SURVEY

On several aspects of J-inner functions in Schur analysis

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Abstract

The aim of this paper is to give a presentation of several subjects of Schur analysis with some historical information. The class of J-inner functions plays a key role in this new mathematical field which is situated at the seam of various mathematical disciplines (operator theory, scattering theory, complex function theory, prediction theory for stochastic processes, spectral theory for differential operators). This article shows the importance of J-inner functions in Schur analysis. We shall concentrate on the Soviet roots of the theory and discuss Potapov's factorization theory and Arov's investigations on Darlington synthesis. Furthermore we present some of Arov's results on interrelations between a certain subclass of J-inner functions and generalized bitangential Schur-Nevanlinna-Pick interpolation.

0 Introduction

In the last 25 years one could observe an intensive research interest in matricial and operator-theoretical versions of classical moment problems and related questions in interpolation theory. This was mainly initiated by the fundamental papers of V.M. Adamjan, D.Z. Arov and M.G. Kreĭn [AAK68a]–[AAK71b]. In the first 30 years of our century, the original scalar versions of these problems attracted

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the attention of several of the most important mathematicians of that time. Several fundamental papers appeared in that period and they have exerted a strong influence on further developments which have led to the present state-of-the-art (see Carathéodory [Car07]–[Car29], Carathéodory/Fejér [CF11], Hausdorff [Hau21], Hellinger [Hel22], Herglotz [Her83], Pick [Pic16]–[Pic20], R. Nevanlinna [Nev19]–[Nev29], F. Riesz [Rie11]–[Rie18], M. Riesz [Rie22], Schur [Sch12], [Sch17], Szegő [Sze20], Toeplitz [Toe11], Weyl [Wey35]).

We mention in particular an ingenious algorithm which was developed by Schur [Sch17] and generalized by R. Nevanlinna [Nev29]. This algorithm can be considered as the starting point of a series of developments which led to the growth of a new mathematical field which is situated at the seam of various mathematical disciplines (operator theory, scattering theory, complex function theory, prediction theory for stochastic processes, spectral theory for differential operators). This field is now designated as *Schur analysis*. In the last decade the lively research in Schur analysis has materialized in the publication of numerous monographs which customize modern developments for a wide audience (see Bakonyi/Constantinescu [BC92], Ball/Gohberg/Rodman [BGR90], Constantinescu [Con96], Dubovoj/Fritzsche/Kirstein [DFK92], Dym [Dym89], Foiaş/Frazho [FF90], Gohberg/Goldberg/Kaashoek [GGK90], Helton [Hel87], Rosenblum/Rovnyak [RR85]).

It is remarkable that in these monographs, essentially the same problems are studied but the developed methods for the solution are absolutely different from each other. This illustrates that the considered problems are highly multifaceted. In this paper we want to point out the available resources which can lead to a better understanding of these problems and which provide the tools for their solution. It is impossible to give in the context of this paper all the details of these diverse disciplines. The reader will almost certainly not understand all the alusions made, but there is a great chance that most readers will find some piece of the cake that he or she really likes (and knows about). As the authors we take the humble task of describing only the simplest possible cases and we give the links and references to the literature concerning the more advanced topics which were treated in the different approaches, each with its own terminology and tools.

In most cases, the underlying problem can be reduced to an interpolation problem. Therefore we take these interpolation problems as a starting point and use them as a tread in the further exposition. As Schur analysis developed, it became clear that the scalar interpolation problems had to be generalized to matrix versions. The synthesis of the various approaches for matricial interpolation problems created an apparatus for the treatment of complicated classes of interpolation problems whose complexity and generality goes far beyond the original scalar problems. During a Schur analysis conference in Leipzig in October 1989, V.E. Katsnelson emphasized this orientation as an important research field for the next decade and so far, his prediction has largely become true. For more elaborate cross connections and the synthesis of several approaches to interpolation problems, the reader is referred to Alpay/Bolotnikov [AB93], [AB94], Alpay/Bolotnikov/Peretz [ABP95], Dubovoj/Fritzsche/Kirstein [DFK92], Fritzsche/Fuchs/Kirstein [FFK92], Fritzsche/Kirstein [FK87b], [FK87a], Katsnelson/Kheifets/Yuditskii [KKY87].

Let us first introduce some classical interpolation problems in the complex plane which are at the basis of this Schur analysis.

1 Classical interpolation problems

Several of the classical interpolation problems and moment problems can be formulated as follows. Let Δ and Ω be domains in $\mathbb{C}$. We denote by $\mathcal{F}(\Delta, \Omega)$ the class of complex functions which are analytic in Δ and which take values in $\Omega \cup \partial\Omega$. Usually these domains are the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ or a half plane like $\mathbb{C}_+ = \{z : \operatorname{Im}(z) > 0\}$. Since there is a conformal mapping from $\mathbb{D}$ to $\mathbb{C}_+$, these problems are essentially the same and one could restrict the discussion to the case $\Delta = \Omega = \mathbb{D}$. For that choice, we have to consider the class $\mathcal{F}(\mathbb{D}, \mathbb{D})$ which is called the *Schur class* and denoted as $\mathcal{B}(\mathbb{D})$. It is a unit ball in H^∞ (for the unit disk). Also the class $\mathcal{P}(\mathbb{C}_+) = \mathcal{F}(\mathbb{C}_+, \mathbb{C}_+)$ is a special case one often refers to. It is called the Nevanlinna class in [Ac69], but we shall call it the *Nevanlinna-Pick* class in order not to confuse with the *Nevanlinna class* $\mathcal{N}$ from harmonic analysis. By the *Carathéodory class* $\mathcal{C}(\mathbb{D})$ one usually means the functions analytic in $\mathbb{D}$ and taking values in the right half plane instead of the upper half plane. Of course these classes are basically the same since one can apply a Cayley transform mapping the disk into a half plane or visa versa. Let $\mathcal{I}(\Gamma)$ denote the set of complex functions which satisfy a certain set of interpolation conditions described by the data Γ. For example, we can consider a set of couples $\Gamma = \{(z_k, w_k) \in \mathbb{C}^2 : k \in K\}$ where the z_k are supposed to be different and define $\mathcal{I}(\Gamma)$ as the set of complex functions w satisfying $w(z_k) = w_k$ for all $k \in K$. Also, if some of the interpolation points coincide, we can give conditions for the derivatives. Thus, given a set of triples $\Gamma = \{(z_k, \mathbf{w}_k, \alpha_k)\}_{k\in K}$ where α_k are non-negative integers and $\mathbf{w}_k = (w_{0k}, w_{1k}, \ldots, w_{\alpha_k,k}) \in \mathbb{C}^{\alpha_k+1}$, then the set $\mathcal{I}(\Gamma)$ can be described as the set of complex functions w satisfying (the superscript in $w^{(\alpha)}$ means the derivative of order α)

$$w^{(\alpha)}(z_k) = w_{\alpha,k}, \quad \alpha = 0, 1, \ldots, \alpha_k; \quad k \in K.$$

A general complex interpolation problem can now be described as follows.

General scalar interpolation problem: Given is a set of interpolation data Γ defining the set $\mathcal{I}(\Gamma)$ of interpolating functions. Find necessary and sufficient conditions so that $\mathcal{I}(\Gamma) \cap \mathcal{F}(\Delta, \Omega) \neq \varnothing$. If this solution set is not empty, find conditions for which the problem is determinate, i.e., for which the solution is unique and if the problem is indeterminate, describe all the solutions, or find one which is optimal in some sense (for example which has minimal norm). In the case when there are only a finite number of interpolation conditions, there usually exists a solution which is a rational function. Then it makes sense to find for example a solution of minimal degree.

In a classical *scalar Nevanlinna-Pick problem*, one considers an interpolation problem in the Schur class $\mathcal{B}(\mathbb{D})$ where the different points $\{z_k\}_{k\in K}$ are all in $\mathbb{D}$. Pick gave necessary and sufficient condition for solvability when the set K is finite and R. Nevanlinna generalized this to a countable set K. Schur considered the same problem, but with all the interpolation points z_k coinciding at the origin. This means that he looked for a function in $\mathcal{B}(\mathbb{D})$ which has a power series expansion at the origin, starting with a given polynomial. By a conformal mapping, the *Schur problem* can be transformed into a *Carathéodory coefficient problem.* Here, one solves an interpolation problem in $\mathcal{C}(\mathbb{D})$ with all $z_k = 0$. Carathéodory and Fejér considered an extra question where it was required to find among all solutions the one with minimal norm. The Schur problem is also known to be closely related to the *trigonometric moment problem*. Here one has to find a positive measure on the unit circle which has prescribed moments $m_k = \int t^{-k} \mathrm{d}\mu(t)$, $k \in \mathbb{Z}$. The relation between the two problems is that if w is a solution of the Carathéodory coefficient problem and μ is a solution of the corresponding trigonometric moment problem then w is the Riesz-Herglotz transform of the measure μ: $w(z) = \int (t+z)/(t-z) d\mu(t)$. When some of the points z_k coincide, but not all at the same point, then we have a multipoint generalization of the trigonometric moment problem. It is not difficult to imagine that the solutions of such a multipoint moment problem are related to the solutions of multipoint Carathéodory problems, again by a Riesz-Herglotz transform . See [BGVHN97]. The relationship between these interpolation problems and moment problems should not come as a complete surprise if one realizes that functions from the classes $\mathcal{F}(\Delta, \Omega)$ usually have integral representations involving a positive measure and finding the interpolating function is basically the same as finding its representing measure.

A somewhat different situation occurs when the points z_k are chosen on the boundary $\partial\Delta$. These problems are often referred to as *Loewner problems* or boundary Nevanlinna-Pick problems. The best known situation occurs when the problem is solved in $\mathcal{P}(\mathbb{C}_+)$ and all z_k are at ∞. We then have to find $w \in \mathcal{P}(\mathbb{C}_+)$ with a given asymptotic expansion at ∞. This is known to be equivalent with the *Hamburger moment problem*: Find a positive measure on the real line such that it has prescribed moments $m_k = \int t^k \mathrm{d}\mu(t)$, $k = 0, 1, \ldots$ (or a truncated version thereof). As in the trigonometric case, this equivalence is also a direct consequence of the integral representation of functions in $\mathcal{P}(\mathbb{C}_+)$. Again, when not all the interpolation points coincide at ∞, but when there are several points of confluence, one obtains multipoint moment problems [Nud94]. However, when a finite number of interpolation conditions are prescribed at a point on the boundary, then it is natural to give an even number of conditions. To explain this, we have to introduce first the notion of pseudocontinuation. We define it for the circular case, but similar notions do exist for the real line.

Let g be a function which belongs to the *meromorphic Nevanlinna class* of the unit disk $\mathcal{NM}(\mathbb{D})$ — that is the class of functions meromorphic in $\mathbb{D}$ that can be written as the ratio of two bounded holomorphic functions.

Then one says that g *admits a pseudocontinuation* (outside unit disc) if there exists a function $g^{\#}$ which is defined outside unit disk such that the radial boundary values of g and $g^{\#}$ coincide almost everywhere on the unit circle (with respect to the Lebesgue measure). It is obvious that a function of the meromorphic Nevanlinna class admits at most one pseudocontinuation. Note that if a pseudocontinuable function is also analytically continuable through some open arc of the unit disc, then this analytic continuation coincides with the pseudocontinuation. The concept of pseudocontinuity goes back to H.S. Shapiro [Sha66] and appears implicitly in Tumarkins paper [Tum61] on weighted rational approximation.

