)$*

(c) *$L \subseteq N(M)$, $d'(M, M') \leq m - 1$ for all $M' \in \mathcal{M}' \cap \mathrm{Res}(L)$.*

Proofs of proposition 2.5 and 2.6. Both are easily proved by the polar space interpretation. In proposition 2.5, p and M are represented in Δ by maximal totally singular subspaces P and M of V, belonging to $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Thus $P \cap M$ has even dimension and (a), (b) and (c) represent the cases $\dim(P \cap M) \leq 4$, 2 or 0, respectively. For example, when $P \cap M$ contains a subspace of dimension 4, this means that in Γ, p and M are both incident with a subspace $D \in \mathcal{D}_{n-4}$, and,

as $\mathrm{diam}(D) = m-2$, $D \cap M \leq \Delta_{m-2}(p)$. But incidence of D and M means, $D \cap M$ has codimension 4 in M, from which (a) follows. The proofs of (b) and (c) in the other two cases are similar.

Proposition 2.5 has a similar proof. For $(a, M_1, M_2) \in \mathcal{P} \times \mathcal{M}' \times \mathcal{M}'$, interpreted as totally singular n-subspaces $(A, M_1, M_2) \in \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_2$, we have the following:

$$\begin{array}{ll} a \in N(M_1) & \text{if and only if } A \cap M_1 \text{ contains a 2-space} \\ d'(M_1, M_2) \leq m-1 & \text{if and only if } M_1 \cap M_2 \text{ contains a 3-space.} \end{array} \tag{2}$$

(As n is odd, $\dim(A \cap M)$ is even and $\dim(M_1 \cap M_2)$ is odd.) A line L is interpreted as a totally singular 2-space L'. Now using (2) it is easy to deduce conclusions (a), (b) and (c) from the respective cases $\dim(L' \cap M') \geq 2$, $\dim(L' \cap M') = 1$ and $L' \cap M' = 0$, where M' is the totally singular subspace of $\mathcal{M}_2$ corresponding to $M \in \mathcal{M}'$. □

Theorem 2.7 *Assume* Γ *is a half-spin geometry of odd type with* $\mathrm{diam}(\Gamma) = m$

(i) *For any* $M \in \mathcal{M}'$, $N(M)$ *is a geometric hyperplane of* Γ

(ii) *Let* p *and* q *be distinct points. Then there is an element* $M_1 \in \mathcal{M}'$ *such that* $p \notin N(M_1)$ *while* $q \in N(M_1)$.

(iii) *For any* M_1, $M_2 \in \mathcal{M}'$, $N(M_1) = N(M_2)$ *implies* $M_1 = M_2$.

Proof. (i) This follows immediately from proposition 2.6.

(ii) Let $(p = p_0, \ldots, p_m)$ be a geodesic of maximal length running through q, so $q = p_k$ for some k, $1 \leq k \leq m$. We let M' be an element of $\mathcal{M}'$ containing the line on p_{m-1} and p_m. Then as $d(p, p_m) = m$, we see from proposition 2.5 that $\Delta_{m-1}(p) \cap M'$ has codimension 2 in M'. It is therefore possible to select a line L within M' on p_m so that $L \cap \Delta_{m-1}(p) = \emptyset$. Applying a duality automorphism σ, this says that all elements M'' of $\mathcal{M}' \cap \mathrm{Res}(L^\sigma)$ satisfy $d'(M'', p^\sigma) = m$ so L^σ and p^σ are in the relationship (a) of proposition 2.6. But then by the proposition, $|L^\sigma \cap N(p^\sigma)| = 1$. Applying σ^{-1}, this asserts that there is a *unique* element M'' in $\mathcal{M}' \cap \mathrm{Res}(L)$ such that $p \in N(M'')$. It is thus possible to find M_1 in $\mathcal{M}' \cap \mathrm{Res}(L)$ so that p is *not* in $N(M_1)$. But as $d(q, p_m) \leq m-1$, $\Delta_{m-1}(q) \cap M_1 \neq \emptyset$ so $q \in N(M_1)$. Thus M_1 has the desired properties.

