Equivalence of Crossed Coproducts

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1 Introduction

The concept of "crossed coproduct" appeared as a dual version of the usual crossed product for Hopf algebras and it was used in several papers (for instance, in [8] it gives rise, together with the crossed product, to the so-called "bicrossproduct"). In [4] were studied cleft coextensions, a dual notion for that of cleft extension, and it was proved that a cleft coextension is isomorphic to a crossed coproduct (and, another caracterization, a cleft coextension is a Galois coextension with normal basis).

In this paper, we continue the study performed in [5] and [4] on crossed coproducts and cleft coextensions. Our main source of inspiration was Doi's paper [7]; our results are dual to those obtained by Doi. A few remarks are in order:
1) In his paper, Doi uses the cohomology groups introduced by Sweedler in [11]; we use here the dual objects, also introduced by Doi in [6].
2) In Doi's paper, the centre of an algebra was used. Following the philosophy of dualization, we were led, naturally, to the use of a dual object, the "cocentre" of a coalgebra. This object was introduced recently, in [13], and is slightly more complicated than its dual version.

The main results of this paper are the following:
1) If H is a Hopf algebra and C a coalgebra, then there exists a bijection between the set of isomorphism classes of H-cleft coextensions of C and the set of the equivalence classes of crossed cosystems for H over C.
2) if H is a commutative Hopf algebra, C a coalgebra , $Z(C)$ the cocentre of C, D/C an H-cleft coextension , $\phi : D \to H$ a fixed cosection, (ψ, α) the corresponding crossed cosystem, then there exists a bijection between the cohomology group $Coalg - H^2(Z(C), H)$ and the set of the equivalence classes of all those crossed cosystems for H over C which have ψ as a weak coaction.

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2 Preliminaries

Throughout k is a fixed field. All coalgebras, algebras, vector spaces and unadorned $\otimes$, Hom, etc., are over k. We refer to [10] for details on coalgebras and Hopf algebras.

We recall now some constructions from [5] and [4].

Definition 2.1. *Let H be a Hopf algebra and C a coalgebra. A k-linear map $\psi : C \to H\otimes C$, $\psi(c) = \sum c^1 \otimes c^2$ is called a weak coaction if the following conditions are satisfied:*

$$\sum c^1 \otimes (c^2)_1 \otimes (c^2)_2 = \sum (c_1)^1 (c_2)^1 \otimes (c_1)^2 \otimes (c_2)^2 \tag{1}$$

$$\sum c^1 \varepsilon_C(c^2) = \varepsilon_C(c) 1_H \tag{2}$$

$$\sum \varepsilon_H(c^1) c^2 = c \tag{3}$$

for any $c \in C$.

In the above conditions, let $\alpha : C \to H \otimes H$ be a k-linear map, with notation $\alpha(c) = \sum \alpha_1(c) \otimes \alpha_2(c)$, satisfying the following conditions:

(CU) $\sum \varepsilon_H(\alpha_1(c))\alpha_2(c) = \varepsilon_C(c)1_H = \sum \alpha_1(c)\varepsilon_H(\alpha_2(c))$

(C) $\sum (c_1)^1 \alpha_1(c_2) \otimes \alpha_1((c_1)^2)(\alpha_2(c_2))_1 \otimes \alpha_2((c_1)^2)(\alpha_2(c_2))_2 =$

$$= \sum \alpha_1(c_1)(\alpha_1(c_2))_1 \otimes \alpha_2(c_1)(\alpha_1(c_2))_2 \otimes \alpha_2(c_2)$$

(TC) $\sum (c_1)^1 \alpha_1(c_2) \otimes ((c_1)^2)^1 \alpha_2(c_2) \otimes ((c_1)^2)^2 =$

$$= \sum \alpha_1(c_1)((c_2)^1)_1 \otimes \alpha_2(c_1)((c_2)^1)_2 \otimes (c_2)^2$$

for any $c \in C$. Then we can construct the crossed coproduct $C \rtimes_\alpha H$, which is a coalgebra, with $C \otimes H$ as the underlying linear space and the structures

$$\Delta_\alpha : C \otimes H \to C \otimes H \otimes C \otimes H$$

$$\Delta_\alpha(c \otimes h) = \sum c_1 \otimes (c_2)^1 \alpha_1(c_3) h_1 \otimes (c_2)^2 \otimes \alpha_2(c_3) h_2$$

and

$$\varepsilon_\alpha : C \otimes H \to k$$

$$\varepsilon_\alpha(c \otimes h) = \varepsilon_C(c)\varepsilon_H(h)$$

Definition 2.2. *If $C \rtimes_\alpha H$ is a crossed coproduct and α is convolution invertible, we shall say that (ψ, α) is a crossed cosystem for H over C.*

Definition 2.3. *If H is a Hopf algebra and C a coalgebra, a right H-coextension of C is a pair (D, p), where D is a right H-module coalgebra, $p : D \to C$ a surjective coalgebra map, and $Ker(p) = DH^+$, where $H^+ = Ker(\varepsilon_H)$. We shall denote a coextension by D/C.*