Thus if a function $w \in \mathcal{P}(\mathbb{C}_+)$ has a pseudocontinuation, then it can be extended to lower half plane $\mathbb{C}_-$ by the relation $w(z) = \overline{w(\overline{z})}$, for $z \in \mathbb{C}_-$. Therefore, if an interpolation condition $w(z_k) = w_k$ is satisfied for $z_k \in \mathbb{C}_+$, then the pseudocontinued function immediately satisfies $w(\overline{z}_k) = \overline{w(z)}$ in the lower half plane as well. Thus if z_k is moved to the boundary, there will be two interpolation conditions in the boundary point: one coming from $\mathbb{C}_-$ and one coming form $\mathbb{C}_+$.

Among other things, current research in Schur analysis is concerned with the solution of the matrix generalizations of all the scalar interpolation problems that we have briefly sketched above. For example, consider $\mathcal{B}^{p\times q}(\mathbb{D})$ as the generalization of the Schur class of $p \times q$ matrix valued functions analytic in $\mathbb{D}$ and which are contractive, i.e., for which $S(z)^*S(z) \leq I_q$ in $\mathbb{D}$ where for Hermitian matrices A and B, the inequality $A \leq B$ means that $B - A$ is Hermitian positive semidefinite. The interpolation data are $\Gamma = \{(z_k, \mathbf{x}_k, \mathbf{y}_k)\}_{k\in K}$ where the z_k are different complex numbers, $\mathbf{x}_k \in \mathbb{C}^p$ and $\mathbf{y}_k \in \mathbb{C}^q$. $\mathcal{I}(\Gamma)$ is then the set of $\mathbb{C}^{p\times q}$-valued functions w which satisfy the directional or tangential Nevanlinna-Pick interpolation conditions $\mathbf{x}_k^* w(z_k) = \mathbf{y}_k^*$ for all $k \in K$. Of course, this is a one-sided formulation. We could have imposed the interpolation conditions in the form $w(z_k)\mathbf{y}_k = \mathbf{x}_k$. More difficult are bidirectional or bitangential formulations where left as well as right conditions are imposed simultaneously. We refer to Section 8 for more details.

For the solution of such problems, the J-contractive, J-unitary and J-inner functions play a central role. To introduce these notions in their simplest possible form, we give some elements about Blaschke products and the Nevanlinna-Pick algorithm in the next section.

2 Blaschke products and the Nevanlinna-Pick algorithm

Let us start by defining a Blaschke factor. Assume $a \in \mathbb{D}$ then we define a *Blaschke factor* ζ_a as a complex-valued function

$$\zeta_a(z) := \eta(a)\frac{a - z}{1 - \overline{a}z}\ , \tag{1}$$

where $\eta(a) = \overline{a}/|a| \in \mathbb{T}$ if $a \neq 0$, and if $a = 0$, then we set $\eta(0) = -1$ so that $\zeta_0(z) = z$. This function is a conformal map of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ onto itself such that the unit circle, its exterior and its interior are stable under this transformation.

Let $A = \{a_k\}_{k\in K}$ be a sequence of points in the unit disk where K is a finite or infinite set of integers. If the condition

$$\sum_{a\in A}(1 - |a|) < \infty \tag{2}$$

is satisfied, then the product

$$b(z) = \prod_{a\in A}\zeta_a(z) \tag{3}$$

converges to a non-zero analytic function [Wal60]. Hereby the condition (2) is called *Blaschke condition* and the product (3) is also called the *Blaschke product* which is based on the sequence A. Any Blaschke product is an inner function because it has unitary boundary values almost everywhere on the unit circle (with respect to the linear Lebesgue-Borel measure). Let f be a holomorphic contractive complex-valued function which does not vanish identically in the unit disc. Then the set of the roots $A = \{a_k\}_{k\in K}$ of this function satisfies the Blaschke condition and we can construct a Blaschke product b with this set of points. The well-known factorization theorem of F. Riesz and V.I. Smirnov says that every analytic and contractive function f admits a factorization

$$f = b \cdot s \tag{4}$$

where b is its Blaschke product (which is built on the roots of the function f) and s is a contractive holomorphic function with no roots in the unit disc. The function s is singular, i.e., both functions s and s^{-1} are holomorphic. The representation (4) is called a *multiplicative representation.* Note that this representation is unique.

The Nevanlinna-Pick algorithm is based on the simple fact that if w is a complex function, $a \in \mathbb{D}$ and $\rho = w(a)$, then $w \in \mathcal{B}(\mathbb{D})$ if and only if either $|\rho| < 1$ and

$$u(z) = \frac{1}{\zeta_a(z)} \frac{\rho - w(z)}{1 - \overline{\rho} w(z)} \in \mathcal{B}(\mathbb{D}),$$

or $|\rho| = 1$, in which case w is a constant. Inverting this formula we get that w is a Schur function that takes the value $\rho \in \mathbb{D}$ in the point $a \in \mathbb{D}$ if and only of it is of the form

$$w(z) = \frac{u(z)\zeta_a(z) - \rho}{u(z)\zeta_a(z)\overline{\rho} - 1},$$

for arbitrary $u \in \mathcal{B}(\mathbb{D})$. If we assume that the w and u are written as the ratio of two analytic functions in $\mathbb{D}$: $w = \Delta_{10}/\Delta_{20}$, and $u = \Delta_{11}/\Delta_{21}$, then we can express the previous relation as

$$[\Delta_{10} \;\; \Delta_{20}]^T = \theta(z)[\Delta_{11} \;\; \Delta_{21}]^T$$

(the ^{T} means transpose) with

$$\theta(z) = U_\rho J Z_a(z)$$

where

$$Z_a(z) = \begin{bmatrix} \zeta_a(z) & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad U_\rho = \gamma \begin{bmatrix} 1 & \rho \\ \overline{\rho} & 1 \end{bmatrix},$$

with $\gamma = (1 - |\rho|^2)^{-1/2}$ a normalizing constant.

If we also assume that the function values w_k at the different interpolation points z_k are given as ratios of complex numbers: $w_k = w_{1k}^{(0)}/w_{2k}^{(0)}$, for $k = 0, 1, \ldots$, then we can iterate this procedure and define

$$[\Delta_{1,k-1} \;\; \Delta_{2,k-1}]^T = \theta_k(z)[\Delta_{1k} \;\; \Delta_{2k}]^T$$

with

$$\theta_k(z) = U_{\rho_k} J Z_{z_k}(z)$$

and $\rho_k = w_{1,k}^{(n-k)}/w_{2,k}^{(n-k)}$ with

$$[w_{1i}^{(k)} \;\; w_{2i}^{(k)}]^T = \theta_k(z_i)^{-1}[w_{1i}^{(k-1)} \;\; w_{2i}^{(k-1)}]^T, \quad i = 0, 1, \ldots, n-k.$$

If Δ_{1n}/Δ_{2n} is an arbitrary function from $\mathcal{B}(\mathbb{D})$, then Δ_{10}/Δ_{20} will be a function from $\mathcal{B}(\mathbb{D})$ as well and it will take the values w_k at the points z_k for $k = 0, 1, \ldots, n$.

Thus, assuming that all $\rho_k \in \mathbb{D}$, we have constructed the solutions for a Nevanlinna-Pick problem where only a finite number of interpolation conditions are given at different points inside the unit disk. The algorithm can be adapted to the case where some (or all) of the interpolation points z_k coincide. For example, when they are all equal to zero, then the Nevanlinna-Pick algorithm reduces to the Schur algorithm. In the case of the Schur algorithm, the numbers ρ_k are called Schur parameters or reflection coefficients. In the Schur as well as in the Nevanlinna-Pick case, there is a solution of the problem with $N \leq \infty$ interpolation conditions when all ρ_k are in $\mathbb{D}$. If we have a situation where $|\rho_k| < 1$, $k = 1, \ldots, n-1$, $|\rho_n| = 1$ and $\rho_{n+1} = \cdots = \rho_N = 0$, then the problem is determinate and has a unique solution which is a rational function of degree n. It is the interpolant obtained by the previous procedure when setting $\Delta_{1n}/\Delta_{2n} = 0$. In general all the solutions which are interpolating the data $\{z_k, w_k\}_{k=1}^n$ are given by

$$[\Delta_{10} \quad \Delta_{20}]^T = \theta_1(z)\theta_2(z)\cdots\theta_n(z)[\Delta_{1n} \quad \Delta_{2n}]^T.$$

where $\Delta_{1n}/\Delta_{2n} \in \mathcal{B}(\mathbb{D})$. A rational solution of minimal degree is obtained by setting $[\Delta_{1n} \quad \Delta_{2n}] = [0 \quad 1]$. Thus that solution is contained in the second column of the matrix $W_n(z) = \theta_1(z)\cdots\theta_n(z)$.

Matrices like $W_n(z)$ play a prominent role in Schur analysis and they are the main concern of this survey paper. If we set

$$W_n = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

then we can give all the solutions, satisfying n interpolation conditions in the form of a linear fractional transform:

$$\frac{W_{11}g + W_{12}}{W_{21}g + W_{22}}, \quad g = \frac{\Delta_{1n}}{\Delta_{2n}} \in \mathcal{B}(\mathbb{D}).$$

What is so special about these matrix functions $W(z)$? To lift the answer to this question to a higher level of generality, we first have a look at the generalization of the matrix J. The matrix J is a special case of a *signature matrix.* A square $m \times m$ complex matrix J is called a signature matrix if $J^2 = I_m$ and $J = J^*$ (J^* is the adjoint of J). For any signature matrix J, we say that a square matrix M is *J-unitary* if $M^*JM = J$ and it is called *J-contractive* if $M^*JM < J$ and *J-expansive* if $M^*JM > J$. If M is a function of a complex variable z, and M is analytic and J-contractive in $\mathbb{D}$ while it is J-unitary on $\partial\mathbb{D}$, then we have an indefinite analog of the Schur class, the difference being that the (possibly indefinite) signature matrix J replaces the identity I in the definition of the Schur class. Such a matrix will be said to belong to the Potapov class. Before we give the general definition of this class, let us briefly return to our 2×2 example of the matrix W in the Nevanlinna-Pick algorithm. Obviously, the matrix J is a simple indefinite signature matrix. The U_ρ part of $\theta(z)$ is a constant matrix which is J-unitary. For the matrix $Z_a(z)$ we have

$$Z_a(z)^* J Z_a(z) = \begin{bmatrix} |\zeta_a(z)|^2 & 0 \\ 0 & -1 \end{bmatrix}$$

and because we assumed that $a \in \mathbb{D}$, we have by the mapping property of the Moebius transform $\zeta_a(z)$ that the matrix $Z_a(z)$ is J-unitary on the unit circle and

J-contractive in $\mathbb{D}$ while it is J-expansive outside the closed unit disk. Note that, also in the general situation, the above properties are stable under multiplication. This implies that not only a matrix of the form $\theta_k(z)$, but also the product $W = \theta_1(z)\cdots\theta_n(z)$ will be in the Potapov class.

We now give a general definition of the Potapov class, which (for reasons that should have become clear from the scalar example of Nevanlinna-Pick interpolation) we restrict to the meromorphic functions. Let $J_p \in \mathbb{C}^{p\times p}$ and $J_q \in \mathbb{C}^{q\times q}$ be general signature matrices. The *Potapov class* $\mathfrak{P}_{J_p,J_q}$ is defined as the set of all meromorphic $p \times q$ matrix-valued functions W which satisfy the condition

$$J_q - W^*(z)J_pW(z) \geq 0 \tag{5}$$

for all z where W is analytic. A matrix which fulfils (5) is also called (J_p, J_q)-*contractive*. Observe, that every element of a matrix-valued function $W \in \mathfrak{P}_{J_p,J_q}$ belongs to the meromorphic Nevanlinna class $\mathcal{NM}$, i.e., every element of W can be represented as a quotient of two bounded holomorphic functions. The $p \times q$ matrix valued Schur class is denoted by $\mathcal{B}^{p\times q}(\mathbb{D})$ and coincides with the Potapov class when J_p and J_q are unit matrices. For square matrices when $J_p = J_q = J$, we denote the Potapov class as $\mathfrak{P}_J$.

