

Rational L.S. Category of Function Space Components for F_0 -Spaces

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Abstract

We study the rational L.S. category of path components of function space components for F_0 -spaces to determine which are finite and which infinite. Our results include both new cases and extensions of a famous conjecture of S. Halperin concerning the rational homotopy of F_0 -spaces.

1 Introduction

Let X be a finite, simply connected complex with finite-dimensional rational homotopy and evenly graded rational cohomology. We refer to the collection of such spaces X as the class of F_0 -spaces. Halperin studied this class of spaces extensively in [4] proving, among other things, that F_0 -spaces are all rational Poincaré duality spaces with rational cohomology a complete intersection algebra. The following long-standing conjecture concerning F_0 -spaces is due to Halperin, as well.

Conjecture (Halperin [4]) *The rational Serre spectral sequence collapses for every \mathbb{Q} -orientable fibration of the form $X \hookrightarrow E \rightarrow B$ with X an F_0 -space.*

The Halperin conjecture has been confirmed for homogeneous spaces $X = G/H$ with $\text{rank}H = \text{rank}G$ by Shiga and Tezuka [9], when the rational cohomology of X is generated by three or fewer elements by Lupton [6] and when the polynomial relations in the rational cohomology of X are monomials by Markl [7].

Received by the editors August 1997.

Communicated by Y. Félix.

1991 *Mathematics Subject Classification* : Primary 55P62, 55P15, 58D99.

Key words and phrases : Rational homotopy theory, Haefliger Model, Halperin conjecture.

Thomas [12] and Meier [8] have observed that the Halperin conjecture can be viewed as a problem concerning the rational homotopy of function spaces. Let $M(X, Y)$ denote the space of all continuous functions from X to Y with the compact-open topology. Given a map $f : X \rightarrow Y$ let $M_f(X, Y)$ denote the component of $M(X, Y)$ containing f .

Theorem 1.1. ([8, 12]) *The Halperin conjecture is true for an F_0 -space X if and only if the rational homotopy groups of the identity component $M_1(X, X)$ are oddly graded.* ■

Theorem 1.1 provides a link between the Halperin conjecture and the theory of rational L.S. category. Given a space S , let $\text{cat}_0(S)$ denote the rational L.S. category of S , as defined and studied by Félix and Halperin in [2]. Note that $M_1(X, X)$ is a topological monoid with multiplication given by composition of functions. The only rational H -space with finite rational category is a product of odd spheres. Thus

Theorem 1.2. *The Halperin conjecture is true for an F_0 -space X if and only if $\text{cat}_0(M_1(X, X)) < +\infty$.* ■

By Theorem 1.2, the Halperin conjecture can be interpreted as a special case of a general problem in the rational homotopy theory of function spaces: namely, given a map $f : X \rightarrow Y$ between F_0 -spaces X and Y , determine whether the rational category of $M_f(X, Y)$ is finite or infinite. The purpose of this paper is to study this problem with an eye toward confirming new special cases of the Halperin conjecture. More generally, our goal is to discover the extent to which the Halperin conjecture “extends” to other function space components. In [1], Y. Félix made the observation that “the category of [function space components] is often infinite”. Our results indicate that this observation applies, as well, to function space components for F_0 -spaces. However, as we shall see, there are important examples where the Halperin conjecture is generic and the identity component is just one of a class of components all of finite rational category.

2 Haefliger’s Model for F_0 -spaces

We describe Haefliger’s model [3] for components of the free and based function spaces $M(X, Y)$ and $M(X, Y)_*$ when X and Y are F_0 -spaces. In this section and throughout, all homology, cohomology and homotopy groups are assumed to be rational.

In [4], Halperin showed that the rational cohomology of an F_0 -space X is of the form $H^*(X) = \Lambda(x_1, \dots, x_n)/(P_1, \dots, P_n)$ where the x_i are of even degree and the $P_i = P_i(x_1, \dots, x_n)$ form a regular sequence of polynomials in the free algebra $\Lambda(x_1, \dots, x_n)$. His result implies the Sullivan minimal model (\mathcal{M}_X, d_X) for an F_0 -space X is a *two-stage* model. That is, $\mathcal{M}_X = \Lambda(V_0) \otimes_{d_X} \Lambda(V_1)$ where $d_X(V_0) = 0$ and $d_X(V_1) \subset \Lambda(V_0)$. Here $V_0 = \mathbb{Q}(x_1, \dots, x_n)$ is the evenly graded space of generators, $V_1 = \mathbb{Q}(y_1, \dots, y_n)$ is the oddly graded space of relations and $d_X(y_i) = P_i$.