Definition 2.4. *An H-coextension D/C is called a cleft coextension if there exists a k-linear map $\phi : D \to H$, convolution invertible and which is moreover a right H-module homomorphism (such a map is called a cointegral).*

Lemma 2.5. *If D/C is an H-cleft coextension, then there exists a cointegral ϕ' : $D \to H$ which is counitary, i.e. $\varepsilon_H \circ \phi' = \varepsilon_D$.*

Definition 2.6. *A unitary cointegral is called a cosection of D.*

Remark 2.7. *If $C \rtimes_\alpha H$ is a crossed coproduct, then the map*

$$\pi : C \rtimes_\alpha H \to C, \ \ \pi(c \otimes h) = \varepsilon_H(h)c$$

is a surjective coalgebra homomorphism.

Proposition 2.8. *Let D/C be an H-coextension. Then the following statements are equivalent:*
(i) D/C is a cleft coextension.
(ii) D is isomorphic to a crossed coproduct $C \rtimes_\alpha H$, with the cocycle α convolution invertible, such that, if we identify D to $C \rtimes_\alpha H$, the map $p : D \to C$ equals the map π defined in the previous remark.
More exactly, let $\phi : D \to H$ be a crossed cosection, let

$$\psi : C \to H \otimes C, \ \ \psi(\overline{c}) = \sum \phi(c_1)\phi^{-1}(c_3) \otimes \overline{c}_2$$

$$\alpha : C \to H \otimes H, \ \ \alpha(\overline{c}) = \sum \phi(c_1)\phi^{-1}(c_3)_1 \otimes \phi(c_2)\phi^{-1}(c_3)_2$$

where, for $c \in C$, we denoted $\overline{c} = p(c)$.
Then ψ and α are well defined, (ψ, α) is a crossed cosystem for H over C (we shall say that it corresponds to ϕ) and D is isomorphic to $C \rtimes_\alpha H$, such that, if we identify D to $C \rtimes_\alpha H$, then $p = \pi$.

Definition 2.9. *Let H be a Hopf algebra and C a coalgebra. Two crossed cosystems (ψ, α) and (φ, β) are called equivalent (and we shall write $(\psi, \alpha) \sim (\varphi, \beta)$) if there exists a k-linear map $v : C \to H$, convolution invertible, with $\varepsilon_H \circ v = \varepsilon_C$, such that:*

$$\sum c_{-1} \otimes c_0 = \sum v(c_1)(c_2)^1 v^{-1}(c_3) \otimes (c_2)^2 \tag{4}$$

$$\sum \beta_1(c) \otimes \beta_2(c) = \sum v(c_1)(c_2)^1 \alpha_1(c_3) v^{-1}(c_4)_1 \otimes v((c_2)^2)\alpha_2(c_3)v^{-1}(c_4)_2 \tag{5}$$

for any $c \in C$, where we denoted $\varphi : C \to H \otimes C$, $\varphi(c) = \sum c_{-1} \otimes c_0$, $\psi : C \to H \otimes C$, $\psi(c) = \sum c^1 \otimes c^2$.

Remark 2.10. *The above relation is an equivalence relation.*

We recall now from [13] some facts about the cocentre of a coalgebra. If D is a coalgebra, it can be defined the cocentre $(Z(D), 1^d)$ of D, where $Z(D)$ is a cocommutative coalgebra and $1^d : D \to Z(D)$ is a surjective coalgebra map, which satisfies the equality

$$\sum 1^d(d_1) \otimes d_2 = \sum 1^d(d_2) \otimes d_1$$

for all $d \in D$. The cocentre satisfies the following universal property: for any coalgebra H and any coalgebra map $f : D \to H$, which satisfies the condition $\sum f(d_1) \otimes d_2 = \sum f(d_2) \otimes d_1$ for all $d \in D$, there exists a unique coalgebra map $g : Z(D) \to H$ such that $f = g \circ 1^d$ (see [13], Cor.2.3). From this universal property, the cocentre of a coalgebra is unique up to isomorphism.

3 Equivalence of crossed coproducts

In what follows, H will be a Hopf algebra.

Lemma 3.1. *Let D/C be an H-cleft coextension, ϕ and γ two cosections for D, (ψ,α) and (φ,β) the crossed cosystems corresponding to ϕ and γ respectively. Then we have $(\psi,\alpha)\sim(\varphi,\beta)$.*

Proof: Define $u: D\rightarrow H$, $u=\gamma*\phi^{-1}$ We prove first that $DH^+\subset Ker(u)$, and for this it is enough to show that $u(ch)=0$ for $c\in D$ and $h\in H^+$. D is a right H-module coalgebra , so we have that $\sum(ch)_1\otimes(ch)_2=\sum c_1h_1\otimes c_2h_2$. From [4], Lemma 2.3, we know that $\phi^{-1}(ch)=S(h)\phi^{-1}(c)$. Hence we obtain $u(ch)=0$, by applying the above formulae, $h\in H^+$, and the fact that γ is a right module homomorphism. We can define now $v: C\rightarrow H$, $v(\overline{c})=\sum\gamma(c_1)\phi^{-1}(c_2)$. With the same proof we can define $w: C\rightarrow H$, $w(\overline{c})=\sum\phi(c_1)\gamma^{-1}(c_2)$. It is easy to see that $w=v^{-1}$ in $(Hom(C,H),*)$. Then $(\varepsilon_H\circ v)(\overline{c})=\varepsilon_H(\phi^{-1}(c))$, because $\varepsilon_H\circ\gamma=\varepsilon_D$.