Due to a classical result of Fatou, the function $W \in \mathfrak{P}_J$ has almost everywhere radial boundary values on the unit circle (with respect to the linear Lebesgue-Borel measure on the unit circle). If $\underline{W}$ denotes such a radial boundary value of W, then it follows for Lebesgue-almost all z on the unit circle that

$$J - \underline{W}^*(z)J\underline{W}(z) \geq 0\ . \tag{6}$$

Let us now return to interpolation problems. It will not require a big leap of faith from the reader to accept that in various matricial versions of the interpolation problems discussed above, such matrix valued functions from the Potapov class will play an important role. For example, one can write the solutions of a matricial Nevanlinna-Pick problem in the form

$$f = (W_{11}g + W_{12})(W_{21}g + W_{22})^{-1}\ , \quad g \in \mathcal{B}^{p\times q}(\mathbb{D}) \tag{7}$$

where the matrix functions W_{ij} can be arranged in an array

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{8}$$

where W_{11} has size $p \times p$ and W_{22} has size $q \times q$. This matrix will be in the Potapov class of square matrix valued functions of size $p+q$ and the corresponding signature matrix has the form

$$J = j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}. \tag{9}$$

The matrix W can be constructed from the interpolation data in a way which is a natural generalization of the scalar Nevanlinna-Pick algorithm.

Every meromorphic matrix-valued function (8) which describes the solution set of the considered problem via (7) is called an associated *resolvent matrix*. This concept of the resolvent matrix was created by M.G. Kreĭn [Kre44]–[Kre49]. In the 40s he developed with M.S. Livšic and M.A. Naĭmark a certain approach to the solution of interpolation problems which can be described as follows. The basic idea consists in a transformation of the original problem in an equivalent problem of extension of a certain isometric or symmetric operator to a unitary (respectively, selfadjoint) operator in a suitable Hilbert space. The interpolation problem and the extension problem are equivalent because there is a bijective correspondence between the solution set of the interpolation problem and the set of all unitary (respectively, selfadjoint) extensions of the original operator. This bijective correspondence is realized by the so-called Kreĭn formula for the family of all generalized resolvents of the original operator. Under a certain condition of regularity, this formula can be rewritten as a linear fractional transformation and this transformation is characterized by the resolvent matrix.

The treatment of various matricial versions of classical interpolation problems showed that the associated resolvent matrices are functions of the Potapov class which are *J-unitary*, i.e., which fulfil the condition (6) with equality (Lebesgue-almost everywhere) on the unit circle. This subclass of the Potapov class is very important and is called the class of *J-inner* functions. Within the class of J-inner functions we can distinguish between regular and singular ones. If a J-inner function and its inverse function are holomorphic then it is called *singular*. Recent investigations of D.Z. Arov [Aro88]–[Aro90] on Schur-Nevanlinna-Pick interpolation led him to the following subclasses of J-inner functions. A J-inner function W is called *A-singular* (or *Arov-singular*) if both functions W and W^{-1} belong to the Smirnov class (which is a certain subalgebra of the holomorphic Nevanlinna class whose elements fulfil some conditions in growth[1]). Note that an A-singular J-inner function is singular. The concept of A-singularity was created by Katsnelson to distinguish this class from singular functions. A J-inner function W is called *left* (respectively, *right*) *Arov-regular* if W has the following property: If $W = W_r W_s$ (respectively, $W = W_s W_r$) is an arbitrary representation of W with some J-inner function W_r and some Arov-singular J-inner function W_s then W_s is necessarily constant. The set of Arov-regular J-inner functions plays a key role in the theory of Schur-Nevanlinna-Pick interpolation. This will be explained to some extend in Section 8.

Potapov has studied the J-contractive functions in great detail. One of his main concerns was to factor a rational J-contractive matrix as a product of "elementary" J-contractive factors. This problem is inspired by the problem of Darlington synthesis, a problem from electrical engineering where this factorization will help to realize an electrical network, called an m-port as a cascade of elementary m-ports. To motivate the reader, we give some elements about Darlington synthesis in the next section.

[1] If λ is the Lebesgue measure of the unit circle, then the Smirnov class $\mathcal{N}_+(\mathbb{D})$ is the class of functions $g \in \mathcal{N}(\mathbb{D})$ for which $\int \log^+ |\underline{g}(z)|\lambda(\mathrm{d}z) = \lim_{r\uparrow 1} \int \log^+ |g(rz)|\lambda(\mathrm{d}z)$ where $\log^+ x = \max\{\log x, 0\}$, $x \geq 0$, $\underline{g}$ is the boundary function.

3 Darlington synthesis and scattering theory

The electrical background of the problem of Darlington realization of a general m-port can be found in the book by Belevitch [Bel70].

An *electrical network* is a finite number of interconnected elements like resistances, capacitances, inductances, current or voltage generators etc. Such a network can have terminals, i.e., "loose ends" to which some other network can be connected or where some measurements can be taken. A couple of such terminals is called a *port* and a network is called an m-port if it has $2m$ terminals which are paired in m ports. A port is characterized by 2 variables, called port variables, which could be e.g., the voltage and the current over that port. Such port variables are functions which are interconnected by differential equations, which describe the properties of the electrical elements. Taking the Laplace transform we get an algebraic relation between functions of the complex variable z in the transform (frequency) domain. If, in this z-domain, the voltage and current for port i are represented by V_i and I_i, then $P = \sum_{i=1}^{n} I_i^* V_i$ is called the *power* dissipated in the n-port. Setting for an

Figure 1: A 2-port

m-port

$$V = [V_1 \;\; V_2 \;\; \cdots \;\; V_m]^T \quad \text{and} \quad I = [I_1 \;\; I_2 \;\; \cdots \;\; I_m]^T,$$

then $P = I^* V$. If the network has no internal generators, then there is a linear relation between V and I given by

$$V = RI$$

where the $m \times m$ matrix R is called the *impedance* matrix.

A *passive* m-port has an active power that is non-negative in the right half plane, i.e., $\text{Re } P(z) \geq 0$ for $\text{Re } z > 0$. If we have in addition $\text{Re } P(z) = 0$ for $\text{Re } z = 0$, then the m-port is called *lossless*. Since for a passive network $I^*(\text{Re } R)I \geq 0$ in the right half plane, it follows that $R(z)$ is a square $m \times m$ matrix valued function which satisfies

$$\text{Re } R(z) \geq 0 \quad \text{for} \quad \text{Re } z > 0.$$

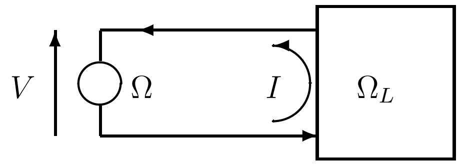
Such a matrix is called passive. For a lossless network we have moreover that

$$\text{Re } R(z) = 0 \quad \text{for} \quad \text{Re } z = 0.$$

Such a matrix is called lossless. Obviously, a passive matrix valued function is for the right half plane what a square matrix valued I_m-contractive function is for the unit disk. It is not difficult in fact to transform one into the other. Similarly, a lossless function is like a square I_m-inner function. Up to a rotation from the right half plane to the upper half plane, the passive matrix functions correspond to a square matrix version of the Nevanlinna-Pick class, which we have defined in Section 1. In fact, when we had started with a digital (i.e. a discrete) network then we should have applied the z-transform instead of the Laplace transform and we would immediately have obtained the formulation of the prolem in the unit disk. This problem in the unit disk corresponds to a scattering problem, which is equivalent to a problem of transmission lines in electrical engineering. Since we started with a continuous problem, we will now line out how it can be transformed to the unit disk in the form of a scattering problem.

If a 1-port contains internal current or voltage generators, we have an inhomogeneous relation between voltage V and current I: $V = RI + E$ where V and I represent some combinations of internal variables of the network. Physically, this formula means that it can be realized by placing a voltage generator in series with an impedance R. This is in fact Thevenin's theorem. When $I = 0$, then $V = E$. Thus E is the open-circuit voltage of the 1-port. If all the internal generators are neutralized, i.e., voltage generators replaced by short-circuits and current generators by open circuits, then there is no equivalent external E and $V = RI$. The impedance $R = V/I$ is called the internal impedance of the 1-port. Now consider

Figure 2: Generator with load



a voltage generator with internal impedance Ω and which is loaded by an external load Ω_L, then there will be a current

$$I = \frac{V}{\Omega + \Omega_L}$$

in that port. If one takes $\Omega_L = \overline{\Omega}$, then the total power dissipated in the load is maximal [Bel70, p. 159]. In that case the current is

$$I_0 = \frac{V}{2\,\mathrm{Re}\,\Omega},$$

so that the relative difference for the currents with an open-circuit (load 0) and a load Ω_L is

$$s = \frac{I_0 - I}{I} = \frac{\Omega_L - \Omega}{\Omega_L - \Omega}.$$

This s is called the *reflectance* of Ω_L relative to Ω. Choosing the normalized variables $i = I\sqrt{\Omega}$, $v = V/\sqrt{\Omega}$ and $\omega = \Omega_L/\Omega$, we get

$$s = \frac{\omega - 1}{\omega + 1} \quad \text{or equivalently} \quad \omega = \frac{1+s}{1-s}.$$

Note that $\omega = v/i$ because $\Omega = V/I$. It follows that

$$\mathrm{Re}\,\omega = \frac{1 - |s|^2}{|1 - s|^2},$$

which shows that Ω, hence also ω, being passive is equivalent with $|s(z)| \le 1$ for Re $z > 0$, and if ω is lossless, then $|s(z)| = 1$ for Re $z = 0$. Replacing the current i and the voltage v by two other variables:

$$x = \frac{v+i}{2} \quad \text{and} \quad y = \frac{v-i}{2},$$

we find from $s = (v-i)/(v+i)$ that $y = sx$. The x is called the *incoming wave* variable and y the *outgoing wave* variable. All this has been explained for a 1-port, but the same kind of transformations can be done for each port of an m-port. If the wave variables for port i are (x_i, y_i) and if we define $X = [x_1 \ \ x_2 \ \ \ldots \ \ x_m]^T = (V+I)/2$, and $Y = [y_1 \ \ y_2 \ \ \ldots \ \ y_m]^T = (V-I)/2$, then we obtain a relation $Y = SX$ which can replace $V = RI$. The matrix is called the *scattering matrix* of the m-port. It is related to the (internal) impedance matrix of the m-port by

$$S = (R + I_m)^{-1}(R - I_m).$$

Thus

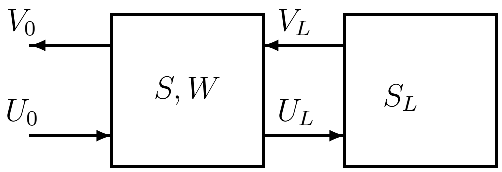
$$P = I^*V = (X-Y)^*(X+Y) = X^*(I_m - S^*S)X.$$

So, we have that for a passive m-port, $S(z)^*S(z) \le I_m$ in Re $z > 0$ and if the m-port is lossless, then $S(z)^*S(z) = I_m$ for Re $z = 0$. One refers to these properties as passiveness respectively losslessnes of the scattering matrix S.

This can be interpreted as a scattering problem. A scattering medium has an input wave U_0 and a reflected wave V_0 at the surface of the medium. The transfer function $S_0 = V_0/U_0$ is called the scattering function of the medium.