(iii) Suppose M_1 and M_2 are members of $\mathcal{M}'$ with $M_1 \neq M_2$. Let σ be a twisting automorphism. Then $p = M_1^\sigma$ and $q = M_2^\sigma$ are distinct points of $\mathcal{P}$ and so by part (ii), there exists an element M' in $\mathcal{M}'$ with $p \notin N(M')$ and $q \in N(M')$. Let us apply σ^{-1} and let $x = \sigma^{-1}(M')$. Then x is an element of $N(M_2)$ but not $N(M_1)$. Thus $M_1 \neq M_2$ implies $N(M_1) \neq N(M_2)$. The contrapositive of this is the assertion of (iii). The proof is complete. □

Theorem 2.8 *Let* Γ *be a half-spin geometry of odd type. Suppose* M_1, M_2 *and* M *are members of* $\mathcal{M}'$ *such that* $M_1 \cap M_2 = L$, *a line, and* $N(M) \supseteq N(M_1) \cap N(M_2)$. *Then* M *contains* L.

Proof. Suppose M does not contain L. We claim there is a point of $N(M_1) \cap N(M_2)$ not in $N(M)$. Applying a twisting automorphism σ (which transposes $\mathcal{P}$ and $\mathcal{M}'$ but stabilizes all other varieties), and setting $n_1 = M_1^\sigma$, $n_2 = M_2^\sigma$, $t = M^\sigma$ and $L' = n_1 n_2 = L^\sigma$, we see that t is not in L. We must then find an element $B \in \mathcal{M}'$ such that $L' \subseteq N(B)$ but $t \notin N(B)$.

Let D be an element of $\mathcal{D}_{n-1}$ containing L' but not t (D exists by proposition 2.3(vi)). We set $M_t = t^\perp \cap D$, an element of $\mathcal{M}'(D)$. By proposition 2.3(v), D admits a duality automorphism and so the collinearity graphs on D and $\mathcal{M}'(D)$ are isomorphic. Since each point of D is at distance $m = \mathrm{diam}(D)$ from some point, there must exist an element A in $\mathcal{M}'(D)$ at distance m from M_t in the collinearity graph on $\mathcal{M}'(D)$. We let $B = \hat{A}$ and $T = \langle t, M_t\rangle = \hat{M}_t$, the unique element of $\mathcal{M}'$ containing A and M_t in $\mathcal{M}'(D)$ (proposition 2.2(iv)(a)). Then by duality, just as D is a convex subspace of Γ, so D is also convex as a subspace of $(\mathcal{M}', \mathcal{L})$. Thus A and M_t being at distance m in $\mathcal{M}'(D)$ implies $d'(B, T) = m$. But this means D is the unique member of $\mathcal{D}_{n-1}$ incident with both B and T and the hyperplane $M_t = D \cap T$ consists of all elements x of T for which $\Delta_{m-1}(x) \cap B \neq \emptyset$. Thus we have $\Delta_{m-1}(t) \cap B = \emptyset$ and so t is not in $N(B)$.

But for each point x of D we see (as D is even type) that $\Delta_{m-1}(x) \cap A$ is a hyperplane of A. Thus $x \in N(B)$, and in fact $D \subseteq N(B)$. In particular $L' \subseteq N(B)$.

Thus $B \in \mathcal{M}'$ has the desired properties, and the proof is complete. □

3 Proof of theorem 1

We begin with an embedding $e : \Gamma \to \mathbb{P}(V)$ where $\Gamma = (\mathcal{P}, \mathcal{L})$ is a half spin geometry of type D_n, $n \geq 4$, and V is a vector space of dimension at least 2^{n-1}. From the definition of embedding we have $\langle e(\mathcal{P})\rangle = V$.

We handle first the case $n = 4$. In this case Γ is a non-degenerate polar space since $D_{4,4}$ is isomorphic to $D_{4,1}$ as Lie incidence geometries. The theorems of Buekenhout-Lefèvre and Dienst [1, 4, 5] show that e must be the natural embedding – that is, $e(\mathcal{P})$ is the quadric of maximal Witt index in $\mathbb{P}(V) \simeq \mathrm{PG}(7, F)$, and $\dim V = 8$. If H is a hyperplane of Γ, then either $H = p^\perp$ or H is itself a non-degenerate polar subspace of Γ. In the former case, $\langle e(H)\rangle \subseteq e(p)^\perp$ where "$\perp$" is with respect to the sesquilinear forms associated with the quadratic form on V defining the quadric $e(p)$. But $e(H)$ consists of *all* singular 1-spaces in $e(p)^\perp$ and as these generate $e(p)^\perp$, the containment of the previous sentence is an equality. Thus H arises from an embedding in this case.

