# Invariance of $q$-completeness with corners under finite holomorphic surjections

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## 1 Introduction

It was shown by Narasimhan [6] using purely cohomological methods that a complex space $Y$ is Stein if, and only if, its normalization $Y^\star$ is Stein; more generaly, if $\pi : X \longrightarrow Y$ is a finite surjective holomorphic map, then $X$ is Stein if, and only if, $Y$ is Stein.

In the same circle of ideas, a more general invariance theorem was given in [10]; namely:

**Theorem** *Let $\pi : X \longrightarrow Y$ be a finite holomorphic surjection of complex spaces. Then $X$ is cohomologically $q$-complete if, and only if, $Y$ is cohomologically $q$-complete.*

We recall that a complex space $Z$ is said to be "cohomologically $q$-complete" if the cohomology group $H^i(Z, \mathcal{F})$ vanishes for every integer $i \geq q$ and every coherent analytic sheaf $\mathcal{F}$ on $Z$. Therefore, by the well-known theorem of Cartan and Serre, Stein spaces correspond to cohomologically 1-complete spaces.

In this paper we deal with a more geometrical aspect of an extension of Narasimhan's result, which is obtained for $q = 1$; namely we prove the following

**Theorem 1** *Let $\pi : X \longrightarrow Y$ be a finite holomorphic surjection of complex spaces. Then $X$ is $q$-complete with corners (resp. $q$-convex with corners) if, and only if, $Y$ is $q$-complete with corners (resp. $q$-convex with corners).*

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In particular we get:

**Corollary 1** *A complex space is q-complete with corners (resp. q-convex with corners) if, and only if, its normalization is so.*

**Corollary 2** *A complex space is q-complete with corners if, and only if, every irreducible component of it is q-complete with corners.*

Finally, we obtain an extension of a well-known result of Lelong [4].

**Proposition 1** *Let $M$ be a purely n-dimensional Stein manifold and $D \subset M$ an open set such that for every hypersurface $H$ of $M$ the intersection $H \cap D$ is q-complete with corners. Assume that $1 \leq q < n-1$. Then $D$ is q-complete with corners.*

## 2 Preliminaries

Throughout this paper all complex spaces are assumed to be reduced and with countable topology.

(•) Let $D \subset \mathbf{C}^n$ be an open set. For $\varphi \in C^2(D, \mathbf{R})$ and $z \in D$ we define the quadratic form on $\mathbf{C}^n$ by:

$$L(\varphi, z)(\xi, \eta) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(z)\xi_i \bar{\eta}_j,\ \xi, \eta \in \mathbf{C}^n.$$

We say that $\varphi$ is *q-convex* (on $D$) if its Levi form $L(\varphi, z)(\xi) := L(\varphi, z)(\xi, \xi), \xi \in \mathbf{C}^n$, has at least $n-q+1$ positive eigenvalues for every $z \in D$. Equivalently, this means that through each point $z \in D$ passes a $(n-q+1)$-dimensional affine complex subspace $H_z$ of $\mathbf{C}^n$ so that $\varphi_{|_{H_z}}$ is strongly plurisubharmonic (or 1-convex according with our setting) on a neighborhood of $z$ in $H_z$.

Let $X$ be a complex space. A function $\varphi \in C^2(X, \mathbf{R})$ is said to be *q-convex* if each point $x \in X$ has an open neighborhood $U$ together with an embedding $\iota : U \longrightarrow \widehat{U}$ where $\widehat{U} \subset \mathbf{C}^n$ is an open set, and a $q$-convex extension function $\widehat{\varphi} \in C^2(\widehat{U}, \mathbf{R})$ of $\varphi_{|_U}$. (This definition is independent of the local embeddings.)

We say that $\psi \in C^o(X, \mathbf{R})$ is *q-convex with corners* if each point $x \in X$ admits an open neighborhood $U$ for which there are finitely many $q$-convex functions $f_1, \ldots, f_k$ on $U$ such that

$$\psi_{|_U} = \max\{f_1, \ldots, f_k\}.$$

A complex space $X$ is said to be *q-convex with corners* if there is an continuous exhaustion function $\varphi : X \longrightarrow \mathbf{R}$ which is $q$-convex with corners on $X \setminus K$ for some compact subset of $X$. If we can take $K$ the empty set, then $X$ is said to be *q-complete with corners*. The normalization is chosen such that Stein spaces are precisely the spaces 1-complete with corners.