One may now decompose the scattering medium in two layers. On the left, U_0 is the incoming wave and V_0 is the reflected wave. At the interface of the left layer and the right layer, a wave U_L is transmitted from the left to the right layer and a wave V_L is reflected by the right layer to the left layer. Suppose the scattering function of the right layer is S_L then $V_L = S_L U_L$. The right layer acts as a load applied to

Figure 3: Scattering medium with load

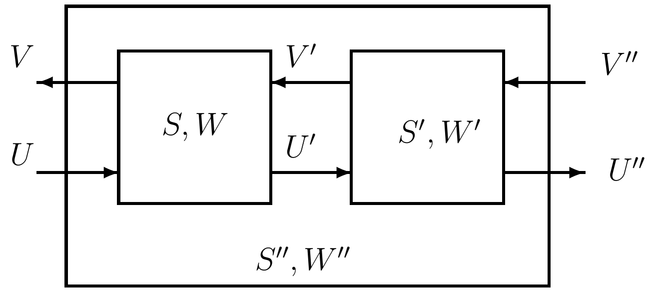


a 2-port which represents the left layer. Considering $X = [U_0 \ \ V_L]^T$ as input and $Y = [U_L \ \ V_0]^T$ as output for this 2-port, we have

$$Y = SX$$

where S is a (in this case a 2 by 2) scattering matrix. Such a scattering matrix has however the following drawback. Suppose we want to subdivide the scattering medium in more layers. Consider two adjacent layers as in Figure 4. The left one has

Figure 4: A cascade of 2 layers



a scattering matrix S and the right one a scattering matrix S'. Using the notation of the figure, we have

$$\begin{bmatrix} U' \\ V \end{bmatrix} = S \begin{bmatrix} U \\ V' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U'' \\ V' \end{bmatrix} = S' \begin{bmatrix} U' \\ V'' \end{bmatrix}$$

while for the cascade of both we assume a scattering matrix S'', thus

$$\begin{bmatrix} U'' \\ V \end{bmatrix} = S'' \begin{bmatrix} U \\ V'' \end{bmatrix}.$$

The relation between S, S' and S'' is given by the Redheffer product [Red62]

$$S'' = S * S' = \begin{bmatrix} S'_{11} + S'_{12} S_{11} \Gamma S'_{12} & S'_{12} S_{11} \Gamma S'_{22} S_{12} + S'_{12} S_{12} \\ S_{21} \Gamma S'_{21} & S_{21} \Gamma S'_{22} S_{12} + S_{22} \end{bmatrix}$$

with $\Gamma = (1 - S'_{22} S_{11})^{-1}$. If $1 - S'_{22} S_{11}$ is not identically zero, this will exist for all values of z, except for at most a countable number of values. This is a rather complicated expression. For the cascading of layers, it is much easier to work with *chain scattering* matrices. Scattering matrices give relations between inputs and outputs where each port is considered to have an input and an output. For the cascading, it is more natural to consider both terminals of the left port as an input to the medium and both terminals of the right port as an output. The relation between these inputs and outputs are given by the chain scattering matrix, which we denote by W. Thus

$$\begin{bmatrix} U' \\ V \end{bmatrix} = S \begin{bmatrix} U \\ V' \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} U' \\ V' \end{bmatrix} = W \begin{bmatrix} U \\ V \end{bmatrix}.$$

It is then much easier to cascade because the cascade of two layers with chain scattering matrices W and W' has a scattering matrix $W'' = WW'$ which is given by the ordinary matrix product. To derive the relation between S and W for a 2-port, we introduce the projectors

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P^\perp = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We then have

$$\begin{bmatrix} U' \\ V' \end{bmatrix} = (PS + P^\perp) \begin{bmatrix} U \\ V' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U \\ V \end{bmatrix} = (P^\perp S + P) \begin{bmatrix} U \\ V' \end{bmatrix}.$$

Thus

$$\begin{bmatrix} U' \\ V' \end{bmatrix} = (PS + P^{\perp})(P^{\perp}S + P)^{-1} \begin{bmatrix} U \\ V \end{bmatrix}$$

so that

$$W = (PS + P^{\perp})(P^{\perp}S + P)^{-1} = \begin{bmatrix} S_{11} - S_{12}S_{22}^{-1}S_{21} & S_{12}S_{22}^{-1} \\ -S_{22}^{-1}S_{21} & S_{22}^{-1} \end{bmatrix} \tag{10}$$

if S_{22} is not identically zero. The latter transformation is often called the Potapov-Ginzburg transformation. If the scattering matrix is passive (resp. lossless) then $S^*S \leq I_2$ in Re $z > 0$ (resp. $S^*S \leq I_2$ in Re $z > 0$ and $S^*S = I_2$ on Re $z = 0$) and this translates into the chain scattering matrix being J-contractive (resp. J-inner) for the right half plane where

$$J = P - P^{\perp} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Similar observations hold for an m-port which is passive or lossless: its chain scattering matrix is a J-contractive or J-inner matrix valued function for a more general signature matrix J.

After this very elementary introduction in the tools and terminology, it should be clear that to understand complex m-port electrical networks, or to analyse a complex scattering medium, it is important to thoroughly understand the structure and properties of J-contractive and J-inner matrix valued functions. If one wants to realize the network, or model the scattering medium as a layered system, then one should be able to factor such a J-contractive function as a product of elementary J-contractive functions. Each such factor should be rational and of lowest possible degree. Indeed, the degree of a rational transfer function somehow describes the complexity of the system, computationally but also physically. The simplest possible section should have only one pole, which could be inside, outside or on the border of the region of interest which is the unit disk or a half plane. Such factors are now called Blaschke-Potapov factors. So far we only allowed poles outside this region, since we assumed for stability reasons (i.e., passiveness) that the function was analytic inside. An elementary m-port corresponding to such an elementary factor with pole outside the region is often referred to as a Schur section. However, under certain conditions, it is also possible to extract factors with poles on the boundary. Such a section is called a Brune section [Bru31]. In that case, there is of course the relation with the boundary Nevanlinna-Pick or Loewner interpolation problems and multipoint moment problems.

We note that these scattering problems can also be formulated as linear prediction problems. This was in fact the approach taken by Dewilde/Dym [DD81a, DD81b, Dew84, DD84]. These prediction and filtering problems were initiated by Wiener [Wie49], and by the generalization of Wiener/Massani [WM57]. Here some least squares problem has to be solved and it was Levinson [Lev47] who formulated an efficient algorithm to solve the normal equations. The recursion used in this algorithm turned out to be equivalent to the Szegő recursion for polynomials orthogonal on the unit circle [Sze67]. The Schur algorithm was rediscovered in the fifties in the context of seismic signal processiong where it turned out that the algorithm could be given a full physical interpretation and the Schur parameters are often called reflection coefficients because of this interpretation. Burg in his Ph.D. thesis at Stanford [Bur75] explained the connection between spectral analysis of time series, the maximum entropy method and Szegő's theory.

What has been said here in the context of the Schur algorithm can be be generalized to the Nevanlinna-Pick algorithm. A generalization of the theory of Szegő's orthogonal polynomials has been generalized to orthogonal rational functions with prescribed poles. In prediction terms, this means that the autoregressive filters are replaced by autoregressive moving average (ARMA) filters with given transmission zeros. The study of these orthogonal rational functions was initiated by M.M. Djrbashian in the sixties [Djr62b, Djr62a, Djr66a, Djr66b, Djr67, Djr90]. It was continued in a number of papers by Bultheel/González-Vera/Hendriksen/Njåstad. The current state of the art is collected in [BGVHN97].

These ideas made their way firmly into electrical engineering, signal processing and robust control since then. A recent monograph by Kimura [Kim97] clearly illustrates the role of chain scattering matrices in optimal control.

In the remaining sections we shall be mainly concerned with a survey of the history and the different approaches taken by several (groups of) researchers in their attempt to tackle the study of general interpolation problems and in particular of general J-inner functions.

4 Operator theoretic interpolation problems and dilation theory

Not only matricial, but also operator versions of classical interpolation problems play an important role in this context because they too are connected with applications in the field of electrical circuits (see Efimov/Potapov [EP73]) and questions of system theory (see Arov [Aro79b], [Aro79c]). This trail of research culminated in the *commutant lifting theorem* by Nagy and Foiaş which was initiated by results of Sarason for the solution of the Nevanlinna-Pick problem. We shall briefly sketch these ideas in this section.

Potapov's algorithm for the solution of matricial interpolation problems was generalized to operator theoretical interpolation problems by Ivacenko/Sakhnovic [IS87b]. The main tool in their approach is based on dilation theory. To explain this, we have to introduce the concept of dilation first.

Let H and H_1 be Hilbert spaces. A linear transformation T from H into H_1 is a *contraction* if

$$\|Th\|_{H_1} \leq \|h\|_H$$

for all $h \in H$, i.e. $\|T\| \leq 1$. Assume that H and H_1 are subspaces of some Hilbert space K. Then an operator U (defined on K) is called *a dilation of* T if

$$T^n h = P_H U^n h, \quad n = 1, 2, \ldots$$

for all $h \in H$ where P_H denotes the orthogonal projection from K onto H. Two dilations of T, say U and $\widetilde{U}$ on K and $\widetilde{K}$, respectively, are called *isomorphic* if there exists a unitary transformation ϕ from $\widetilde{K}$ onto K such that $\phi h = h$ for all $h \in H$, and $\widetilde{U} = \phi^{-1} U \phi$. A dilation V is called *isometric* if $V^* V = I$. Observe that for every contraction T on a Hilbert space H, there is an isometric dilation V on some Hilbert space K_+ which is *minimal* in the sense that

$$K_+ = \bigvee_{n=0}^{\infty} V^n H.$$

Note that this minimal isometric dilation of T is determined up to an isomorphism. Therefore it is called *the minimal isometric dilation of* T.

Let T and T' be contractions on some Hilbert spaces H and H' respectively, and let $I(T,T')$ denote the set of all contractions $A : H \to H'$ intertwining T and T', i.e. $T'A = AT$. Let V on K_+ and V' on K'_+ be the minimal isometric dilations of T and T', respectively. If A belongs to $I(T,T')$ then $B : K_+ \to K'_+$ is called *a contractive intertwining lifting of* A if B belongs to $I(V,V')$ and B is a lifting of A, i.e., if

$$P_{H'}B = AP_H$$

(P_H and $P_{H'}$ denote projections). See Figure 5.

Figure 5: Intertwining lifting

$$\begin{tikzcd}
K_+ \arrow[rrr, "B"] \arrow[dr, "P_H"'] \arrow[ddd, "V"'] & & & K'_+ \arrow[dl, "P_{H'}"] \arrow[ddd, "V'"] \\
& H \arrow[r, "A"] \arrow[d, "T"'] & H' \arrow[d, "T'"] & \\
& H \arrow[r, "A"'] & H' & \\
K_+ \arrow[rrr, "B"'] \arrow[ur, "P_H"'] & & & K'_+ \arrow[ul, "P_{H'}"]
\end{tikzcd}$$

Now we are able to formulate the fundamental commutant lifting theorem which, in its general form, goes back to Sz.-Nagy/Foiaş [SNF68], [SNF70].

Theorem. If A belongs to $I(T,T')$ then there exists a contractive intertwining lifting B of A.

Since these original papers, a set of alternate proofs of this theorem were published (see Douglas/Muhly/Pearcy [DMP68], Parrott [Par78], Arocena [Aro83]). Foiaş/Frazho give in [FF90] several proofs for the commutant lifting theorem where each proof illuminates different features of this theorem. One proof discusses the uniqueness question in the commutant lifting theorem. Another one establishes the connection to Ando's dilation theorem for two commuting contractions, while a further one is based on Arocena's coupling of contractions. There also exist algorithms to construct all the commuting intertwining liftings of which the theorem gives the existence, and these algorithms are somehow generalizations of the Nevanlinna-Pick algorithm. Here of course we can make the connection with the previous sections of this text. To do this, we give a result due to Sarason [Sar67] which was the immediate source of inspiration for the formulation of the commutant lifting theorem and which was formulated while he was investigating interpolation problems of Nevanlinna-Pick type.