We know $\varepsilon_H\circ\phi=\varepsilon_D$; multiplying by convolution with $\varepsilon_H\circ\phi^{-1}$, we obtain $\varepsilon_D(c)=\varepsilon_H(\phi^{-1}(c))$ for each $c\in D$, hence $\varepsilon_H\circ v=\varepsilon_C$. Now, for any $c\in D$, we have (denoting $\psi(\overline{c})=\sum\overline{c}^1\otimes\overline{c}^2$) that

$\sum\gamma(\overline{c}_1)(\overline{c}_2)^1v^{-1}(\overline{c}_3)\otimes(\overline{c}_2)^2=$

$\sum\gamma(c_1)\phi^{-1}(c_2)\phi(c_3)\phi^{-1}(c_5)\phi(c_6)\gamma^{-1}(c_7)\otimes\overline{c}_4=$

$\sum\gamma(c_1)\gamma^{-1}(c_3)\otimes\overline{c}_2=\varphi(\overline{c})$

and

$\sum v(\overline{c}_1)(\overline{c}_2)^1\alpha_1(\overline{c}_3)v^{-1}(\overline{c}_4)_1\otimes v((\overline{c}_2)^2)\alpha_2(\overline{c}_3)v^{-1}(\overline{c}_4)_2=$

$\sum\gamma(c_1)\phi^{-1}(c_2)\phi(c_3)\phi^{-1}(c_5)\phi(c_6)\phi^{-1}(c_8)_1\phi(c_9)_1\gamma^{-1}(c_{10})_1\otimes$

$$\otimes v(\overline{c}_4)\phi(c_7)\phi^{-1}(c_8)_2\phi(c_9)_2\gamma^{-1}(c_{10})_2=$$

$\sum\gamma(c_1)\gamma^{-1}(c_4)_1\otimes v(\overline{c}_2)\phi(c_3)\gamma^{-1}(c_4)_2=$

$\sum\gamma(c_1)\gamma^{-1}(c_5)_1\otimes\gamma(c_2)\phi^{-1}(c_3)\phi(c_4)\gamma^{-1}(c_5)_2=$

$\sum\gamma(c_1)\gamma^{-1}(c_3)_1\otimes\gamma(c_2)\gamma^{-1}(c_3)_2=\beta(\overline{c})$

Hence, v gives the equivalence between (ψ,α) and (φ,β).

Corollary 3.2. *Each H-cleft coextension D/C determines a unique equivalence class of crossed cosystems for H over C, which will be denoted by (D/C).*

Let now H be a Hopf algebra, C a coalgebra,

$$\psi: C\rightarrow H\otimes C,\ \ \psi(c)=\sum c^1\otimes c^2$$

a weak coaction, $C\rtimes_\alpha H$ a crossed coproduct.

We know that $\pi: C\rtimes_\alpha H\rightarrow C$, $\pi(c\otimes h)=\varepsilon_H(h)c$ is a surjective coalgebra homomorphism. Let E be a coalgebra , let $\theta: E\rightarrow C$ be a coalgebra homomorphism, $\gamma: E\rightarrow H$ convolution invertible, with $\varepsilon_H\circ\gamma=\varepsilon_E$ and

$(a)\quad\sum\theta(e)^1\otimes\theta(e)^2=\sum\gamma(e_1)\gamma^{-1}(e_3)\otimes\theta(e_2)$

(b) $\sum \alpha_1(\theta(e)) \otimes \alpha_2(\theta(e)) = \sum \gamma(e_1)\gamma^{-1}(e_3)_1 \otimes \gamma(e_2)\gamma^{-1}(e_3)_2$
for any $e \in E$.