**Examples.** 1) An arbitrary finite intersection of $q$-complete with corners open sets of a given complex space stays $q$-complete with corners.

2) If $A \subset \mathbf{P}^n$ is an analytic set such that every irreducible component has dimension $\geq n-q$, then $\mathbf{P}^n \setminus A$ is $q$-complete with corners (see [7]).

3) If $X$ is a Stein manifold and $D \subset X$ an open set such that $H^i(D, \mathcal{O}_X)$ vanishes for all $i \geq q$, then $D$ is $q$-complete with corners (see, for instance [11]).

An open set $D$ of a complex space $X$ is said to be *locally q-complete with corners* if every point of the boundary $\partial D$ of $D$ in $X$ admits an open neighborhood $U$ such that $U \cap D$ is $q$-complete with corners.

($\bullet$) Now let $M$ be a purely $n$-dimensional complex manifold, $q$ a positive integer less than $n$, and $D \subset M$ an open set. Following [3] we say that $D$ is *pseudoconvex of order* $n-q$ if the next property holds:

For every $\xi \in \partial D$ and every coordinate neighborhood $(U, (z_1, \ldots, z_n))$ which contains $\xi$ as the origin one has:

($\clubsuit$) If for some $R > 0$, the set $\{(z_1, \ldots, z_q, 0, \ldots, 0) \in U\,;\, 0 < \sum_{i=1}^q |z_i|^2 < R\}$ is contained in $D$, then there is $r > 0$ such that for every $(w_{q+1}, \ldots, w_n)$ with $|w_j| < r$, $q+1 \leq j \leq n$, the set $\{(z_1, \ldots, z_q, w_{q+1}, \ldots, w_n) \in U\,;\, \sum_{i=1}^q |z_i|^2 < R\}$ contains at least one point of $M \setminus D$.

Next we relate this property with the local $q$-completeness with corners. (This characterization will be useful in proving proposition 1.)

**Lemma 1** *Keeping the notations from above, the following statements are equivalent.*

*(a)* $D$ *is* locally $q$-complete with corners*.*

*(b)* $D$ *is* pseudoconvex of order $n-q$*.*

*(c) For every open set* $U \subset M$ *and* $(n-q)$*-convex function* $\varphi$ *on* $U$*, the restriction* $\varphi_{|U \setminus D}$ **does not** *attain a maximal value.*

*Proof.* The equivalence of (a) and (b) is shown in [11] or [5]. We show now that (a) $\Rightarrow$ (c). We do this by *reductio ad absurdum.* Thus there are: a point $\xi_o \in \partial D$, an open neighborhood $U$ of $\xi_o$ such that $U \cap D$ is $q$-complete with corners, and a $(n-q)$-convex function $\varphi$ on $U$ such that $\varphi|_{U \setminus D} \leq \varphi(\xi_o)$.

Take $H \ni \xi_o$ be a complex submanifold of $X$ of dimension $q+1$ such that $\varphi|_H$ is 1-convex (we may shrink $U$, if necessary). Since $H \cap D \cap U$ is $q$-complete with corners, we may assume without any loss of generality that $q = n-1$, and by some coordinate changes that $\xi_o = 0$. Therefore $\varphi$ is 1-convex. Now, by developping $\varphi$ into Taylor series around 0, we get

$$\varphi(z) = \varphi(0) + \mathrm{Re} f(z) + L(\varphi, 0)z + o(\|z\|^2),$$

on a small enough open ball $B \subset\subset U$ centered at 0 with $f$ holomorphic (in fact given by a holomorphic polynomial of degree at most 2) on a neighborhood of the closure of $B$ in $U$.

Note that for every $c > 0$, the set $\{z \in B\,;\, f(z) = c\}$ is contained in $D$.

Now let $\psi : U \cap D \longrightarrow \mathbf{R}$ be $(n-1)$-convex with corners and exhaustive and $\lambda \in \mathbf{R}$ be given as the maximum of $\psi$ on the compact set $\{z \in \partial B, |f(z)| \leq 1\}$ of

$D \cap U$. If we apply the maximum principle for $(n-1)$-convex functions with corners on the analytic sets $\{f = c\}$ (which are of dimension $\geq n-1$), we deduce that the set $\{z \in B\,;\, f(z) = c, 0 < c < 1\}$ is contained in the compact set $\{z \in U \cap D; \psi(z) \leq \lambda\}$; hence it is relatively compact. But this is absurd !