To formulate Sarason's result, we introduce the concept of a function interpolating an operator. Let U be the shift operator in L^2 and let ψ be a nonconstant inner function. Further, let S be the projection of the shift operator U onto $H^2 \ominus \psi H^2$. For a function ϕ belonging to H^∞, let M_ϕ be the operator which corresponds to the multiplication with ϕ in L^2. Let $\phi(S)$ denote the projection of M_ϕ onto $H^2 \ominus \psi H^2$. If an operator T (on the Hilbert space $H^2 \ominus \psi H^2$) can be written as $\phi(S)$ for a ϕ in H^∞, then we say that ϕ *interpolates* T. Note that the operators $\phi(S)$ are exactly the operators that commute with S. The converse case is given in the following result.

Theorem. If T is an operator on H that commutes with the shift operator S, then there is a function ϕ in H^∞ such that ϕ interpolates T and $\|\phi\|_\infty = \|T\|_H$.

Sarason proved this result in his paper [Sar67]. Furthermore, he indicated the perspectives of the application of lifting theorems in interpolation theory. He considered the Carathéodory problem in the language of contractive intertwining lifting problems in the following way: The Carathéodory problem corresponds to the case where ψ is a power of z, i.e., $\psi(z) = z^{n+1}$. Then the considered subspace $H^2 \ominus \psi H^2$ is an $(n+1)$-dimensional space and has an orthonormal basis $e_k(z) = z^k$, $k = 0, \dots, n$. The operator S is the shift with respect to the basis of $H^2 \ominus \psi H^2$, i. e., $Se_k = e_{k+1}$. Then an operator on $H^2 \ominus \psi H^2$ commutes with S if and only if its matrix (with respect to the basis $\{e_k\}$) has the representation

$$\begin{bmatrix} c_0 & 0 & 0 & \dots & 0 \\ c_1 & c_0 & 0 & \dots & 0 \\ c_2 & c_1 & c_0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ c_n & c_{n-1} & c_{n-2} & & c_0 \end{bmatrix}. \tag{11}$$

Moreover, a function in H^∞ interpolates the operator with the matrix representation (11) if and only if its power series begins with $c_0 + c_1 z + \dots + c_n z^n$, i. e., if and only if it belongs to the solution set of the Carathéodory interpolation problem. This argumentation led Sarason [Sar67] to a condition which is equivalent to the following well-known Carathéodory-Toeplitz condition for the solvability of the Carathéodory coefficient problem: There exists a function in the Carathéodory class $\mathcal{C}(\mathbb{D})$ whose power series begins with the terms $c_0 + c_1 z + \dots + c_n z^n$ if and only if the matrix (11) has a nonnegative real part as an operator on an $(n+1)$-dimensional Hilbert space. There is more than that in [Sar67], since one also finds there the characterization of all solutions of the Carathéodory coefficient problem which have a minimal H^∞-norm. This makes the immediate link with the Adamjan, Arov and Kreĭn papers and minimal Hankel norm approximation.

Sarason also described the Nevanlinna-Pick interpolation problem in an operator-theoretical language. This problem corresponds to the case where ψ coincides with a Blaschke product whose simple zeros are the interpolation points $z_0, \ldots, z_n$. The subspace $H^2 \ominus \psi H^2$ is the $(n+1)$-dimensional subspace spanned by the functions

$$g_k(z) = \frac{1}{1 - \overline{z}_k z}\ , \quad k = 0, \ldots, n. \tag{12}$$

These functions $g_0, \ldots, g_n$ are eigenvectors of S^* with respective eigenvalues $\overline{z}_0, \ldots, \overline{z}_n$. This implies that an operator T commutes with S if and only if $g_0, \ldots, g_n$ are eigenvectors of T^*. To get the Nevanlinna-Pick problem, this is exactly what is needed because then T should be the operator which is defined on $H^2 \ominus \psi H^2$ by

$$T^* g_k = \overline{w}_k g_k\ , \quad k = 0, \ldots, n\ , \tag{13}$$

where $\{w_k\}_{k=0}^n$ are the complex function values corresponding to the given distinct interpolation points $z_0, \ldots, z_n$. In this case, a function ϕ in H^∞ interpolates T if and only if $\phi(z_k) = w_k$ for $k = 0, \ldots, n$, thus if and only if it is a solution of the Nevanlinna-Pick problem. With an appropriate transformation from the disk to the right half plane, Sarason got a condition equivalent to the one given by Pick.

The lifting problem has also a strong connection with the characterization of shift invariant subspaces. This characterization problem was given by Beurling in the scalar case and generalized by Lax [Lax59], Halmos [Hal61], and Masani [Mas62]. Later Lax and Phillips built up their scattering theory [LP67] as an alternative highway to study these problems. Masani together with Wiener worked out their prediction theory [WM57]. A very general approach to the Beurling-Lax theorem was given later by Ball/Helton [BH83] (see below).

The solution of an interpolation problem with dilation theory always leads to a parametrization of all the contractive intertwining dilations of a certain operator. This problem in particular, was also investigated by Ando/Ceausescu/Foiaş [ACF77], Ceausescu/Foiaş [CF78] and Arsene/Ceausescu/Foiaş [ACF80]. They parametrized the contractive intertwining dilations with a sequence of contractive parameters (which is called *choice sequence*). This choice sequence is the operator theoretic generalization of the sequence of Schur parameters. The Schur parameters characterized a Schur function and the choice sequence characterizes a contraction. In the operator theoretic problem, the (generalization of the) Schur algorithm does indeed generate a choice sequence of Schur parameters $\{\Gamma_n\}_{n\in\mathbb{N}}$. In this sequence, $\Gamma_0 \in \mathcal{L}(H_0, K)$ is a contraction from a Hilbert space H_0 into a Hilbert space K and for $n \geq 1$, $\Gamma_n \in \mathcal{L}(H_n, \mathcal{D}_{\Gamma^*_{n-1}})$ is a contraction from a Hilbert space H_n into $\mathcal{D}_{\Gamma^*_{n-1}}$. Here we used for an operator Γ between Hilbert spaces the notation Γ^* to denote its adjoint, $D_\Gamma = (I - \Gamma^*\Gamma)^{1/2}$ is its defect operator and $\mathcal{D}_\Gamma$ is its defect space, that is the closure of the range of D_Γ. $\mathcal{L}(H, K)$ represents the linear operators between the Hilbert spaces H and K.

This parametrization forms a basis on which it is possible to treat several operator extension problems and operator interpolation problems. Especially applications in prediction theory and entropy optimization were considered in this context (Arsene/Constantinescu [AC85], [AC87], Arsene/Ceausescu/Constantinescu [ACC88] and Constantinescu [Con85]–[Con87]).

A disadvantage of the lifting approach is the fact that it can not be used for boundary value interpolation problems. Inspired by Nudelman, Rosenblum/Rovnyak

[RR80], [RR85] formulated a generalized operator interpolation problem which contains the classical results of Schur, Carathéodory, Nevanlinna, Pick and Loewner as special cases.

A further operator theorectical approach for interpolation problems was created by Ball/Helton [BH83]–[BH88] (see also Sarason [Sar85], [Sar87]). They had the original idea to embed interpolation problems into a context of Kreĭn spaces. Hereby the associated operator extension problems were transformed into extension problems of subspaces of the Kreĭn space. The Method of Ball and Helton can also be used to treat boundary value interpolation problems and indefinite interpolation problems. An essential part of their approach is a useful indefinite generalization of the theorem of Beurling/Lax/Halmos/Masani about shift invariant subspaces. Furthermore Ball and Helton got the most far-reaching results concerning the treatment of one- and two-sided tangential interpolation problems which were considered earlier by Fedčina [Fed72]–[Fed75b] in connection with the papers of Adamjan, Arov and Kreĭn. As we mentioned before, such tangential interpolation prolems are characterized by the fact that not the interpolation data themselves but their projection into certain given directions are prescribed. For more details see Section 8 below.

5 Potapov factorization of J-contractive functions

Let us now concentrate on the study of J-inner matrix functions which are at the heart of all the theory and the different approaches that were sketched in the previous sections.

The work of V.P. Potapov is at the origin of the vast literature that is now available on matrix valued J-inner functions. For some history and a survey of early and more recent results one should consult the Potapov memorial volumes [GoS94, DFKK97] published in the Operator Theory Series edited by I. Gohberg.

The class $\mathfrak{P}_J$ as we defined it above was introduced by M.S. Livšic and V.P. Potapov [LP50] in 1950. Livšic obtained basic results about the spectral analysis of non-unitary (respectively, non-selfadjoint) operators with minimal degree of non-unitarity (respectively, non-selfadjointness). These investigations were inspired by system theory and the results are often formulated in terms of special dissipative systems. He showed that the transfer function (or chain scattering matrix) of such a system belongs to $\mathfrak{P}_J$. This is what we also obtained in the simple situation of Section 3. Because such systems are uniquely determined by their transfer function, all questions of analysis and synthesis of these systems can be translated into corresponding analytic questions for the transfer function. As we have seen, an important question is the problem of decomposing such a system into a cascade of "easy pieces" and this corresponds to the factorization of functions of the Potapov class into a product of simple functions of this class. Unfortunately, this is a difficult problem. What are these simple functions? The complete description of the solution of this factorization problem is the main result of V.P. Potapov's fundamental paper [Pot55], where he got a direct generalization of the well-known factorization theorem of F. Riesz and V.I. Smirnov for bounded holomorphic functions on the unit disc. Already the description of the rational elementary factors of the Potapov class, i.e., the matricial generalization of an elementary Blaschke factor, was far from being trivial. It was shown that such rational elementary factors (which are now called *Blaschke-Potapov elementary factors*) are not only determined by a singularity. They depend also on a J-projector.

If P is an $m \times m$ idempotent complex matrix such that JP is non-negative Hermitian, then the *Blaschke-Potapov J-elementary factor of the first kind* has the form

$$B_{w,P} := I_m + (\zeta_w - 1)P \tag{14}$$

where ζ_w is a usual scalar Blaschke factor. This Blaschke-Potapov J-elementary factor is holomorphic and its inverse function $B_{w,P}^{-1}$ has a simple pole in w. It follows easily that $B_{w,P}$ is J-contractive and has a J-unitary boundary value almost everywhere on the unit circle (with respect to the Lebesgue measure). If Q is an $m \times m$ idempotent complex matrix such that $-JQ$ is non-negative Hermitian, then

$$C_{w,P} := I_m + (\frac{1}{\zeta_w} - 1)Q \tag{15}$$

is said to be a *Blaschke-Potapov J-elementary factor of the second kind*. It is readily checked that the inverse function of a Blaschke-Potapov J-elementary factor of the first kind is such a factor of the second kind. Observe that, in the positive definite case ($J = I_m$), there is no Blaschke-Potapov J-elementary factor of the second kind. In the negative definite case, only the Blaschke-Potapov J-elementary factor of the second kind exists.