In the second case H is a non-degenerate polar subspace. But then, applying the Buekenhout-Lefèvre-Dienst theory to H, this time, we see that $e|_H : H \to \mathbb{P}\langle e(H)\rangle$ is again a natural embedding of H which is a dominated embedding in the sense of Tits [10]. Lemma 8.6 of Tits [10] then shows $\mathbb{P}\langle e(H)\rangle$ is a projective hyperplane of $\mathbb{P}(V)$ and $e(H)$ exhausts all singular points in this hyperplane. Thus in this case H also arises from the embedding.

This proves the result for $n = 4$, so we may now assume

$$n > 4 \tag{3}$$

and apply induction on n.

Now choose opposite subspaces D_1 and D_2 of $\mathcal{D}_{n-1}$. Now, as each point of $\mathcal{P}$ not in $D_1 \cup D_2$ lies on a transverse line connecting D_1 and D_2, it follows that the subspace of Γ generated by $D_1 \cup D_2$ is all of $\mathcal{P}$. It follows that $V = \langle e(\mathcal{P})\rangle = \langle e(D_1) \cup e(D_2)\rangle$, and by hypothesis $\dim V \geq 2^{n-1}$. Now D_i is a half spin space of type D_{n-1} and e restricted to D_i yields an embedding $e_i : D_i \to \mathbb{P}(\langle e(D_i)\rangle)$, $i = 1, 2$. By induction, $\dim(\langle e(D_i)\rangle) \leq 2^{n-2}$. Yet, as $\langle e(D_1)\rangle$ and $\langle e(D_2)\rangle$ span V, $\dim(V) > 2^{n-1}$ would imply that at least one of the $\langle e(D_i)\rangle$ would have dimension exceeding 2^{n-2}, a contradition. Thus we see $\dim V = 2^{n-1}$ exactly, $\dim(\langle e(D_i)\rangle) = 2^{n-2}$ exactly, and that V is a direct sum $V = \langle e(D_1)\rangle \oplus \langle e(D_2)\rangle$.

We are to prove, that for a hyperplane H of Γ, $\langle e(H)\rangle$ is a vector space hyperplane of V. Set $H_i = H \cap D_i$, $i = 1, 2$. Then either $H = D_i$ or else H_i is a geometric hyperplane of D_i.

If $H_i = D_i$ for both $i = 1, 2$, then $\mathcal{P} = \langle D_1, D_2\rangle \subseteq H$, against H being a hyperplane of Γ.

Suppose, for one of the i (say $i = 1$), we have $D_i = H_i$. Then $D_1 \subseteq H$ while H_2 is a hyperplane of D_2. Moreover, if $h \in H - (D_1 \cup D_2)$, and L_h is the unique transversal line on h, then $\{h_1\} = L_h \cap D_1$ is contained in H, so $L_h \subseteq H$ so $\{h_2\} = L_h \cap D_2$ is contained in H_2. Thus H is the subspace generated by $D_1 = H_1$ and H_2 and $\langle e(H)\rangle = \langle e(D_1)\rangle \oplus \langle e(H_2)\rangle$. Now as we have seen, the first summand has dimension 2^{n-2} and the second summand has dimension $2^{n-2} - 1$, since, by induction, it is a vector space hyperplane of $\langle e(D_2)\rangle$. Thus $\langle e(H)\rangle$ has dimension $2^{n-1} - 1$ and so is a vector space hyperplane of V, our desired conclusion.

Thus we may assume

$$H_i = H \cap D_i \quad \textit{is a hyperplane of } D_i,\ i = 1, 2. \tag{4}$$

Then each $\langle e(H_i)\rangle$ has dimension $2^{n-2} - 1$, $\langle e(H_1)\rangle \oplus \langle e(H_2)\rangle$ has dimension $2^{n-1} - 2$, and contains the embedded 1-space $e(h)$ for every point $h \in H - (D_1 \cup D_2)$ whose unique transversal line L_h is contained in H. Therefore, $\langle e(H)\rangle$ is generated by $e(H_1)$, $e(H_2)$ and 1-spaces $e(p)$ where p ranges over the set

$$X = \{x \in H - (D_1 \cup D_2) \mid L_x \cap D_i \not\subseteq H, \quad i = 1, 2\}$$

where L_x, as usual, denotes the unique transversal line on x. We shall achieve our goal that $\langle e(H)\rangle$ is a vector space hyperplane, if we can show

$$\langle e(H_1)\rangle \oplus \langle e(H_2)\rangle \oplus \langle e(x)\rangle$$

is the same vector space (hyperplane) independently of the choice of $x \in X$.