Proposition 3.3. *In the above situation , the map* $\Theta : E \to C \bowtie_\alpha H$*,*
$\Theta(e) = \sum \theta(e_1) \otimes \gamma(e_2)$ *is a coalgebra homomorphism , and* $\pi \circ \Theta = \theta$*.*

Proof: $\Theta(e_1) \otimes \Theta(e_2) = \sum \theta(e_1) \otimes \gamma(e_2) \otimes \theta(e_3) \otimes \gamma(e_4)$
The comultiplication on $C \bowtie_\alpha H$ is

$\Delta(c \otimes h) = \sum c_1 \otimes (c_2)^1\alpha_1(c_3)h_1 \otimes (c_2)^2 \otimes \alpha_2(c_3)h_2$, so:

$\sum \Theta(e)_1 \otimes \Theta(e)_2 = \sum \theta(e_1) \otimes \theta(e_2)^1\alpha_1(\theta(e_3))\gamma(e_4)_1 \otimes \theta(e_2)^2 \otimes \alpha_2(\theta(e_3))\gamma(e_4)_2$

(because θ is a coalgebra homomorphism)

$= \sum \theta(e_1) \otimes \gamma(e_2)\gamma^{-1}(e_4)\gamma(e_5)\gamma^{-1}(e_7)_1\gamma(e_8)_1 \otimes \theta(e_3) \otimes \gamma(e_6)\gamma^{-1}(e_7)_2\gamma(e_8)_2$

(using *(a)* and *(b)*)

$= \sum \theta(e_1) \otimes \gamma(e_2) \otimes \theta(e_3) \otimes \gamma(e_4)$

Then $\varepsilon(\Theta(e)) = \sum \varepsilon(\theta(e_1))\varepsilon(\gamma(e_2)) = \varepsilon_H(\gamma(e)) = \varepsilon_E(e)$, so Θ is a coalgebra homomorphism. Finally,
$\pi(\Theta(e)) = \sum \varepsilon_H(\gamma(e_2))\theta(e_1) = \sum \varepsilon_E(e_2)\theta(e_1) = \theta(e)$.

Definition 3.4. *Let* H *be a Hopf algebra ,* C *a coalgebra ,* $\psi : C \to H \otimes C$ *a left weak coaction. Let* E *be a coalgebra,* $\pi : E \to C$ *a surjective coalgebra homomorphism . We shall say that* ψ *is an* E*-inner coaction if there exists* $\gamma : E \to H$ *, convolution invertible , such that*
$\sum \pi(e)^1 \otimes \pi(e)^2 = \sum \gamma(e_1)\gamma^{-1}(e_3) \otimes \pi(e_2)$ *for any* $e \in E$*.*

Remark 3.5. *If* $E = C$ *and* $\pi = id$ *, we obtain the notion of "inner coaction".*

Example 3.6. Let H be a Hopf algebra , C a coalgebra , (ψ, α) a crossed cosystem for H over C ; let $E = C \bowtie_\alpha H$, $\pi : E \to C$, $\pi(c \otimes h) = \varepsilon(h)c$, $\gamma : E \to H$, $\gamma(c \otimes h) = \varepsilon(c)h$.
We know from [4], Proposition 2.1., that γ is convolution invertible and

$$\gamma^{-1}(c \otimes h) = \sum S(\alpha_1^{-1}(c)h)\alpha_2^{-1}(c).$$

We show that ψ is a $C \bowtie_\alpha H$-inner coaction.

$\sum \pi(c \otimes h)^1 \otimes \pi(c \otimes h)^2 = \varepsilon(h) \sum c^1 \otimes c^2$

$\sum \gamma((c \otimes h)_1)\gamma^{-1}((c \otimes h)_3) \otimes \pi((c \otimes h)_2) =$

$\sum (c_1)^1\alpha_1(c_2)h_1\gamma^{-1}([(c_1)^2 \otimes \alpha_2(c_2)h_2]_2) \otimes \pi([(c_1)^2 \otimes \alpha_2(c_2)h_2]_1) =$

$\sum (c_1)^1\alpha_1(c_2)h_1\gamma^{-1}((((c_1)^2)_2)^2 \otimes \alpha_2(((c_1)^2)_3\alpha_2(c_2)_2h_3)\otimes$

$$\otimes\pi(((c_1)^2)_1 \otimes (((c_1)^2)_2)^1\alpha_1(((c_1)^2)_3)\alpha_2(c_2)_1h_2) =$$

$\sum (c_1)^1\alpha_1(c_2)h_1\gamma^{-1}(((c_1)^2)_2 \otimes \alpha_2(c_2)h_2) \otimes ((c_1)^2)_1 =$

$\varepsilon(h) \sum (c_1)^1\alpha_1(c_2)S(\alpha_2(c_2))S(\alpha_1^{-1}(((c_1)^2)_2))\alpha_2^{-1}(((c_1)^2)_2) \otimes ((c_1)^2)_1 =$

$\varepsilon(h)\sum(c_1)^1(c_2)^1\alpha_1(c_3)S(\alpha_2(c_3))S(\alpha_1^{-1}((c_2)^2))\alpha_2^{-1}((c_2)^2)\otimes(c_1)^2$

(using the definition of the weak coaction for c_1)

$=\varepsilon(h)\sum c^1\otimes c^2$

where the last equality follows after some computations, but applying first for $c=c_2$ the following relation (which is Lemma 1.4. in [4]):

$$\sum c^1\otimes\alpha_1^{-1}(c^2)\otimes\alpha_2^{-1}(c^2)=$$

$$=\sum\alpha_1(c_1)(\alpha_1^{-1}(c_2))_1\alpha_1^{-1}(c_3)\otimes(\alpha_2(c_1))_1(\alpha_1^{-1}(c_2))_2\alpha_2^{-1}(c_3)\otimes(\alpha_2(c_1))_2\alpha_2^{-1}(c_2)$$