Let $\{w_k\}_{k\in K}$ be a sequence of points of the unit disk and let $\{P_k\}_{k\in K}$ be a sequence of idempotent complex matrices such that JP_k is non-negative Hermitian for all $k \in K$. Assume that

$$\sum_{k\in K} (1 - |w_k|)\|P_k\| < \infty \ . \tag{16}$$

Furthermore let B_{w_k,P_k} be the corresponding Blaschke-Potapov J-elementary factors of the first kind. Then the products

$$B_l := \overrightarrow{\prod_{k\in K}} B_{w_k,P_k} = B_{w_{i_1},P_{i_1}} B_{w_{i_2},P_{i_2}} \ \ldots \tag{17}$$

and

$$B_r := \overleftarrow{\prod_{k\in K}} B_{w_k,P_k} = \ldots \ B_{w_{i_2},P_{i_2}} B_{w_{i_1},P_{i_1}} \tag{18}$$

where $K = \{i_1, i_2, \ldots\}$, converge absolutely and uniformly on each compact set inside the unit disc. Hereby the order of the factors is essential because the multiplication of matrices is non-commutative. The condition (16) is called *Blaschke-Potapov condition* and the product B_l (resp. B_r) is called a *left* (resp. *right*) *Blaschke-Potapov product of the first kind with respect to J*. A Blaschke-Potapov product of the second kind is defined in a similar way. A matrix-valued function $B^{(l)}$ (respectively, $B^{(r)}$) is called a *left* (respectively, *right*) *Blaschke-Potapov product with respect to J* if it admits a product representation with Blaschke-Potapov J-elementary factors of first *or* second kind. For further convergence properties of Blaschke-Potapov products we refer the reader to Potapov [Pot55] and Ginzburg [Gin58]. See also [GiS94] for a survey.

Since a Blaschke-Potapov factor of the first kind is a holomorphic J-contractive matrix-valued function in the unit disc, a Blaschke-Potapov product with respect to J is a J-contractive meromorphic function. This implies that every such Blaschke-Potapov product has radial boundary values almost everywhere on the unit circle. Arov proved that these boundary values are J-unitary. That means that the restriction of a left or right Blaschke-Potapov product B to the domain of analyticity of B is a J-inner function. We can say more: the restriction of an arbitrary left (resp. right) Blaschke-Potapov product B to the domain of analyticity of B is a left (resp. right) Arov-regular J-inner function. In the rest of this paper we use the symbol $B^{\square}$ to denote the restriction of B to its domain of analyticity in the unit disc.

In 1955 V.P. Potapov [Pot55] formulated and proved his fundamental theorem about the existence and uniqueness of multiplicative representations of J-contractive matrix-valued meromorphic functions. This result can be considered as a generalization of the F. Riesz-Nevanlinna-Smirnov factorization for bounded holomorphic functions in the unit disc. We formulate this as a theorem.

Theorem. Let J be an $m \times m$ signature matrix and let $W \in \mathfrak{P}_J$. Further let $\mathfrak{P}_{J;s}$ be the set of all singular functions which belong to $\mathfrak{P}_J$. Then the following holds true.

(a) There are a left Blaschke-Potapov product B_l with respect to J and a function Σ_{ls} belonging to $\mathfrak{P}_{J;s}$ such that $W = B_l^{\square} \cdot \Sigma_{ls}$.

(b) If $W = \widetilde{B_l}^{\square} \cdot \widetilde{\Sigma_{ls}}$ is another factorization of W with some left Blaschke-Potapov product $\widetilde{B_l}$ with respect to J and some function $\widetilde{\Sigma_{ls}} \in \mathfrak{P}_{J;s}$ then there is a J-unitary matrix U such that $\widetilde{B_l} = B_l U$ and $\widetilde{\Sigma_{ls}} = U^{-1}\Sigma_{ls}$. If W is J-inner then $\widetilde{\Sigma_{ls}}$ is J-inner.

(c) If U is an arbitrary J-unitary matrix then $\widetilde{B_l} := B_l U$ and $\widetilde{\Sigma_{ls}} := U^{-1}\Sigma_{ls}$ are a left Blaschke-Potapov product with respect to J and a function belonging to $\mathfrak{P}_{J;s}$, respectively, such that $W = \widetilde{B_l}^{\square} \cdot \widetilde{\Sigma_{ls}}$.

During his investigation on Schur-Nevanlinna-Pick interpolation, Arov [Aro73], [Aro74a], [Aro75] got another factorization for the special case, that W is a J-inner function. For that we refer to Section 8.

Further details on left and right Potapov products were obtained by V.E. Katsnelson in a series of papers [Kat89], [Kat90], [Kat93]. He got remarkable relations between left and right Blaschke-Potapov products (with respect to an arbitrary signature matrix J). Of course, in the definite case, i.e., if the signature matrix J coincides with I or $-I$, then a left Blaschke-Potapov product is a right Blaschke-Potapov product at the same time. However, in the indefinite case, i.e. when the signature matrix J is not definite, it can happen that the so-called singular part in the multiplicative representation of a left Blaschke-Potapov product is not constant. Hence there are matrix functions which are a left Blaschke-Potapov product but not a right Blaschke-Potapov product. Furthermore it proves to be possible to give a function-theoretical description of the class of all singular functions which occur in the right multiplicative representations of left Blaschke-Potapov products. We include two of his theorems to illustrate these results.

Theorem. Let J be an $m \times m$ signature matrix which is different from I_m and $-I_m$.

(a) Let B_l be a left Blaschke-Potapov product with respect to J. Further let W_s be an Arov-singular J-inner function and let W_{rr} be a right Arov-regular J-inner function such that

$$B_l^{\square} = W_s W_{rr} \ .$$

Then there is a right Blaschke-Potapov product $\widetilde{B_r}$ with respect to J such that

$$W_{rr} = \widetilde{B_r}^{\square} \ .$$

(b) Let B_r be a right Blaschke-Potapov product with respect to J. Further let V_s be an Arov-singular J-inner function and let V_{lr} be a left Arov-regular J-inner function such that

$$B_r^{\square} = V_{lr} V_s \ .$$

Then there is a left Blaschke-Potapov product $\widetilde{B_l}$ with respect to J such that

$$V_{lr} = \widetilde{B_l}^{\square} \ .$$

Theorem. Let J be an $m \times m$ signature matrix which is different from I_m and $-I_m$.

(a) Let B_l be a left Blaschke-Potapov product with respect to J. If Σ_{rs} is a function that belongs to $\mathfrak{P}_J$ and if $\widetilde{B_r}$ is a right Blaschke-Potapov product with respect to J such that

$$B_l^{\square} = \Sigma_{rs} \widetilde{B_r}^{\square}$$

then Σ_{rs} is an Arov-singular J-inner function.

(a) Let B_r be a right Blaschke-Potapov product with respect to J. If Σ_{ls} is a function that belongs to $\mathfrak{P}_J$ and if $\widetilde{B_l}$ is a left Blaschke-Potapov product with respect to J such that

$$B_r^{\square} = \widetilde{B_l}^{\square} \Sigma_{ls}$$

then Σ_{ls} is an Arov-singular J-inner function.

V.E. Katsnelson's investigations are essentially based on the treatment of problems of weighted approximation for special classes of meromorphic matrix-valued functions. This approach uses a couple of deep results, namely a generalization of Frostman's result about the value distribution of holomorphic functions (which goes back to W. Rudin) and a theorem of D.Z. Arov about approximation of pseudocontinuable functions.

If we consider Potapov's Riesz-Nevanlinna-Smirnov type of theorem which we formulated above as the main result of a first phase in the development of J-theory, then we could say that in the second area of J-theory, V.P. Potapov and his collaborators A.V. Efimov and I.V. Kovalishina dealt with problems which had their origin in the theory of electrical circuits (see Efimov/Potapov [EP73], Kovalishina [Kov66], Melamud [Mel72], Tovmasjan [Tov71]). In this framework, a detailed discussion about several constellations of poles of J-elementary factors was initiated. Moreover, the inner structure of J-orthogonal projectors was discovered. V.P. Potapov [Pot69] improved the procedure for separating a J-elementary factor with one pole from a given matrix-valued function which belongs to $\mathfrak{P}_J$. It was a fundamental observation of Potapov to see that the Schur algorithm is equivalent to the multiplicative decomposition of a J-elementary factor (with a pole of order n at infinity) in a product of n J-elementary factors (with a pole of order 1). For a detailed treatment of this procedure, the reader can also be referred to [DFK92, Chapter 4.3] or to the papers by Dewilde and Dym [DD81a, DD81b, Dew84, DD84].)

In this way, the important role of J-elementary factors was exposed in the framework of matricial versions of classical interpolation and moment problems. The third phase of J-theory began in which general interpolation problems were considered. We discuss this in the next section.

6 Non-negative definite kernels, fundamental matrix inequalities, fundamental identities and the generalized Nehari problem

V.P. Potapov developed a powerful approach to matricial interpolation problems. This approach is based on a generalization of a classical lemma of H.A. Schwarz and a modification of this result which goes back to G. Pick. He converted the original problem in an equivalent matricial inequality (the *fundamental matrix inequality* or FMI of the problem). In the case that the so-called information block of this inequality is non-degenerate, he created a clever factorization method which allows the determination of the solution set of the matrix equality and consequently also of the original interpolation problem (see Kovalishina/Potapov [KP74]–[KP89], Dubovoj [Dub82], Galstjan [Gal77], Golinskii [Gol83b], [Gol83a], Djukarev/Katsnelson [DK81], Djukarev [Dju82], Dubovoj/Fritzsche/Kirstein [DFK92, Chapter 5], and Katsnelson [Kat97]).

As an example, we consider the Nevanlinna-Pick problem for the upper half plane. If $z_1, z_2, \ldots$ is a set of distinct points in the upper half plane and $w_1, w_2, \ldots$ a corresponding set of function values with positive real part, then we have to find a Nevanlinna-Pick function $w \in \mathcal{P}(\mathbb{C}_+)$ such that $w(z_i) = w_i$, $i = 1, 2, \ldots$. Potapov showed that w is a solution of this problem if and only if

$$\begin{bmatrix} \left[\dfrac{w_i - w_k^*}{z_i - z_k}\right]_{i,k=1}^n & \left[\dfrac{w(z) - w_i}{z - z_i}\right]_{i=1}^n \\ \left[\dfrac{w^*(z) - w_k^*}{z^* - z_k^*}\right]_{k=1}^n & \left[\dfrac{w(z) - w^*(z)}{z - z^*}\right]_{i=1}^n \end{bmatrix} \geq 0.$$

This is called the FMI for this Nevanlinna-Pick problem. The left-upper part of this matrix is the information block (which is in this case the Pick matrix). This matrix is positive definite if the information matrix and its Schur complement is positive definite. The positivity of the information matrix is the well known Pick condition. The second condition on the Schur complement leads to a factorization problem which can be solved by a Nevanlinna-Pick algorithm. Indeed, when this algorithm is performed, a matrix W from the Potapov class is obtained and the Schur complement inequality can be reformulated as

$$[w^*(z) \;\; 1](W^*)^{-1}JW^{-1}\begin{bmatrix} w(z) \\ 1 \end{bmatrix} \geq 0,$$

where J is some appropriate signature matrix. The Nevanlinna-Pick algorithm (or other algorithms which can solve this problem) correspond to an explicit or implicit Cholesky or inverse Cholesky factorization of the information matrix. Since this matrix is highly structured, it can be factored in a very efficient way. As we described above, the Nevanlinna-Pick algorithm generates a matrix W which is in the Potapov class. All solutions of the FMI can be described by

$$w = (W_{11}g + W_{12})(W_{21}g + W_{22})^{-1}, \quad g \in \mathcal{P}(\mathbb{C}_+).$$

Similar relations can be written down for other interpolation problems or generalizations.

Further it should be remarked that V.E. Katsnelson [Kat81], [Kat85] studied several continuous analogs of classical interpolation problems of Hamburger-Nevanlinna type. These are mainly problems of integral representations for certain non-negative definite kernels. Such problems can be reformulated as equivalent interpolation problems for holomorphic functions with non-negative imaginary part in the upper half plane (the Nevanlinna-Pick class $\mathcal{P}(\mathbb{C}_+)$) which fulfil certain conditions in growth. In the beginning of the 80s R. Arocena, M. Cotlar and C. Sadosky initiated the investigation of non-negative definite kernels of mixed Toeplitz-Hankel type (which are also called *generalized Toeplitz kernels*).