The simple rational structure of F_0 -spaces X and Y implies a Sullivan model for components of $M(X, Y)$ can be constructed directly from a generalized Postnikov decomposition of Y (see [10]). Let $f : X \rightarrow Y$ be a given map and write the minimal model of Y as $\mathcal{M}_Y = \Lambda(V_0) \otimes_{d_Y} \Lambda(V_1)$, where V_0 is evenly graded and V_1 is oddly graded. Then by [10, Theorem 3.1] there exists a two-stage (non-minimal) model (\mathcal{A}_f, d_f) for the function space $M_f(X, Y)$ of the form $\mathcal{A}_f = \Lambda(Z_0) \otimes_{d_f} \Lambda(Z_1)$, where

$$Z_0^n = \bigoplus_{i=0}^{\infty} H_{2i}(X) \otimes V_0^{2i+n} \quad \text{and} \quad Z_1^n = \bigoplus_{i=0}^{\infty} H_{2i}(X) \otimes V_1^{2i+n}.$$

The same argument gives a model (\mathcal{A}_f^*, d_f^*) for the based function space component $M_f(X, Y)_*$ of the form $\mathcal{A}_f^* = \Lambda(\overline{Z}_0) \otimes_{d_f^*} \Lambda(\overline{Z}_1)$, where here

$$\overline{Z}_0^n = \bigoplus_{i>0}^{\infty} H_{2i}(X) \otimes V_0^{2i+n} \quad \text{and} \quad \overline{Z}_1^n = \bigoplus_{i>0}^{\infty} H_{2i}(X) \otimes V_1^{2i+n}.$$

To describe the differentials d_f and d_f^* , fix an additive basis $\{a_k | k \in I\}$ for $H^*(X)$. Let $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ be bases for V_0 and V_1 , respectively, and suppose $d_Y(y_i) = P_i(x_1, \dots, x_n)$. The space Z_1 is spanned by elements of the form $b \otimes y_i$ where $b \in H_*(X)$ and $|b| < |y_i|$. If $|b| > 0$ as well, then $b \otimes y_i \in \overline{Z}_1$. We view $H_*(X)$ as the dual space to $H^*(X)$ and adopt Haefliger’s notation, letting $b \cap a \otimes P = b(a) \cdot P$ for $P \in \Lambda(Z_0)$. By the proof of [10, Theorem 3.2]

$$d_f(b \otimes y_i) = b \cap P_i \left(\sum_{|a_k| < |x_1|} a_k \otimes a_k^* \otimes x_1 + f^*(x_1) \otimes 1, \right. \tag{1}$$

$$\left. \dots, \sum_{|a_k| < |x_n|} a_k \otimes a_k^* \otimes x_n + f^*(x_n) \otimes 1 \right),$$

where $a_k^* \in H_*(X)$ is dual to a_k and $f^*(x_i)$ is viewed as an element of $H^*(X)$. Note that the elements $a_k \otimes a_k^* \otimes x_i$ lie in the space $H^*(X) \otimes \Lambda(Z_0)$. Thus multiplication is given by $(a_k \otimes a_k^* \otimes x_i) \cdot (a_l \otimes a_l^* \otimes x_j) = a_k a_l \otimes (a_k^* \otimes x_i)(a_l^* \otimes x_j)$. If $0 < |b| < |y_i|$ then

$$d_f^*(b \otimes y_i) = b \cap P_i \left(\sum_{0 < |a_k| < |x_1|} a_k \otimes a_k^* \otimes x_1 + f^*(x_1) \otimes 1, \right. \tag{2}$$

$$\left. \dots, \sum_{0 < |a_k| < |x_n|} a_k \otimes a_k^* \otimes x_n + f^*(x_n) \otimes 1 \right).$$

The following “normalization” lemma is useful for dealing with non-minimal two-stage models.