We define a graph $\mathfrak{X} = (X, \approx)$ with vertex set X, by asserting that vertex x is adjacent to vertex y (denoted $x \approx y$) if and only if

$$\langle e(H_1), e(H_2), e(x)\rangle = \langle e(H_1), e(H_2), e(y)\rangle$$

as vector space hyperplanes of V.

Our first observation is the following:

$$\text{No point } x \text{ of } X \text{ satisfies } e(x) \leq \langle e(H_1)\rangle \oplus \langle e(H_2)\rangle. \tag{5}$$

If this were true, we could put $e(x) = \langle w_1 + w_2 \rangle$ where w_i is a vector in $\langle e(H_1) \rangle$, $i = 1, 2$. But then $w_i \in \langle e(D_i) \rangle$, $i = 1, 2$, and so $\langle w_1, w_2 \rangle$ is a 2-space containing $e(x)$ meeting each $\langle e(D_i) \rangle$ non-trivially. But $e(L_x)$ is also such a 2-space, meeting each $\langle e(D_i) \rangle$ at $\langle e(x_i) \rangle$, $i = 1, 2$. Since $V = \langle e(D_1) \rangle \oplus \langle e(D_2) \rangle$ is a direct sum, we have $\langle x_i \rangle = \langle w_i \rangle$, $i = 1, 2$. Thus $\langle x_i \rangle \leq \langle e(H_i) \rangle$. But by induction, the hyperplane H_1 *arises* from the embedding $e|_D$, which means that every embedded point – such as $e(x_i)$ – found inside the hyperplane $\langle e(H_i) \rangle$, must be the image of a point belonging to H_i, $i = 1, 2$. Thus one deduces $x_i \in H$, which contradicts the definition of X.

Our second observation is the following:

$$\text{If } x \text{ and } y \text{ are collinear points of } X \text{ then } x \approx y. \tag{6}$$

To see this, set $\{x_i\} = L_x \cap D_i$, and $\{y_i\} = L_y \cap D_i$, $i = 1, 2$. Suppose x is not collinear with y_2. Then $d(x, y_2) = 2$, and by proposition 2.1(iii), the convex closure of x and y_2 in Γ is a subspace $S \in \mathcal{D}_4$, which contains y_1 and $L_y = yy_2$. Then by proposition 2.2(viii) $S \cap D_i = M_i \in \mathcal{M}_3$. Since $y_2 \in M_2 - H$, $M_2 \cap H = \Pi_2$ is a projective plane. Now from the discussion when $n = 4$, $\dim(\langle e(S) \rangle) = 8$ and as S is not in H $\dim(\langle e(S \cap H) \rangle) = 7$. Then if M_1 were contained in H, $\langle e(S \cap H) \rangle = \langle e(M_1) \rangle \oplus \langle e(\Pi_2) \rangle$, the direct sum of a 4-space and a 3-space, and this contradicts $e(x)$ not in $\langle e(H_1) \oplus \langle e(H_2) \rangle$ by (5). Thus $M_1 \cap H = \Pi_1$ is also a projective plane, so the 7-space $\langle e(S \cap H) \rangle$ would contain the two 7-spaces $W_x = \langle e(\Pi_1) \rangle \oplus \langle e(\Pi_2) \rangle \oplus e(x)$ and $W_y = \langle e(\Pi_1) \rangle \oplus \langle e(\Pi_2) \rangle \oplus e(y)$. (Note that (5) is used to justify the second direct sum sign.) Since $\dim(\langle e(\Pi_i) \rangle) = 3$ for $i = 1, 2$ (for there is only one way to embed a projective space) we have $W_x = \langle e(S \cap H) \rangle = W_y$, whence $\langle e(H_1), e(H_2) \rangle \oplus e(x) = \langle e(H_1), e(H_2) \rangle \oplus e(y)$ and $x \approx y$.