Now we turn to the implication (c) $\Rightarrow$ (b). Here we follow the preprint of Ueda [9]. Again we use the *reductio ad absurdum* method. Thus for some point $\xi_o \in \partial D$ there exists a coordinate neighborhood $(U, (z_1, \ldots, z_n))$ which contains $\xi_o$ as the origin such that if we set $z' = (z_1, \ldots, z_q)$ and $z'' = (z_{q+1}, \ldots, z_n)$ one has:

There is a positive number $R$ with $\{(z', 0) \in U; 0 < \|z'\| \leq R\} \subset D$, and there exist a sequence $\{z''_\nu\}$ which converges to 0 in $\mathbf{C}^{n-q}$ such that $\Gamma_\nu := \{(z', z''_\nu)\,;\, \|z'\| \leq R\} \subset D$.

Set $\Gamma_o := \{(z', 0)\,;\, \|z'\| \leq R\}$, $\Delta' := \{z' \in \mathbf{C}^q\,;\, \|z'\| < R\}$, $\Delta'' := \{z'' \in \mathbf{C}^{n-q}\,;\, \|z''\| < r\}$ with $r > 0$ small enough such that $\{(z', z'')\,;\, \|z'\| = R, \|z''\| \leq r\} \subset D$, and $\Delta := \Delta' \times \Delta''$. Set $\Omega_\nu := \Delta \setminus \Gamma_\nu$. Define $\varphi_\nu$ on $\Omega_\nu$ by

$$\varphi_\nu(z', z'') = \|z'\|^2 + \|z'' - z''_\nu\|^{-2}.$$

Clearly $\varphi_\nu$ has $q+1$ positive eigenvalues, hence it is $(n-q)$-convex. Now remark that $L_\nu := \partial(\Omega_\nu \setminus D)$ is contained in $\overline{\Delta}' \times \partial\Delta''$ (This is obvious!) and $\varphi_\nu|_{L_\nu} \leq R^2 + 1/r^2$, for all $\nu$. On the other hand, the sequence $\{\sup_{\xi \in \Gamma_o \setminus D} \varphi_\nu(\xi)\}_\nu$ diverges to infinity. Thus, if $\nu$ is large enough, $\varphi_\nu|_{\Omega_\nu \setminus D}$ attains a maximum. This concludes the proof of the lemma.

**The proof of propositon 1.**

In virtue of Peternell's result [7], one has to show that $D$ is locally $q$-complete with corners. Assume this is not true. Hence by the above lemma there are: a point $\xi_o \in \partial D$, an open neighborhood $U$ of $\xi_o$, and a $(n-q)$-convex function $\varphi$ on $U$ such that $\varphi|_{U \setminus D} \leq \varphi(\xi_o)$.

Take $H \ni \xi_o$ be a complex submanifold of $X$ of dimension $q+1$ such that $\varphi|_H$ is 1-convex (we may shrink $U$, if necessary). Now we employ the lemma 1 again to the $q$-complete with corners open set $D \cap H$ of the Stein manifold $H$, and we derive easily a contradiction.

**Remark**. *Mutatis mutandis*, the statement of proposition 1 remains true if we replace $M$ by a purely $n$-dimensional projective manifold.

## 3 The proof of theorem 1

We show firstly the "if" part. Let $\psi : Y \longrightarrow \mathbf{R}$ be $q$-convex with corners and exhaustive. Then there exists an open covering $\{V_i\}_{i \in I}$ of $Y$ such that: $\{\overline{V}_i\}_{i \in I}$ is locally finite, each $\overline{V}_i$ is a Stein compactum, and there are $q$-convex functions with corners $\psi_{ij}, j = 1, \ldots, k_i$, defined on a neighborhood of $\overline{V}_i$ with

$$\psi(y) = \max\{\psi_{ij}(y)\,;\, i \text{ and } j \text{ such that } y \in V_i\}$$

and if $y \in \partial V_i$ then $\psi_{ij}(y) < \psi(y)$ for all $j = 1, \ldots, k_i$. This can be easily satisfied.