Remark 3.7. *If D/C is a right H-coextension for C, we shall denote in the sequel by $\pi:D\to C$ the surjective coalgebra homomorphism with $Ker(\pi)=DH^+$.*

Definition 3.8. *Let D/C and D'/C two right H-coextensions. We shall say that they are isomorphic if there exists a right H-module coalgebra isomorphism $f:D\to D'$ such that $\pi'\circ f=\pi$. We shall denote by $[D/C]$ the equivalence class of D/C.*

Proposition 3.9. *Two H-cleft coextensions D/C and D'/C are isomorphic if and only if $(D/C)=(D'/C)$; thus the assignement $[D/C]\to(D/C)$ determines a bijection between the isomorphism classes of H-cleft coextensions of C and the equivalence classes of crossed cosystems for H over C.*

Proof: Let $f:D\to D'$ a module coalgebra isomorphism with $\pi'\circ f=\pi$, let $\phi':D'\to H$ a co-section of D', let $\phi=\phi'\circ f$; obviously ϕ is a right comodule homomorphism , $\varepsilon_H\circ\phi=\varepsilon_C$ and ϕ is convolution invertible with inverse $\phi^{-1}=\phi'^{-1}\circ f$, hence ϕ is a cosection for D.
Let (ψ,α) and (ψ',α') be the crossed cosystems corresponding to ϕ and ϕ' respectively , i.e. for any $c\in D$ we have

$$\psi:C\to H\otimes C,\ \ \psi(\pi(c))=\sum\phi(c_1)\phi^{-1}(c_3)\otimes\pi(c_2)$$

$$\alpha:C\to H\otimes H,\ \ \alpha(\pi(c))=\sum\phi(c_1)\phi^{-1}(c_3)_1\otimes\phi(c_2)\phi^{-1}(c_3)_2$$

(and the corresponding relations for ψ' and α').
Since f is surjective, for any $c'\in D'$ there exists $c\in D$ with $f(c)=c'$, hence

$\psi'(\pi'(c'))=\sum\phi'(f(c_1))\phi'^{-1}(f(c_3))\otimes\pi'(f(c_2))=$

$\sum\phi(c_1)\phi^{-1}(c_3)\otimes\pi(c_2)=\psi(\pi(c))$

But $\pi'(c')=\pi(c)$, hence $\psi=\psi'$; with an analogous proof, we obtain $\alpha=\alpha'$, therefore $(D/C)=(D'/C)$.

Conversely, let ϕ,ϕ' cosections for D and D' respectively, let (ψ,α) and (φ,β) the corresponding crossed cosystems. From $(D/C)=(D'/C)$ we obtain $(\psi,\alpha)\sim(\varphi,\beta)$, so the relations (4) and (5) are satisfied.
Let $\gamma:D\to H$, $\gamma=(v\circ\pi)*\phi$. It is easy to see that γ is convolution invertible with inverse $\gamma^{-1}(c)=\sum\phi^{-1}(c_1)v^{-1}(\pi(c_2))$, and $\varepsilon_H\circ\gamma=\varepsilon_D$.
From (4) we obtain $\varphi(x)=\sum v(x_1)(x_2)^1v^{-1}(x_3)\otimes(x_2)^2$ for any $x\in C$, where $\psi(x)=\sum x^1\otimes x^2$. Then, if we take $c\in D$ with $\pi(c)=x$, we obtain

$$\varphi(\pi(c))=\sum\gamma(c_1)\gamma^{-1}(c_3)\otimes\pi(c_2)$$

for any $c \in C$, because $\psi(\pi(c)) = \sum \phi(c_1)\phi^{-1}(c_3) \otimes \pi(c_2)$.
In the same way, from (5) we obtain :

$$\beta(\pi(c)) = \sum v(\pi(c_1))\phi(c_2)\phi^{-1}(c_4)\alpha_1(\pi(c_5))v^{-1}(\pi(c_6))_1 \otimes$$

$$\otimes v(\pi(c_3))\alpha_2(\pi(c_5))v^{-1}(\pi(c_6))_2$$

$$= \sum v(\pi(c_1))\phi(c_2)\phi^{-1}(c_4)\phi(c_5)\phi^{-1}(c_7)_1 v^{-1}(\pi(c_8))_1 \otimes$$

$$\otimes v(\pi(c_3))\phi(c_6)\phi^{-1}(c_7)_2 v^{-1}(\pi(c_8))_2$$

(from $\alpha(\pi(c)) = \sum \phi(c_1)\phi^{-1}(c_3)_1 \otimes \phi(c_2)\phi^{-1}(c_3)_2$, for c_5 instead of c)

$$= \sum v(\pi(c_1))\phi(c_2)\phi^{-1}(c_5)_1 v^{-1}(\pi(c_6))_1 \otimes v(\pi(c_3))\phi(c_4)\phi^{-1}(c_5)_2 v^{-1}(\pi(c_6))_2$$

$$= \sum \gamma(c_1)\gamma^{-1}(c_3)_1 \otimes \gamma(c_2)\gamma^{-1}(c_3)_2$$

for any $c \in D$.