Thus we can assume x is collinear with y_2. Then x_2 is collinear with y_2 as $x^\perp \cap D_2$ is a clique. By a similar argument y is collinear with x_1 and so x_1 is collinear with y_1. So we are reduced to the case that $L_x \cup L_y$ generates a singular subspace M of Γ, belonging to $\mathcal{M}_3$. Thus $L_i = x_i y_i = M \cap D_i$, is a line carrying a unique point h_i of H, $i = 1, 2$. Also $H \cap M$ is a plane Π, and so $\dim(\langle e(\Pi) \rangle) = 3$ and so

$$e(h_1) \oplus e(h_2) \oplus e(x) = \langle e(\Pi) \rangle = e(h_1) \oplus e(h_2) \oplus e(y)$$

and $x \approx y$ follows from this. This completes the proof of (6).

Our proof that $\langle e(H) \rangle$ is a vector space hyperplane will be complete upon showing that $\mathfrak{X} = (X, \approx)$ is a connected graph. By way of contradiction we suppose x and y are chosen in distinct connected components of $\mathfrak{X}$ with $d(x, y) = d$ minimal.

Let $Z = \langle x, y \rangle_\Gamma$ be the convex closure of x and y in Γ. So, by proposition 2.1(iii), and the fact that $d > 1$, Z is a half-spin geometry of even type D_{2d}, $2d \geq 4$. (The possibility that $Z = \mathcal{P}$ is not excluded.)

Now by proposition 2.3(vi), there exists a subspace $D_3 \in \mathcal{D}_{2d-1}$ contained in Z and containing the point x, such that $D_3 \cap D_1 = \emptyset = D_3 \cap D_2$. Then, by proposition 2.3(ix), there exist embeddings $\pi_i : D_3 \to D_i$ where π_i takes each point v of D_3 to the single point of $L_v \cap D_i$, where L_v is the unique transversal line on v.

Now

$$H \cap D_3 = \pi_2^{-1} \left((H_2 \cap \pi_2(D_3)) + (X \cap D_3) \right) \tag{7}$$

so

$$X \cap D_3 = H \cap D_3 - H_2' \quad \text{where} \quad H_2' = \pi_2^{-1}\left(H \cap \pi_2\left(D_3\right)\right). \tag{8}$$

Note that as $x \in D_3$, and $\pi_2(D_3)$ and D_3 are isomorphic, that H_2' is a hyperplane of D_3. Also, $H \cap D_3 = D_3$ or is a hyperplane of D_3.

Now, by the minimality of d and the fact that $D_3 \subseteq \Delta_{d-1}(x)$, we have

$$\Delta_{d-1}(y) \cap D_3 \cap X = \emptyset. \tag{9}$$

So, from the partition in (6),

$$\Delta_{d-1}(y) \cap D_3 \cap H \subseteq H_2' \tag{10}$$

But D_3 is a half-spin space of odd type and diameter $d-1$, so, for each point u in $N_{D_3}(M_1)$, where $M_1 = y^\perp \cap D_3 \in \mathcal{M}'(D_3)$, we must have $\Delta_{d-2}(u) \cap M_1 \neq \emptyset$ (see the definition preceding proposition 2.6). Thus

$$\Delta_{d-1}(y) \cap D_3 \supseteq N_{D_3}(M_1). \tag{11}$$

(Note that by proposition 2.7(i), $N_{D_3}(M_1)$ is a hyperplane of D_3. Also the left side is a hyperplane of D_3 since $D_3 \subseteq Z$ of even type and by proposition 2.4(ii), $\Delta_{d-1}(y) \cap Z$ is a hyperplane of Z. Since Veldkamp lines exist for D_3, we must have equality in (11).) Now from (7)-(9) we have

$$N_{D_3}(M_1) \cap H \subseteq H_2'. \tag{12}$$

At this point we exploit lemma 1.2: D_3 has the internal geometric property that it possesses Veldkamp lines. Since x is an element of $H \cap D_3 - H_2'$, there are then only two possibilities accounting for (10).

$$\begin{array}{ll} \text{(i)} & D_3 \subseteq H \text{ and } N_{D_3}(M_1) = H_2', \\ \text{or} & \\ \text{(ii)} & D_3 \not\subseteq H \text{ and } N_{D_3}(M_1) \cap H = H_2' \cap H. \end{array} \tag{13}$$