Lemma 2.1. *Let $\mathcal{A} = \Lambda(V_0) \otimes_d \Lambda(V_1)$ be a two-stage model for a nilpotent complex X . Suppose there exists a finite-dimensional subspace W_0 of V_0 such that $\Lambda(W_0)$ is contained in the image of d . Then there exists a model (\mathcal{A}', d') for X of the form $\mathcal{A}' = \Lambda(V'_0) \otimes_{d'} \Lambda(V'_1)$ where V'_0 is a complementary subspace to W_0 and V'_1 is a subspace of V_1 .*

Proof. The elements of least degree in W_0 are the images of elements of V_1 under d . We replace (\mathcal{A}, d) by a quasi-isomorphic model in which these least degree elements and an isomorphic subspace of their preimages in V_1 have been removed. The corresponding subspace W_0 in the new model has higher connectivity. The result thus follows by induction. ■

Given a model (\mathcal{A}, d) for a simply connected space X with $\mathcal{A} = \Lambda(V)$ let \mathcal{A}^+ denote the elements of positive degree and observe that $V \cong \mathcal{A}^+/\mathcal{A}^+ \cdot \mathcal{A}^+$. The projection $\mathcal{A}^+ \rightarrow \mathcal{A}^+/\mathcal{A}^+ \cdot \mathcal{A}^+$ defines a differential $Q(d)$ on V and we have $H^*(V, Q(d)) \cong \pi_*(X)$ [11]. Note that this isomorphism exists as well when X is a nilpotent complex and (\mathcal{A}, d) is a two-stage model, as can be seen by first normalizing the model to remove the linear relations in degree one. Solving for the basis elements $a_k^* \otimes x_n$ of Z_0 and \overline{Z}_0 in equations (1) and (2) above we obtain formulas for the linear differentials in the models (\mathcal{A}_f, d_f) and (\mathcal{A}_f^*, d_f^*) .

Lemma 2.2.

$$Q(d_f)(b \otimes y_j) = \sum_{i=1}^n \sum_{|a_k| < |x_i|} b \left(a_k \cdot f^* \left(\frac{\partial P_j}{\partial x_i} \right) \right) a_k^* \otimes x_i$$

$$Q(d_f^*)(b \otimes y_j) = \sum_{i=1}^n \sum_{0 < |a_k| < |x_i|} b \left(a_k \cdot f^* \left(\frac{\partial P_j}{\partial x_i} \right) \right) a_k^* \otimes x_i$$

■

Remark. By Theorem 1.1, the Halperin conjecture is true for an F_0 -space X if and only if $Q(d_1) : Z_1 \rightarrow Z_0$ is surjective. Thus the first formula of Lemma 2.2 provides another interpretation of the Halperin conjecture in terms of the partial derivatives of polynomial relations (compare [9]).

3 First Consequences of the Models

We use Lemma 2.2 and results of [2] to prove

Theorem 3.1. *Let X and Y be F_0 -spaces. Then, for any map $f : X \rightarrow Y$, if $\text{cat}_0(M_f(X, Y)) < \infty$ then $\text{cat}_0(M_f(X, Y)_*) < \infty$.*

Proof. Since Y is simply connected, evaluation at the basepoint of X defines a rational fibration $\xi : M_f(X, Y)_* \hookrightarrow M_f(X, Y) \rightarrow Y$. By [2, Theorem 6.4] $\text{cat}_0(M_f(X, Y)_*) \leq \text{cat}_0(M_f(X, Y)) + k_\xi$ where k_ξ is the invariant of the fibration ξ defined on [2, p. 20]. We must show k_ξ is finite. Let $j : M_f(X, Y)_* \rightarrow M_f(X, Y)$ denote the inclusion and consider the induced map $j^\# : H^*(Z, Q(d_f)) \rightarrow H^*(\overline{Z}, Q(d_f^*))$ where $Z = Z_0 \oplus Z_1$ and $\overline{Z} = \overline{Z}_0 \oplus \overline{Z}_1$. Clearly, j induces a surjection from Z_0 to \overline{Z}_0 . By Lemma 2.2, if $Q(d_f)(b \otimes y_j) \in \overline{Z}_0$ then $Q(d_f)(b \otimes y_j) = Q(d_f^*)(b \otimes y_j)$ and so exact elements in \overline{Z}_0 under $Q(d_f)$ are also exact under $Q(d_f^*)$. Thus $j^\#$ is surjective in even degrees. But this means that, by definition, k_ξ is the dimension of a subspace of $\pi_*(Y)$ – a finite-dimensional space. ■