Now we shall apply Proposition 3.3 for the crossed coproduct $C \rtimes_\beta H$. We take $E = D$, $\theta = \pi$, $\gamma = \gamma$ in Proposition 3.3, and one can see that the relations proved above are just (a) and (b) in Proposition 3.3. Then the map $\Theta : D \to C \rtimes_\beta H$, $\Theta(c) = \sum \pi(c_1) \otimes \gamma(c_2)$ is a coalgebra homomorphism, with $p \circ \Theta = \pi$, where $p : C \rtimes_\beta H \to C$, $p(c \otimes h) = \varepsilon_H(h)c$.

We prove now that Θ is a right H-module homomorphism.
We have first $\pi(ch) = \pi(ch - c\varepsilon(h)1 + c\varepsilon(h)1) = \pi(c(h - \varepsilon(h)1)) + \pi(c)\varepsilon(h) = \pi(c)\varepsilon(h)$, because $c(h - \varepsilon(h)1) \in DH^+ = Ker\pi$. Then

$$\gamma(ch) = \sum v(\pi(c_1h_1))\phi(c_2h_2) = \sum v(\pi(c_1))\phi(c_2h_)$$

$$= \sum v(\pi(c_1))\phi(c_2)h = \gamma(c)h$$

where the last equality holds because ϕ is a right module homomorphism. Hence

$$\Theta(ch) = \sum \pi(c_1h_1) \otimes \gamma(c_2h_2) = \sum \pi(c_1) \otimes \gamma(c_2)h$$

$= (\sum \pi(c_1) \otimes \gamma(c_2))h = \Theta(c)h$, q.e.d.

Now, define $f : C \rtimes_\alpha H \to C \rtimes_\beta H$, $f(x \otimes h) = \sum x_1 \otimes v(x_2)h$
Because v is convolution invertible, f is bijective with inverse

$$g : C \rtimes_\beta H \to C \rtimes_\alpha H, \;\; g(x \otimes h) = \sum x_1 \otimes v^{-1}(x_2)h.$$

We know from [4] that the map

$$F : D \to C \rtimes_\alpha H, \;\; F(c) = \sum \pi(c_1) \otimes \phi(c_2)$$

is a coalgebra isomorphism; it is also a module homomorphism. Then

$$(f \circ F)(c) = \sum \pi(c_1) \otimes v(\pi(c_2))\phi(c_3) = \Theta(c),$$

so Θ is bijective, hence an isomorphism of H-module coalgebras.
Let

$$F' : D' \to C \rtimes_\beta H, \;\; F'(c) = \sum \pi(c_1) \otimes \phi'(c_2),$$

and

$$\mu : D \to D', \;\; \mu = F'^{-1} \circ \Theta.$$

We obtain that μ is a module coalgebra isomorphism. From $\pi' \circ F'^{-1} = p$ and $p \circ \Theta = \pi$, we obtain $\pi' \circ \mu = \pi$, hence D/C and D'/C are isomorphic.
Thus, we proved that the map $[D/C] \to (D/C)$ is well-defined and injective, and we shall prove now that it is surjective. Let (ψ_0, α_0) be a crossed cosystem, $\psi_0(c) = \sum c^1 \otimes c^2$. From [4] we know that $C \bowtie_{\alpha_0} H/C$ is a cleft coextension, and let (ψ, α) be the crossed cosystem associated to this cleft coextension, with the cosection $\gamma : C \bowtie_{\alpha_0} H \to H$, $\gamma(c \otimes h) = \varepsilon(c)h$. For $c \in C$, let
$c \otimes 1 \in C \bowtie_{\alpha_0} H$; then we have $\pi(c \otimes 1) = c$, where

$\pi : C \bowtie_{\alpha_0} H \to C$,$\pi(c \otimes h) = \varepsilon(h)c$. Hence

$\psi(c) = \sum \gamma((c \otimes 1)_1)\gamma^{-1}((c \otimes 1)_3) \otimes \pi((c \otimes 1)_2)$

$= \sum (c_1)^1 \alpha_1(c_2)\gamma^{-1}(((c_1)^2)_2 \otimes \alpha_2(c_2)) \otimes ((c_1)^2)_1$

$= \sum (c_1)^1 \alpha_1(c_2) S(\alpha_2(c_2)) S(\alpha_1^{-1}(((c_1)^2)_2))\alpha_2^{-1}(((c_1)^2)_2) \otimes ((c_1)^2)_1$

$= \sum c^1 \otimes c^2$

where the last equality follows from the proof of the Example 3.6.
Hence $\psi = \psi_0$; in the same way we can prove that $\alpha = \alpha_0$, so that the map is surjective.