Now set $U_i := \pi^{-1}(V_i), i \in I$, and $\varphi_{ij} := \psi_{ij} \circ \pi$. Let $\theta_i$ be 1-convex on a neighborhood of $\overline{U}_i$. For every sequence $\epsilon = \{\epsilon_i\}_i$ of positive numbers we define a function, not necessarily continuous, $\varphi_\epsilon : X \longrightarrow \mathbf{R}$ as follows: For $x \in X$ we put

$$\varphi_\epsilon(x) := \max\{\varphi_{ij}(x) + \epsilon_i\theta_i(x), j = 1, \ldots, k_i, x \in U_i\}.$$

Then $\varphi_\epsilon > \psi \circ \pi$; consequently $\varphi_\epsilon$ is exhaustive. Moreover, if $\epsilon_i > 0$ are sufficiently small, one gets for all $i$ and $x \in \partial U_i$ that

$$\max\{\varphi_{ij}(x), j = 1, \ldots, k_i\} < \varphi_\epsilon(x).$$

In particular we deduce that $\varphi_\epsilon$ is continuous and $q$-convex with corners. (On each $\epsilon_i$ we impose finitely many restrictions.) The proof of the "if" part follows.

To show the "only if" part, we use induction over the dimension $n$ of $X$. The case $n = 0$ being clear, let $n > 0$ and assume the theorem proven for complex spaces $X$ of dimension less than $n$.

Consider $Y' \subset Y$ a thin analytic set such that $\pi : X \setminus X' \longrightarrow Y \setminus Y'$, $X' := \pi^{-1}(Y')$, is a covering map, say, with $k$ sheets. Since $X'$ is $q$-complete with corners, by the induction hypothesis, $Y'$ is also $q$-complete with corners. Then there are: $W \subset Y$ an open neighborhood of $Y'$ and $\psi' : U \longrightarrow \mathbf{R}$ a function $q$-convex with corners on a neighborhood $U$ of $\overline{W}$ such that $\psi'$ restricted to $\overline{W}$ is exhaustive. This can be done similarly as in [2], theorem 1.

Let $\varphi : X \longrightarrow \mathbf{R}$ be $q$-convex with corners and exhaustive. From [8], there is a function $\sigma : X \longrightarrow \mathbf{R} \cup \{-\infty\}$ such that: $\{\sigma = -\infty\} = X'$, $\exp\sigma$ is smooth, and every point $x \in X$ admits an open neighborhood $\Omega$ for which there is a smooth function $h$ on $\Omega$ with $\sigma + h$ is 1-convex on $\Omega \setminus X'$. Therefore, if $\chi \in C^\infty(\mathbf{R}, \mathbf{R})$ is rapidly increasing and convex, $\varphi' := \chi(\varphi) + \sigma$ is $q$-convex with corners on $X \setminus X'$ and exhaustive on $X \setminus \pi^{-1}(V)$, where $V$ is an open neighborhood of $Y'$, $\overline{V} \subset W$. Moreover, notice that the sequence $\{\varphi'(x_\nu)\}$ diverges to $-\infty$ whenever the sequence $\{x_\nu\} \subset X \setminus X'$ converges to some point of $X'$.

Now we define a continuous function $\theta : Y \setminus Y' \longrightarrow \mathbf{R}$ as follows: For every point $y \in Y \setminus Y'$ let $\pi^{-1}(y) = \{x_1, \ldots, x_k\}$; then set

$$\theta(y) = \max\{\varphi'(x_1), \ldots, \varphi'(x_k)\}.$$

Since $\pi$ induces a covering map with $k$ sheets between $X \setminus X'$ and $Y \setminus Y'$, one gets easily that $\theta$ is $q$-convex with corners, $\theta$ restricted to $Y \setminus V$ is exhaustive, and for every sequence $\{y_\nu\}_\nu$ in $Y \setminus Y'$ which converges to a point of $Y'$ the sequence $\{\theta(y_\nu)\}_\nu$ diverges to $-\infty$.

Set $t_o := \min_{\overline{W} \setminus V} \theta$ which is a real number since $\theta$ restricted to $\overline{W} \setminus V$ is exhaustive; then select a smooth function $\mu : \mathbf{R} \longrightarrow \mathbf{R}$ which is strictly increasing and convex such that $\mu(t) = t$ if $t \leq t_o - 1$ and $\mu(\theta) > \psi'$ on $\overline{W} \setminus V$. We define $\psi : Y \longrightarrow \mathbf{R}$ as follows

$$\psi := \begin{cases} \max\{\mu(\theta), \psi'\} & \text{on } W; \\ \mu(\theta), & \text{on } Y \setminus \overline{V}. \end{cases}$$

Then $\psi$ is well-defined, $q$-convex with corners, and exhaustive for $Y$. Therefore $Y$ is $q$-complete with corners; whence the proof of the induction step.

The $q$-convex with corners analogue is to be treated in a similar way and we omit it.

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