We next define two numeric invariants of the rational homotopy of F_0 -spaces X . Let $l(X)$ and $u(X)$ denote, respectively, the bottom and top degrees of the space V_0 of even degree generators of the minimal model of X . In other words,

$$l(X) = \min\{2n \mid \pi_{2n}(X) \neq 0\} \quad \text{and} \quad u(X) = \max\{2n \mid \pi_{2n}(X) \neq 0\}.$$

Note that $l(X)$ is one more than the rational connectivity of X . The following results illustrate the significance of the difference $u(Y) - l(X)$ for the rational homotopy of components of $M(X, Y)$:

Theorem 3.2. *Let X and Y be F_0 spaces with $u(Y) \leq l(X)$. Then, for any map $f : X \rightarrow Y$, $M_f(X, Y)_*$ is rationally a product of odd spheres. When f is trivial we have the decomposition $M_0(X, Y) \simeq_{\mathbb{Q}} M_0(X, Y)_* \times Y$.*

Proof. The first statement follows from the fact that, in the model (\mathcal{A}_f^*, d_f^*) for $M_f(X, Y)_*$, the space \bar{Z}_0 is trivial. Thus the minimal model for $M_f(X, Y)_*$ is just $(\Lambda(\bar{Z}_1), 0)$. For the second statement note that, in the model (\mathcal{A}_0, d_0) for $M_0(X, Y)$, the elements $1^* \otimes x_i$ span Z_0 and the only nontrivial differentials are the polynomials $d_0(1^* \otimes y_i) = P_i(1^* \otimes x_1, \dots, 1^* \otimes x_n)$. ■

Theorem 3.3. *Let X and Y be F_0 -spaces and suppose the polynomial relations in the cohomology of Y are actually monomials. Then the space $M_0(X, Y)$ is formal if and only if $u(Y) \leq l(X)$.*

Proof. If $u(Y) \leq l(X)$, the result follows from Theorem 3.2. Suppose that $u(Y) > l(X)$. Choose nonzero elements $a \in H^{l(X)}(X)$ and $x_i \in V_0$ with $|x_i| = u(Y)$. Then, in the model (\mathcal{A}_0, d_0) , $1^* \otimes x_i$ and $a^* \otimes x_i$ are closed elements of Z_0 which are clearly not exact. By hypothesis, we can find j so that $P_j(x_1, \dots, x_n) = x_i^k$ for some k . Taking 1 and a to be elements of our basis for $H^*(X)$, observe that for any $b \in H_*(X)$

$$d_0(b \otimes y_j) = b \cap (1 \otimes 1^* \otimes x_i + \dots + a \otimes a^* \otimes x_i)^k.$$

In particular, $d_0(1^* \otimes y_j) = (1^* \otimes x_i)^k$ and $d_0(a^* \otimes y_j) = k(1^* \otimes x_i)^{k-1}(a^* \otimes x_i)$. Thus the element

$$(1^* \otimes x_i)(a^* \otimes y_j) - k(a^* \otimes x_i)(1^* \otimes y_j)$$

represents a nontrivial Massey product in the rational cohomology of $M_0(X, Y)$. ■

Remark. Theorem 3.3 implies rational indecomposability results for null-components as studied integrally by Hansen in [5]. For example, it is easy to see that if $X = S^{2k} \times S^{2n}$ then $M_0(X, X)_*$ is a formal space. However, if $k \neq n$, $M_0(X, X)$ is not formal by Theorem 3.3. Thus, for $k \neq n$, $M_0(X, X) \not\simeq_{\mathbb{Q}} X \times M_0(X, X)_*$.

4 Components of Finite Category

We begin this section by establishing the following partial converse to Theorem 3.1.

Theorem 4.1. *Let X and Y be F_0 -spaces with $u(Y) - l(Y) < l(X)$. Suppose the rational homotopy groups of $M_f(X, Y)_*$ are oddly graded. Then $cat_0(M_f(X, Y)) < +\infty$.*

Proof. Our first hypothesis assures us that all elements of \overline{Z}_0 have strictly smaller degree than the elements $1^* \otimes x_j$ of Z_0 . For if $a_k \in H^{>0}(X)$ then

$$|a_k^* \otimes x_i| \leq u(Y) - l(X) < l(Y) \leq |1^* \otimes x_j|.$$

This means that if an element $a_k^* \otimes x_i$ appears as a linear summand in a differential $d_f(b \otimes y)$ then no term $1^* \otimes x_j$ can appear in this same expression. In other words, if $Q(d_f^*)(b \otimes y) \neq 0$ then $d_f(b \otimes y) \in \Lambda(\overline{Z}_0)$.