Definition 3.10. *If D/C is an H-cleft coextension such that there exists a cosection $\phi : D \to H$ which is a coalgebra homomorphism, then ϕ is called an algebraic cosection and the coextension D/C is called H-smash.*

Lemma 3.11. *In the situation of Prop.2.8, we have : ϕ is an algebraic co-section if and only if α is a trivial cocycle, i.e. $\alpha(x) = \varepsilon(x)1_H \otimes 1_H$ for any $x \in C$ (and in this case C is an H-comodule coalgebra).*

Proof: Suppose that ϕ is a coalgebra homomorphism; then

$\alpha(\overline{c}) = \sum [\phi(c_1)\phi^{-1}(c_2)]_1 \otimes [\phi(c_1)_2\phi^{-1}(c_2)]_2 = \varepsilon_D(c)1_H \otimes 1_H = \varepsilon_D(\overline{c})1_H \otimes 1_H$.

Conversely, suppose that α is trivial ; then

$\sum \phi(c_1)\phi^{-1}(c_3)_1 \otimes \phi(c_2)\phi^{-1}(c_3)_2 = \varepsilon_D(c)1_H \otimes 1_H$ for any $c \in D$.

Multiplying by convolution with the map

$$\psi : D \to H \otimes H, \quad \psi(c) = \sum \phi(c)_1 \otimes \phi(c)_2$$

we obtain $\sum \phi(c_1) \otimes \phi(c_2) = \sum \phi(c)_1 \otimes \phi(c)_2$, that is ϕ is a coalgebra homomorphism.

Proposition 3.12. *Let D/C be an H-cleft coextension and (ψ, α) a crossed cosystem associated to D/C; then the following statements are equivalent:*
(i) D/C is H-smash
(ii) (D/C) is the equivalence class of a crossed cosystem (ϕ, β) for which $\beta(c) = \varepsilon(c)1_H \otimes 1_H$ for any $c \in C$.
(iii) There exists $v : C \to H$, k-linear, convolution invertible, with $\varepsilon_H \circ v = \varepsilon_C$, such that

$$\alpha(c) = \sum (c_1)^1 v(c_2) v^{-1}(c_3)_1 \otimes v((c_1)^2) v^{-1}(c_3)_2 \tag{6}$$

for any $c \in C$, where $\psi(c) = \sum c^1 \otimes c^2$.

Proof: $(i) \Rightarrow (ii)$ is obvious, from Lemma 3.11 and Lemma 3.1.
$(ii) \Rightarrow (iii)$ We have $(\psi,\alpha) \sim (\varphi,\beta)$, with $\beta(c)=\varepsilon(c)1_H \otimes 1_H$. Hence, there exists $v : C \to H$, k-linear, convolution invertible, with $\varepsilon_H \circ v = \varepsilon_C$, such that

$$\sum c^1 \otimes c^2 = \sum v(c_1)(c_2)_{-1}v^{-1}(c_3) \otimes (c_2)_0 \tag{7}$$

$$\alpha(c) = \sum v(c_1)(c_2)_{-1}\beta_1(c_3)v^{-1}(c_4)_1 \otimes v((c_2)_0)\beta_2(c_3)v^{-1}(c_4)_2 \tag{8}$$

where $\psi(c) = \sum c^1 \otimes c^2$ and $\varphi(c) = \sum c_{-1} \otimes c_0$.

Since $\beta(c) = \varepsilon_C(c)1_H \otimes 1_H$, (8) becomes:

$\alpha(c) = \sum v(c_1)(c_2)_{-1}v^{-1}(c_3)_1 \otimes v((c_2)_0)v^{-1}(c_3)_2 =$

$\sum v(c_1)(c_2)_{-1}v^{-1}(c_3)v(c_4)v^{-1}(c_5)_1 \otimes v((c_2)_0)v^{-1}(c_5)_2 =$

$\sum (c_1)^1 v(c_2)v^{-1}(c_3)_1 \otimes v((c_1)^2)v^{-1}(c_3)_2$

which is exactly (iii), where for the last equality we used (7).
$(iii) \Rightarrow (i)$ Using the map v given in (iii), define $\gamma : C \bowtie_\alpha H \to H$, $\gamma(c \otimes h) = v^{-1}(c)h$. We have

$\varepsilon_H \circ v = \varepsilon_C \Rightarrow \varepsilon_H \circ v^{-1} = \varepsilon_C \Rightarrow \varepsilon_H \circ \gamma = \varepsilon_{C\bowtie_\alpha H}$

$\gamma((c\otimes h)g) = \gamma(c \otimes hg) = v^{-1}(c)hg = (v^{-1}(c)h)g = \gamma(c\otimes h)g$

hence γ is a right H-module map.
Now we shall prove that γ is a coalgebra map.