Already in 1985 V.E. Katsnelson [Kat85] integrated his results in Potapov's model and he created a modified fundamental inequality with skillful usage of a certain transformation (about these transformations see [Kat97]). Furthermore, he also formulated the associated *fundamental identity* or FI. This extension of Potapov's method goes back to L.A. Sakhnovich (see [IS87b, IS87a, IS94]). He analysed the interpolation problems which were discussed by Potapov and his collaborators, and he recognized that for all these interpolation problems, certain matrices show up which satisfy a special matrix identity which has the form of a Lyapunov or Stein equality for matrices. These matrices are the matrices one can find in the information block of the FMI. In our Nevanlinna-Pick example, it was a Pick matrix, but it can have another structure as well (see below). Since one has to solve implicitly or explicitly a system of linear equations with this kind of matrix, it is important to discover the structure of the matrix, so that it can be exploited to compute solutions efficiently.

A simple example of such a FI can be easily given for the trigonometric moment problem. In this case, the information matrix is the infinite Hermitian Toeplitz matrix of the given moments which has to be positive definite for the problem to be solvable. It is easily seen that a Toeplitz matrix $T = [t_{i-j}]$ with $t_{-k} = t_k^*$, satisfies a so-called displacement identity

$$T - ZTZ^* = GJG^*$$

where Z is the down-shift operator with 1s on the subdiagonal and zeros elsewhere, $J = \mathrm{diag}(1, -1)$, and

$$G = t_0^{-1/2} \begin{bmatrix} t_0 & 0 \\ t_1 & t_1 \\ t_2 & t_2 \\ \vdots & \vdots \end{bmatrix}.$$

A matrix equation of the form $AXB - X = C$ is called a *Stein* equation.

Another well known example is the Hankel matrix of moments which appears in the Hamburger moment problem. Again, this (real) Hankel matrix $H = [h_{i+j-1}]$ should be positive definite. A Hankel matrix satisfies the displacement identity

$$ZH - HZ^T = GJG^T$$

where Z is the down-shift as above, while

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad G = h_0^{-1/2} \begin{bmatrix} h_0 & 0 \\ h_1 & h_1 \\ h_2 & h_2 \\ \vdots & \vdots \end{bmatrix}.$$

A matrix equation of the type $AX - XB = C$ is said to be of Lyapunov type. These displacement equations have led to the notions of quasi-Toeplitz, quasi-Hankel and other kinds of structured matrices which can be dealt with in fast algorithms. Such matrices were discovered from the idea of cascade factorization of a scattering medium, linear prediction of mildly non-stationary processes, or equivalently from the factorization of J-inner functions. The efficient algorithms are based on the construction of successive Schur complements. For some history see [Kai91]. These ideas were promoted by T. Kailath and his coworkers [KKM79b, KKM79a, Say92, KS95]. The basics are all in the book by G. Heinig and K. Rost [HR84]. For a recent survey with many applications see [KS95].

Similar observations were made for other, more complex interpolation problems. The individuality of every interpolation problem is expressed by the associated fundamental identity which has to be discovered. Once this identity is found, an abstract version of Potapov's factorization method admits the determination of the solution set of the fundamental matrix inequality. Katsnelson/Kheifets/Yuditskii [KKY87], [KY94] created an abstract scheme for the solution of interpolation problems which contains basic elements from the theory of unitary extensions of isometrical operators (which is another idea that can be traced back to the work of Arov/Grossman [AG83]).

V.E. Katsnelson's paper [Kat85] also discusses the treatment of generalized Nehari problems. Because this is another important classical problem that has not been treated in this paper, we give some details. Let us start with the simple scalar case.

Nehari Problem. Let $\{d_n\}_{-n\in\mathbb{N}}$ be a sequence of complex numbers. Determine the set $\mathcal{N}(\{d_n\}_{-n\in\mathbb{N}})$ of all Borel-measurable complex-valued bounded functions ϕ which admit $\|\phi\| \leq 1$ and which satisfy the condition

$$d_n = \frac{1}{2\pi} \int_{\mathbb{T}} z^{-n} \phi(z) \lambda(\mathrm{d}z) \tag{19}$$

for $n = -1, -2, \ldots$. In particular characterize the case that $\mathcal{N}(\{d_n\}_{-n\in\mathbb{N}})$ is non-empty.

This scalar classical problem was formulated by Z. Nehari [Neh57] in 1957. Note that it is not quite like a trigonometric moment problem. Only the moments for $-n \in \mathbb{N}$ are prescribed, and ϕ need not be positive, but it has another constraint namely $\|\phi\| \leq 1$. In his paper, Nehari gave a necessary and sufficient condition for the solvability of this problem. He characterized the solvability by the fact that the infinite Hankel matrix $(d_{-j-k+1})_{j,k=1}^{\infty}$ defines a contractive operator. In the papers of Adamjan/Arov/Kreĭn [AAK68a]–[AAK71b] which were published around 1970, there is not only a complete description of the solution set of an indeterminate scalar Nehari problem, but under a certain additional condition, there is even a generalization of this results to the matrix case. The general situation of the matrix case was also considered by V.M. Adamjan [Ada73] in 1973. D.Z. Arov and M.G. Kreĭn [AK81], [AK83] published the solution of the entropy optimization problem which is connected with the matricial Nehari problem. Inspired by the study of upgrades of classical theorems of M. Riesz and Helson/Szegő, the group of Arocena, Cotlar and Sadosky came to an important addition to the theory related to the Nehari problem. Their results gave rise to a lot of well-known results of complex analysis, operator theory and theory of stochastic processes. So let us see what a generalized Nehari problem looks like.

Generalized Nehari Problem. Let F_{11} and F_{22} be non-negative Hermitian $p \times p$ and $q \times q$ measures, respectively, and let $\{\beta_k\}_{k=0}^{\infty}$ be a sequence of complex $p \times q$ matrices. Determine the set $\mathcal{F}(F_{11}, F_{22}, \{\beta_k\}_{k=0}^{\infty})$ of all σ-additive $p \times q$ complex-valued mappings F_{12} which satisfy the conditions

$$\int_{\mathbb{T}} z^{-k} F_{12}(\mathrm{d}z) = \beta_k \;, \quad k \in \mathbb{N}_0 \tag{20}$$

and such that

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{12}^* & F_{22} \end{bmatrix}$$

is a non-negative Hermitian $(p+q) \times (p+q)$ Borel measure on the unit circle. In particular characterize the case that $\mathcal{F}(F_{11}, F_{22}, \{\beta_k\}_{k=0}^{\infty})$ is non-empty.

The generalized Nehari problem also leads directly to the investigation of definite kernels of so-called mixed Toeplitz-Hankel type. For the set $\mathcal{F}(F_{11}, F_{22}, \{\beta_k\}_{k=0}^{\infty})$ to be non-empty, it is necessary and sufficient that such a kernel is positive definite. Note that the generalized Nehari problem coincides with the usual Nehari problem in the case $p = q = 1$ and $F_{11} = F_{22} = \lambda/2\pi$. Katsnelson [Kat85] formulated the generalized Nehari problem using the matricial version of the F. Riesz-Herglotz theorem in the following equivalent way.

Nehari Carathéodory Problem. Let α and β and δ be respectively $p \times p$, $p \times q$, and $q \times q$ matrix-valued holomorphic functions. Determine the set $\mathcal{NC}(\alpha, \beta, \delta)$ of all $q \times p$ matrix-valued holomorphic functions γ such that

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

is a $(p+q) \times (p+q)$ matrix-valued Carathéodory function. In particular characterize the case that $\mathcal{NC}(\alpha, \beta, \delta)$ is non-empty.

B. Fritzsche and B. Kirstein [FK88], [FK90] investigated generalized Nehari problems for matricial Carathéodory and Schur functions. They also gave a prediction theoretical interpretation of these problems. Dubovoj/Fritzsche/Kirstein [DFK93] studied a generalized Nehari problem of Schur type which is connected with Darlington synthesis with Arov-singular J-inner functions. Using a result of Katsnelson [Kat93], A.Ya. Kheifets [Khe95] also gave an answer to an inverse question of the Nehari problem which was formulated by Sarason.

7 Arov's investigations on Darlington realizations generated by J-inner functions

In the end of the 60's Potapov's seminar dealt as well with questions about J-theory and related extension problems as basic conceptions of electrical circuits. At this time D.Z. Arov was one of the most active participants of this seminar which took place in Odessa. Stimulated by the discussions of the Potapov-seminar, D.Z. Arov expanded the fields of his research and turned his attention to dissipative linear systems and questions of the theory of J-inner matrix functions and related continuation problems. Electrical circuits were a useful model which allowed him a better understanding and perception of questions of operator theory, scattering theory and matricial function theory. In this context Arov's fundamental papers [Aro71], [Aro73] and [Aro75] about Darlington synthesis arose. Further he considered relations between Darlington synthesis and other subjects like scattering theory [Aro74b], [Aro74a], [Aro79c], the theory of rational approximation [Aro78] and various questions of the theory of passive linear systems [Aro79b, Aro79a, Aro84].

Arov's considerations were crucially influenced by the paper of Douglas/Shapiro/Shields [DSS70] on cyclic vectors of the backward shift which was published in 1970. They characterized the backward shift in terms of pseudocontinuity, a concept we defined in Section 1. It turned out to be an essential tool to solve the Darlington realization problem.

In the following, we will deal with the special $(p+q) \times (p+q)$ signature matrix $J = j_{pq}$ as given in (9). Let f be a $p \times q$ Schur function. Then we will say that f *admits a Darlington realization* if there exists a j_{pq}-inner function W with block partitioning (8) and a contractive $p \times q$ matrix ε such that

$$f = (W_{11}\varepsilon + W_{12})(W_{21}\varepsilon + W_{22})^{-1} \tag{21}$$

holds for all points of the domain of analyticity of W. One also says that W *generates a Darlington realization of* f. The Darlington realization is called *non-degenerate* (respectively, *canonical*) if ε is strictly contractive (respectively, if ε is the null matrix). If a $p \times q$ Schur function f admits a non-degenerate Darlington realization with some strictly contractive $p \times q$ matrix ε then the j_{pq}-inner function

$$V := W \cdot U_\varepsilon \tag{22}$$

generates a canonical Darlington realization of f where the matrix U_ε is given by

$$U_\varepsilon := \begin{bmatrix} \sqrt{(I_p - \varepsilon\varepsilon^*)^{-1}} & \varepsilon\sqrt{(I_q - \varepsilon^*\varepsilon)^{-1}} \\ \varepsilon^*\sqrt{(I_p - \varepsilon\varepsilon^*)^{-1}} & \sqrt{(I_q - \varepsilon^*\varepsilon)^{-1}} \end{bmatrix}. \tag{23}$$

The reader will certainly recognize it as the J-unitary matrix U_ε we used in the scalar Nevanlinna-Pick algorithm. This matrix U_ε is known as the *Halmos extension* of ε. In particular, if W is an Arov-singular j_{pq}-inner function then V is Arov-singular as well. Note that every Darlington realization of a $p \times q$ Schur function f is non-degenerate if $I_p - f^* f$ is positive Hermitian almost everywhere on the unit circle. This condition is particularly satisfied if f has finite entropy, i.e., if

$$\frac{1}{4\pi}\int_{\mathbb{T}} \log\Big(\det(I_p - \underline{f}\,\underline{f}^*)\Big)\, d\underline{\lambda} > -\infty \tag{24}$$

holds true. Arov [Aro73] proved the important fact that a matrix-valued Schur function f admits a Darlington realization if and only if f admits a pseudocontinuation (see also [DH73]). If a matrix-valued Schur function h admits a non-degenerate Darlington realization, then it also admits a canonical Darlington realization. In this case, one can see that h also has finite entropy.