Our second hypothesis implies that $Q(d_f^*) : \overline{Z}_1 \rightarrow \overline{Z}_0$ is surjective. Combined with the above, we see that the subalgebra $\Lambda(\overline{Z}_0)$ of \mathcal{A}_f is in the image of d_f . By Lemma 2.1 we may replace (\mathcal{A}_f, d_f) by a model (\mathcal{A}'_f, d'_f) of the form $\mathcal{A}'_f = \Lambda(V_0) \otimes_{d'_f} \Lambda(Z'_1)$ where Z'_1 is a subspace of Z_1 and we view V_0 as spanned by the elements $1^* \otimes x_i$. By equation (1), $d_f(1^* \otimes y_j) = P_j(1^* \otimes x_1, \dots, 1^* \otimes x_n)$. Thus \mathcal{A}'_f can be rewritten as $(\mathcal{A}'_f, d'_f) = (\mathcal{M}_Y, d_Y) \otimes_{d'_f} \Lambda(Z''_1)$ where (\mathcal{M}_Y, d_Y) is the minimal model of Y .

Let $(\mathcal{M}_Y^{>m}, d_Y^{>m})$ denote the sub-DGA of (\mathcal{M}_Y, d_Y) generated by elements of product length $> m$. Since Y has finite rational category, by [2, Theorem 4.7] (\mathcal{M}_Y, d_Y) is a retract of $(\mathcal{M}_Y/\mathcal{M}_Y^{>m}, d'_Y)$ for some m . Thus (\mathcal{A}'_f, d'_f) is a retract of the DGA $(\mathcal{M}_Y/\mathcal{M}_Y^{>m}, d'_Y) \otimes_{d'_f} \Lambda(Z''_1)$. Since Z''_1 is oddly graded this latter DGA has finite product length also. Thus (\mathcal{A}'_f, d'_f) is a retract of a DGA with finite product length. A second application of [2, Theorem 4.7] implies $\text{cat}_0(M_f(X, Y)) < +\infty$. ■

Theorems 3.2 and 4.1 imply

Corollary 4.2. *Let X and Y be F_0 -spaces with $u(Y) \leq l(X)$. Then, for any map $f : X \rightarrow Y$, $\text{cat}_0(M_f(X, Y)) < +\infty$.* ■

Specializing to the case $X = Y$ and $f = 1$ we recover a well-known special case of the Halperin conjecture.

Corollary 4.3. *Let X be an F_0 space whose even rational homotopy is concentrated in a single degree. Then the Halperin conjecture is true for X .* ■

When $u(Y) > l(X)$ there is no general finiteness result for the rational category of components of $M_f(X, Y)$, as demonstrated by Theorem 5.1 below. The reason is that the model (\mathcal{A}_f, d_f) is now complicated by the even degree generators $a_k^* \otimes x_i$ with $l(X) \leq |a_k^*| < |x_i|$. When the polynomial relations in $H^*(Y)$ involving generators of degree $> l(X)$ are relatively simple, we can, however, deduce special cases where $\text{cat}_0(M_f(X, Y)) < +\infty$.

Let us fix notation for the remainder of this section as follows: Let V_0 with basis $\{x_1, \dots, x_s, z_1, \dots, z_t\}$ be a space of even degree generators of $H^*(Y)$ where $|x_i| \leq l(X)$ and $|z_i| > l(X)$. Let $P_1, \dots, P_{s+t} \in \Lambda(V_0)$ be the polynomial relations for $H^*(Y)$. Using Theorem 4.1, we prove

Theorem 4.4. *Let X and Y be F_0 -spaces with $u(Y) - l(Y) < l(X)$. Suppose that for each z_j there is a polynomial P_i such that P_i actually lies in $\Lambda(x_1, \dots, x_s, z_j)$ and, for each nonzero $a \in H^{<|z_j|}(X)$, $a \cdot f^*(\frac{\partial P_i}{\partial z_j}) \neq 0$. Then $\text{cat}_0(M_f(X, Y)) < +\infty$.*