Now let L_i be any line of H on y not meeting D_3. (Such lines exist since the H-lines and H-planes on point y form a geometric hyperplane of the point-residual $\Gamma_p = (\mathcal{L}_p, \Pi_p)$, a Grassman space of type $A_{n-1,2}$ while the H-lines meeting D_3 are just those lying in the maximal singular subspace $\hat{M}_1 = \langle M_1, y\rangle_\Gamma$. To avoid overlapping of previous and subsequent notation we assume without loss that lines L_i are indexed by $i > 2$.) Since $y \in X \cap L_i$, we see that $e(L_i) \cap (\langle e(H_1)\rangle \oplus \langle e(H_2)\rangle$ is at most 1-space in V representing one point of L_i. Thus all but one of the points on L_i belong to X. Since L_i contains at least three points, there is a second point y_i $(i > 2)$ in $L_i \cap X$ distinct from y. Since y_i is collinear with y, then by (6), y_i belongs to the same connected component of $\mathfrak{X}$ as y does.

Now assume $L_i \subseteq Z$. Then as $D_3 \in \mathcal{D}_{2d-1}(Z)$, $y_i^\perp \cap D_3 = M_i$, $i > 2$, is an element of $\mathcal{M}'(D_3)$. Then the minimality of d, and the fact that y_i is in the same connected component of $\mathfrak{X}$ as y imply

$$\Delta_{d-1}(y_i) \cap D_3 \cap (H \cap D_3 - H_2') = \emptyset.$$

Then as D_3 has Veldkamp lines, the situation is the same for (M_i, D_3) as for (M_1, D_3) above. Thus

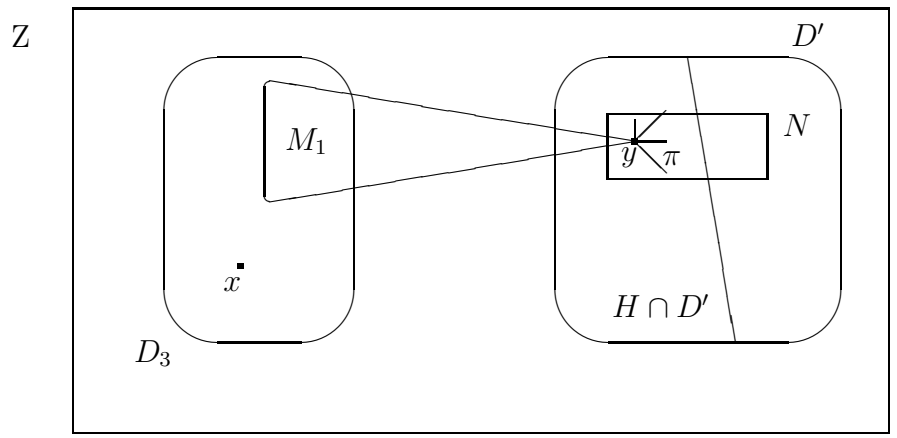
$$\begin{array}{lll} \text{either} & \text{(i)} & D_3 \subseteq H \text{ and } N_{D_3}(M_i) = N_{D_3}(M_1) = H_2' \\ \text{or} & \text{(ii)} & D_3 \not\subseteq H \text{ and } N_{D_3}(M_i) \cap H = H \cap H_2' \forall i \neq 1. \end{array} \tag{14}$$

Moreover, since the line L_i meets D_3 trivially, it is not possible that $M_i = M_1$ (otherwise $\langle L_i, M_1 \rangle$ is a singular subspace of Z of projective dimension $2d$ which is too large!). Thus M_i meets M_1 at a line N_i by proposition 2.3(iii).