In Arov's famous paper [Aro73], he gave a complete description of all canonical Darlington realizations of a given $p \times q$ Schur function which has finite entropy and admits a pseudocontinuation. Furthermore, he got additionally that every matrix-valued pseudocontinuable Schur function with finite entropy admits Darlington realizations generated by j_{pq}-inner functions of Smirnov type as well as by j_{pq}-inner functions of inverse Smirnov type. B. Fritzsche and B. Kirstein [FK93] characterized the situation when a given matrix-valued Schur function (with finite entropy) admits a Darlington realization generated by some Arov-singular j_{pq}-inner function. In this paper they also investigated some kind of minimal Darlington realizations.

The problem of describing all Darlington realizations is equivalent to a block *completion problem* for matrix-valued inner functions. In such a matrix completion problem one has to characterize the set of all matrix-valued inner functions where the right upper (respectively, left lower) block coincides with a given matricial Schur function. Using the Potapov-Ginzburg transform (see (10)) it is easily checked that a Schur function f admits a canonical Darlington realization with respect to some j_{pq}-inner function if and only if f can be embedded into the right upper corner of some matrix-valued inner function. The investigation of such completion problems for matrix-valued inner functions was initiated by Douglas/Helton [DH73] who characterized the solvability of the problem. A detailed analysis of the structure of the set of all solutions was considered in D.Z. Arov's paper [Aro74a] which can be considered as a continuation of his former work on Darlington synthesis (see [Aro71, Aro73]).

8 On J-inner functions and generalized bitangential Schur-Nevanlinna-Pick interpolation

D.Z. Arov's investigations on Darlington synthesis also mark the starting point of a systematic study of J-inner functions. It is surprising that the non-rational J-inner functions did not attract Potapov's attention. For instance, the important fact, that a left or right Blaschke-Potapov product is also a J-inner matrix function, was proved by Arov/Simakova [AS76] in 1976 (more then 20 years after Potapov's fundmental paper [Pot55]). Furthermore, D.Z. Arov continued his investigations about the matricial Nehari problem which he started already with V.M. Adamjan and M.G. Kreĭn [AAK71b]. He discovered the connection between the matricial Nehari problem and the so-called generalized bitangential Schur-Nevanlinna-Pick problem which is a very useful matricial generalization of these interpolation problems, already investigated by Sarason.

Generalized Bitangential Schur-Nevanlinna-Pick Problem. Let S_0 be a $p \times q$ Schur function. Further let b_1 and b_2 be $p \times p$ and $q \times q$ matrix-valued inner functions, respectively. Describe the set $\mathcal{SNP}(b_1, b_2; S_0)$ of all $p \times q$ Schur functions S such that

$$b_1^{-1}(S - S_0)b_2^{-1}$$

belongs to the Hardy space H^∞ of all bounded holomorphic functions in the unit disc.

Arov considered this problem with the aim to treat various matricial interpolation problems in a uniform framework. The specific interpolation problems result from this general formulation by a special choice of b_1, b_2 and S_0. Obviously, in the scalar case, the solution is a Schur function S which interpolates S_0 in the points $A = \{a_k\}_{k \in K}$ if we choose $(b_1 b_2)^{-1} = \prod_{a \in A} \zeta_a$ where ζ_a is a scalar Blaschke factor with zero a. The generalized bitangential Schur-Nevanlinna-Pick interpolation problem is always solvable because $S = S_0$ is obviously a solution. Arov showed that, under a certain condition, the solution set of this interpolation problem can be described by a linear fractional transform of a Schur function and the matrix representing the transform is generated by a matricial J-inner function. As we mentioned above, M.G. Kreĭn called this function a resolvent matrix of the interpolation problem.

Arov's observations led him to the corresponding inverse problem, which is to describe the J-inner functions which are resolvent matrices of a non-degenerate Schur-Nevanlinna-Pick problem. Several subclasses of J-inner functions turned out to be important and a new factorization theory for J-inner functions arose [Aro73, Aro74a, Aro75]. He proved that under some technical conditions, a J-inner function W admits unique factorizations

$$W = R_l S_l = R_r S_r$$

where R_l and R_r are left respectively right Arov-regular J-inner functions and S_l and S_r are Arov-singular J-inner functions. The definitions of left/right Arov-regular and Arov singular J-inner functions have been given above in Section 2. Recall from the discussion given there that the definition of a left (respectively right) Arov-regular J-inner function is taken with reference to Arov-singular J-inner functions with the restriction, that there must not appear a non-constant Arov-singular J-inner left (respectively right) divisor.

Arov [Aro89], [Aro90] showed that the resolvent matrices of a generalized bitangential Schur-Nevanlinna-Pick interpolation problem (given above) can be characterized by left (respectively, right) Arov-regular j_{pq}-inner functions. In the following, this will be explained in detail.

The generalized bitangential Schur-Nevanlinna-Pick problem which is given above will be designated by $P[b_1, b_2; S_0]$. Let Z be the set of the points of the unit disc, for which the product $\det b_1 \det b_2$ does not vanish. For all $z \in Z$, the set

$$\mathcal{K} := \{S(z) : S \in \mathcal{SNP}(b_1, b_2; S_0)\} \tag{25}$$

admits a representation as a matrix ball, i.e., there are matrices $M(z)$, $L(z)$ and $R(z)$ such that

$$\mathcal{K}(z) = \{M(z) + L(z)\varepsilon R(z)\} \tag{26}$$

where ε is an arbitrary contractive matrix. If there does not exist a point z_0 in Z such that $\det L(z_0)$ vanishes, then the product $\det L(z) \det R(z)$ is non-zero for any $z \in Z$. Otherwise, if there does exist such a $z_0 \in Z$, then the problem $P[b_1, b_2; S_0]$ is called *completely indeterminate*. Smuljan [Smu68] showed that this result does not depend on the special representation of $\mathcal{K}$ as a matrix ball.

Let W be a j_{pq}-inner function which is partitioned as in (8). Then a well-known theorem due to Ginzburg [Gin58] implies that both functions

$$S_{11} := W_{11} - W_{12}W_{22}^{-1}W_{21} \tag{27}$$

and

$$S_{22} := W_{22}^{-1} \tag{28}$$

are matrix-valued Schur functions where the functions $\det S_{11}$ and $\det S_{22}$ do not vanish identically. The functions S_{11} and S_{22} also admit inner-outer factorizations. If b_1 and b_2 are matrix-valued inner functions such that the functions $b_1^{-1}S_{11}$ and $S_{22}b_2^{-1}$ (respectively, $S_{11}b_1^{-1}$ and $b_2^{-1}S_{22}$) are outer, then $[b_1, b_2]$ is called a *left* (respectively, *right*) *pair of inner functions associated with* W. Arov [Aro73, Aro75] proved the following: If $[b_1, b_2]$ is a left pair of inner functions associated with a j_{pq}-inner function W, then $[b_1, b_2]$ is also a left pair of inner functions associated with WW_A, where W_A is an Arov-singular j_{qq}-inner function. There is an analogous result for the right case too. Furthermore, Arov used this pairs of inner functions to characterize the case that a given j_{pq}-inner function is a left or right Blaschke-Potapov product with respect to j_{pq}.

Now we turn our attention back to linear fractional transformations which are generated by a matricial j_{pq}-inner function W (see also Section 7). If W is partitioned as in (8) then, for each $p \times q$ matrix-valued Schur function g, the function $\det(W_{21}g + W_{22})$ does not vanish identically. Furthermore,

$$f := (W_{11}g + W_{12})(W_{21}g + W_{22})^{-1} \tag{29}$$

is a matrix valued Schur function. Let $\mathcal{S}_{p\times q} = \mathcal{B}^{p\times q}(\mathbb{D})$ be the set of all $p \times q$ Schur functions in the unit disc. If we set

$$\mathcal{T}_W(\mathcal{S}_{p\times q}) := \{(W_{11}g + W_{12})(W_{21}g + W_{22})^{-1} : g \in \mathcal{S}_{p\times q}\} \tag{30}$$

then Arov's theorem on the interrelations between generalized bitangential Schur-Nevanlinna-Pick interpolation and j_{pq}-inner function is as follows:

Theorem. Let S_0 be a $p \times q$ Schur function and let b_1 and b_2 be $p \times p$ and $q \times q$ matrix-valued inner functions respectively. Suppose that the problem $P[b_1, b_2; S_0]$ is completely indeterminate. Then there is a left Arov-regular j_{pq}-inner function W_l such that

$$\mathcal{T}_{W_l}(\mathcal{S}_{p\times q}) = \mathcal{SNP}(b_1, b_2; S_0)\ .$$

If V_l is an arbitrary j_{pq}-inner function such that

$$\mathcal{T}_{V_l}(\mathcal{S}_{p\times q}) = \mathcal{SNP}(b_1, b_2; S_0)$$

and such that $[b_1, b_2]$ is a left pair of inner functions associated with V_l, then there is a j_{pq}-unitary matrix U such that $V_l = W_l U$. In particular, V_l is then left Arov-regular.

Note that there is also a description of the set of solutions of a generalized bitangential Schur-Nevanlinna-Pick problem in terms of a linear fractional transformation which is generated by a right Arov-regular j_{pq}-inner function. The next result was also proved by Arov. It is a converse statement to the previous theorem.

Theorem. Let W be a j_{pq}-inner function with block partition (8). Further let $S_0 := W_{12}W_{22}^{-1}$ and let $[b_1, b_2]$ be a left pair of inner functions associated with W. Then the problem $P[b_1, b_2; S_0]$ is completely indeterminate and

$$\mathcal{T}_W(\mathcal{S}_{p\times q}) \subseteq \mathcal{SNP}(b_1, b_2; S_0)$$

where equality holds true if and only if W is left Arov-regular.

This result shows that a given j_{pq}-inner function W generates naturally a generalized bitangential Schur-Nevanlinna-Pick problem. If the j_{pq}-inner function W is A-regular, then the solution set can be described as the image of the Schur class under the linear fractional transformation generated by W. The characterization of bitangential Schur-Nevanlinna-Pick problems which are generated by an arbitrary J-inner function is an unsolved problem.

From the beginning of the nineties, D.Z. Arov, B. Fritzsche and B. Kirstein also investigated further inverse problems. The considerations of questions about the inner block structure of resolvent matrices led the team of Arov, Fritzsche and Kirstein to various completion problems for j_{pq}-inner functions. In [AFK93a] they described the set of all j_{pq}-inner functions W such that the right upper (or the left lower) block of W coincides with a given function from the meromorphic Nevanlinna-Pick class. The special subclasses of j_{pq}-inner functions of Smirnov type and inverse Smirnov type and the subclass of Arov-singular j_{pq}-inner functions were also discussed. The question of the description of the set of all j_{pq}-inner functions with a given block row or a given block column is treated in [AFK93b]

9 Conclusion

We have sketched briefly the wide range of problems and disciplines that are currently covered by the term Schur analysis. Since this is only a survey paper, we have only mentioned some of the topics and deliberately ommitted many others. Schur analysis is currently a very active area of research where mathematicians form operator theory, complex function theory, matrix analysis, harmonic analysis, orthogonal polynomials meet engineers and applied mathematicians from electrical engineering, system theory, scattering theory, control theory, signal processing, prediction theory, time series analysis and maybe many more applications to come. We concentrated on the current evolution and tried to highlight the many roots that are found in the work of researchers from the former east block. A second part of this survey narrowed the field to the evolution of the study of J-inner functions, which we consider to be at the heart of this kind of research. We sketched the old and new results that were obtained since the initiation of the field by Potapov and his coworkers.

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