Proof. Since $P_i \in \Lambda(x_1, \dots, x_s, z_j)$ by Lemma 2.2 we see that, in the model (\mathcal{A}_f^*, d_f^*) for $M_f(X, Y)_*$,

$$Q(d_f^*)(b \otimes y_i) = \sum_{0 < |a_k| < |z_j|} b \left(a_k \cdot f^* \left(\frac{\partial P_i}{\partial z_j} \right) \right) a_k^* \otimes z_j.$$

Our hypothesis thus implies $Q(d_f^*) : \overline{Z}_1 \rightarrow \overline{Z}_0$ is surjective. The result follows from Theorem 4.1. ■

Corollary 4.5. *Let X be an F_0 space with $u(X) < 2l(X)$. Suppose the polynomial relations for $H^*(X)$ are of the form $P_i = z_j^k P'_i$ where $k \geq 0$ $|z_j| > l(X)$ and P'_i involves only variables of degree $l(X)$. Then the Halperin conjecture is true for X .*

Proof. Given z_j we may clearly find P_i as above with $k > 0$. Suppose $a \in H^{<|z_j|}(X)$ is nonzero. Since $|a| < |z_j|$ we may view the product $a \cdot \frac{\partial P_i}{\partial z_j}$ as an element of the space $\Lambda(V_0)/(P_1, \dots, \widehat{P_i}, \dots, P_{s+t})$. Now if $a \cdot \frac{\partial P_i}{\partial z_j} = 0$ then $a \cdot P_i = \frac{1}{k} a \cdot z_j \cdot \frac{\partial P_i}{\partial z_j} = 0$. That is, P_i is a zero-divisor in the space $\Lambda(V_0)/(P_1, \dots, \widehat{P_i}, \dots, P_{s+t})$. But this contradicts the fact that P_1, \dots, P_{s+t} is a regular sequence in $\Lambda(V_0)$. ■

To remove the hypothesis $u(Y) - l(Y) < l(X)$ from Theorem 4.4 we must further restrict the polynomials involving elements of degree $> l(X)$.

Theorem 4.6. *Let X and Y be F_0 -spaces. Suppose the monomials $z_1^{n_1+1}, \dots, z_t^{n_t+1}$ are among the polynomial relations in $H^*(Y)$ and that $a \cdot f^*(z_i^{n_i}) \neq 0$ for all nonzero $a \in H^{<|z_i|}(X)$. Then $cat_0(M_f(X, Y)) < +\infty$.*

Proof. We show that the subalgebra $\Lambda(\overline{Z}_0)$ in the model (\mathcal{A}_f, d_f) is in the image of d_f . The result then follows by the proof of Theorem 4.1.

Let $P_i = z_i^{n_i+1}$. By equation (1),

$$d_f(b \otimes y_i) = b \cap \left(\sum_{|a_k| < |z_j|} a_k \otimes a_k^* \otimes x_i + f^*(z_j) \otimes 1 \right)^{n_i+1}.$$

Let $m = \max\{k | k < |z_i| \text{ and } H^k(X) \neq 0\}$. Then if $b \in H_{n_i|z_i|-m}(X)$ by degree considerations

$$d_f(b \otimes y_i) = \sum_{|a_k|=m} (n_i + 1) b(a_k \cdot f^*(z_i^{n_i})) a_k^* \otimes z_i.$$

Thus, by our hypothesis, d_f maps onto the elements of minimal degree in \overline{Z}_0 . As in the proof of Lemma 2.1, an induction argument now implies $\Lambda(\overline{Z}_0)$ is in the image of d_f . ■

The special case $X = Y$ and $f = 1$ here yields a generalization of the result of Markl [7].

Corollary 4.7. *Let X be an F_0 -space. Suppose the monomials $z_i^{n_i+1}$ for $|z_i| > l(X)$ appear among the polynomial relations for $H^*(X)$. Then the Halperin conjecture is true for X .* ■

5 Components of Infinite Category

We now consider the opposite question to that of §4: namely, when do components of $M(X, Y)$ have infinite category? By Theorem 3.1, it suffices to consider the based function space $M(X, Y)_*$. Our main result is

Theorem 5.1. *Let $f : X \rightarrow Y$ be a map between F_0 -spaces. Suppose there exists a nonzero element $a \in H^*(X)$ such that $|a| < u(Y)$, $a^2 = 0$ and $a \cdot f^*(H^{>0}(Y)) = 0$. Then $\text{cat}_0(M_f(X, Y)_*) = +\infty$.*