Now if $N_{D_3}(M_i) = N_{D_3}(M_1)$, then $M_i = M_1$ by proposition 2.7(iii) applied to D_3. But as $n \geq 4$, this contradicts the fact that $M_1 \cap M_i = N_i$ is a line, as established in the previous paragraph. Thus (14)(i) cannot hold. We thus have $D_3 \not\subseteq H$ and (14)(ii) holds for all $i \neq 1$. Moreover, since for $i \neq 1$, $N_{D_3}(M_i)$ and $N_{D_3}(M_1)$ are distinct hyperplanes containing the intersection of (14)(ii), they define the same Veldkamp line. Thus their intersection can be added to the series of equalities of (14). That is,

$$N_{D_3}(M_i) \cap N_{D_3}(M_1) = H \cap H_2' \; \forall i \neq 1. \tag{15}$$

Now on y we can find an element D' in $\mathcal{D}_{2d-1}(Z)$ which is opposite D_3 (proposition 2.3(vi)). Since $2d - 1 \geq 3$, there is in D' an element N of $\mathcal{M}_3(D') \subseteq \mathcal{M}_3$ on y. Since N is a PG$(3, F)$, there exists a projective plane π on y contained in $N \cap H$. We can then let L_i wander through the pencil of lines on y within π. (See figure)



Now, by proposition 2.3(iv), there is an isomorphism

$$\psi : (D', \mathcal{L}(D')) \to (\mathcal{M}'(D_3), \mathcal{L}(D_3))$$

of incidence systems which takes point $d \in D'$ to $d^{\perp} \cap D_3 \in \mathcal{M}'(D_3)$. Now $\psi(y) = M_1$ and $\psi(y_i) = M_i$ $(i \neq 1)$. Then ψ takes the line $L_i = yy_i$ on y (in plane π) to the line $M_i \cap M_1 = N_i$ (again $i \neq 1$). Thus the lines $N_i = \psi(L_i)$ are all distinct as L_i ranges over the pencil of lines on y which lie in π.

But this last assertion yields a contradiction. For let s and t be distinct integers ≥ 2. Then by (14)(ii)

$$\begin{aligned} N_{D_3}(M_1) \cap N_{D_3}(M_s) &= N_{D_3}(M_1) \cap N_{D_3}(M_t) \\ &\subseteq N_{D_3}(M_t). \end{aligned}$$

Then proposition 2.8 implies that M_t contains the line $N_s = M_1 \cap M_s$ whence $N_t = M_1 \cap M_t = N_s$. Thus $N_s = \psi(L_s) = \psi(L_t) = N_t$, while $L_s \neq L_t$ against ψ being a $1-1$ map on lines: $\mathcal{L}(D') \to \mathcal{L}(D_3)$. This contradiction completes the proof. □

References

[1] **F. Buekenhout and C. Lefèvre**. Generalized quadrangles in projective spaces. *Arch. Math*, 25, pp. 540–552, 1974.

[2] **A. Cohen and E. Shult**. Affine polar spaces. *Geom. Dedicata*, 35, pp. 43–76, 1990.

[3] **B. Cooperstein and E. Shult**. Geometric hyperplanes of embeddable Lie incidence geometries. In J. W. P. Hirschfeld, D. R. Hughes, and J. A. Thas, editors, *Advances in Finite Geometries and Designs*, pages 81–92, Oxford University Press, Oxford, 1991.

[4] **K. J. Dienst**. Verallgemeine Vierecke in projectiven Raumen. *Arch. Math.*, 45, pp. 177–186, 1980.

[5] **C. Lefèvre-Percsy**. Polar spaces embedded in a projective space. In P. J. Cameron, J. W. P. Hirschfeld, and D. R. Hughes, editors, *Finite Geometries and Designs*, volume 49 of *London Math. Soc. Lecture Note Series*, pages 216–220, Cambridge University Press, 1981.

[6] **E. Shult**. On Veldkamp lines. Preprint.

[7] **E. Shult**. Geometric hyperplanes of embeddable Grassmannians. *J. Alg.*, 145, pp. 55–82, 1992.

[8] **E. Shult**. On Veldkamp planes. In A. Beutelspacher, F. Buekenhout, F. De Clerck, J. Doyen, J. W. P. Hirschfeld, and J. A. Thas, editors, *Finite Geometry and Combinatorics*, pages 341–354. Cambridge University Press, 1993.

[9] **E. Shult and J. A. Thas**. Hyperplanes of dual polar spaces and the spin module. *Arch. Math.*, 59, pp. 610–623, 1992.

[10] **J. Tits**. *Buildings of Spherical Type and Finite BN-pairs*, volume 386. Springer-Verlag, Berlin, 1974. Lecture Notes in Math.

E. E. Shult
Kansas State University
Department of Mathematics
Cardwell Hall
Manhattan, Kansas 66506-2602
USA.