Proof. Choose an element $x_i \in V_0$, the space of even generators of the minimal model of Y , with $|x_i| = u(Y)$. Then $a^* \otimes x_i \in \overline{Z}_0$. Since $a \cdot f^*(H^{>0}(Y)) = 0$, $a^* \otimes x_i$ does not appear in any summand in the expression for $Q(d_f^*)$ given by Lemma 2.2. Thus $a^* \otimes x_i$ represents an even degree generator of the cohomology of $M_f(X, Y)_*$. In fact, since $a^2 = 0$, by equation (2), $a^* \otimes x_i$ does not appear in any summand of d_f^* . It follows that $a^* \otimes x_i$ represents a cohomology class with infinite cup-length in the cohomology of $M_f(X, Y)_*$. Thus $\text{cat}_0(M_f(X, Y)_*) = +\infty$ by [2, Corollary 4.10]. ■

Félix’s results in [1] applied to F_0 -spaces X and Y imply that the rational category of the spaces of null maps $M_0(X, Y)$ is infinite when $\dim(X)$ is less than the rational connectivity of Y . (Here $\dim(X)$ denotes the highest nontrivial degree in $H^*(X)$.) The following consequence of Theorem 5.1 extends this result to other components.

Corollary 5.2. *Let X and Y be F_0 -spaces with $\dim(X) < u(Y)$. Then, for any map $f : X \rightarrow Y$, $\text{cat}_0(M_f(X, Y)_*) = +\infty$.* ■

Specializing to null-components we can improve Corollary 5.2 to

Theorem 5.3. *Let X and Y be F_0 -spaces with $\frac{1}{2}\dim(X) + l(X) < u(Y)$. Then $\text{cat}_0(M_0(X, Y)_*) = +\infty$.*

Proof. By Theorem 5.1, it suffices to produce a nonzero element $a \in H^*(X)$ with $|a| \leq \frac{1}{2}\dim(X) + l(X)$ and $a^2 = 0$. Let $x \in H^{l(X)}(X)$ be nonzero and let $n > 0$ be the least integer with $x^n = 0$. If $(n - 1)l(X) \leq \frac{1}{2}\dim(X) + l(X)$ we let $a = x^{n-1}$. Otherwise, we may choose m so that

$$\frac{1}{2}\dim(X) < |x^m| \leq \frac{1}{2}\dim(X) + l(X).$$

By degree considerations, $x^{2m} = 0$ and so we let $a = x^m$. ■

Remark. Observe that $\dim(X) = \frac{1}{2}\dim(X) + l(X)$ if and only if $H^*(X)$ is a truncated polynomial algebra generated by an element of height two. It is natural then to consider the spaces $M(\mathbb{C}P^2, Y)$ where $u(Y) = 4$. A direct calculation shows that $\text{cat}_0(M_0(\mathbb{C}P^2, S^4)) < +\infty$ while $\text{cat}_0(M_0(\mathbb{C}P^2, \mathbb{H}P^4)_*) = +\infty$. Thus while the hypotheses of Corollary 5.2 and Theorem 5.3 are not essential, the inequalities themselves cannot be improved in general.

We conclude with an example which illustrates the dichotomy between components of finite and infinite category for the function space of self-maps of a particular F_0 -space.

Theorem 5.4. *Let S be a product of even spheres and $f : S \rightarrow S$ a map. Then $\text{cat}_0(M_f(S, S)_*) < +\infty$ if and only if $f_* : \pi_n(S) \rightarrow \pi_n(S)$ is an isomorphism for all $n > l(S)$.*

Proof. If $f_* : \pi_n(S) \rightarrow \pi_n(S)$ is an isomorphism for all $n > l(S)$ then, in fact, $\text{cat}_0(M_f(S, S)) < +\infty$ by Theorem 4.6. Suppose, conversely, that $f_*(x_i) = 0$ for some even degree $x_i \in \pi_{>l(S)}(S)$. Since $f^*(x_i) = 0$, the elements $a_k^* \otimes x_i \in \overline{Z}_0$ represent nonzero classes in the cohomology of $M_f(S, S)_*$. Since $a_k^2 = 0$ for all k , by equation (2), the only relations among these classes are of the form $(a_k^* \otimes x_i) \cdot (a_j^* \otimes x_i)$ for $a_k \neq a_j$. It follows easily that the cohomology of $M_f(S, S)_*$ has infinite product length. ■

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