$\sum \gamma(c\otimes h)_1 \otimes \gamma(c\otimes h)_2 = \sum v^{-1}(c)_1h_1 \otimes v^{-1}(c)_2h_2$

$\sum \gamma((c\otimes h)_1) \otimes \gamma((c\otimes h)_2) =$

$\sum \gamma(c_1 \otimes (c_2)^1\alpha_1(c_3)h_1) \otimes \gamma((c_2)^2 \otimes \alpha_2(c_3)h_2) =$

$\sum v^{-1}(c_1)(c_2)^1\alpha_1(c_3)h_1 \otimes v^{-1}((c_2)^2)\alpha_2(c_3)h_2 =$

$\sum v^{-1}(c_1)(c_2)^1(c_3)^1v(c_4)v^{-1}(c_5)_1h_1 \otimes v^{-1}((c_2)^2)v((c_3)^2)v^{-1}(c_5)_2h_2$

(using (6))

$= \sum v^{-1}(c_1)(c_2)^1v(c_3)v^{-1}(c_4)_1h_1 \otimes v^{-1}(((c_2)^2)_1)v(((c_2)^2)_2)v^{-1}(c_4)_2h_2$

(using (1))

$= \sum v^{-1}(c_1)(c_2)^1v(c_3)v^{-1}(c_4)_1h_1 \otimes \varepsilon((c_2)^2)v^{-1}(c_4)_2h_2 =$

$\sum v^{-1}(c_1)v(c_2)v^{-1}(c_3)_1h_1 \otimes v^{-1}(c_3)_2h_2 =$

$\sum v^{-1}(c)_1h_1 \otimes v^{-1}(c)_2h_2,$

hence γ is a coalgebra map.
We prove now that γ is convolution invertible. Define $w : C \bowtie_\alpha H \to H$, by $w(c\otimes h) = \varepsilon(h)v^{-1}(c)$. It is easy to see that w is convolution invertible, with inverse $w^{-1}(c\otimes h) = \varepsilon(h)v(c)$. Let $\gamma_0 : C \bowtie_\alpha H \to H$, $\gamma_0(c\otimes h) = \varepsilon(c)h$.
By [4], γ_0 is convolution invertible, and it is easy to see that $\gamma = w * \gamma_0$. Therefore γ is convolution invertible. The conclusion is that γ is an algebraic cosection, hence $C \bowtie_\alpha H/C$ is H-smash. By Proposition 2.8, we have $D \simeq C \bowtie_\alpha H$, therefore D/C is also H-smash.

Remark 3.13. *Let D/C be an H-coextension and let $\phi : D \to H$ be a cosection.*

Then we have

$$\sum \phi(c_1) \otimes \pi(c_2) = \sum \pi(c_1)^1 \phi(c_2) \otimes \pi(c_1)^2$$

where $\psi(\pi(c)) = \sum \pi(c)^1 \otimes \pi(c)^2 = \sum \phi(c_1)\phi^{-1}(c_3) \otimes \pi(c_2)$ *(as in Proposition 2.8).*

Proof: $\sum \pi(c_1)^1 \phi(c_2) \otimes \pi(c_1)^2 =$
$\sum \phi(c_1)\phi^{-1}(c_3)\phi(c_4) \otimes \pi(c_2) = \sum \phi(c_1) \otimes \pi(c_2)$

Remark 3.14. *In the same conditions, the weak coaction* ψ *of* H *on* C *is trivial (i.e.* $\psi(x) = 1 \otimes x$ *for any* $x \in C$*) if and only if* $\sum \phi(c_1) \otimes \pi(c_2) = \sum \phi(c_2) \otimes \pi(c_1)$ *for any* $c \in D$*.*

Proof: Suppose that ψ is trivial. Then $\sum \pi(c)^1 \otimes \pi(c)^2 = 1 \otimes \pi(c)$; we have
$\sum \pi(c)^1 \phi(c_2) \otimes \pi(c_1)^2 = \sum \phi(c_1) \otimes \pi(c_2)$ (the above remark). Hence
$\sum \phi(c_1) \otimes \pi(c_2) = \sum 1_H \phi(c_2) \otimes \pi(c_1) = \sum \phi(c_2) \otimes \pi(c_1)$ q.e.d.
Conversely, we have:
$\sum \pi(c)^1 \otimes \pi(c)^2 = \sum \pi(c_1)^1 \phi(c_2)\phi^{-1}(c_3) \otimes \pi(c_1)^2$
$= \sum \phi(c_1)\phi^{-1}(c_3) \otimes \pi(c_2)$
$= \sum \phi(c_2)\phi^{-1}(c_3) \otimes \pi(c_1)$
(because $\sum \phi(c_1) \otimes \pi(c_2) = \sum \phi(c_2) \otimes \pi(c_1)$)
$= 1_H \otimes \pi(c)$
for any $c \in D$, hence $\psi(x) = 1_H \otimes x$ for any $x \in C$.

Definition 3.15. *A cleft coextension* D/C *is called* H*-twisted if there exists a cosection* $\phi : D \to H$ *such that* $\sum \phi(c_1) \otimes \pi(c_2) = \sum \phi(c_2) \otimes \pi(c_1)$ *for any* $c \in D$*.*

Proposition 3.16. *Let* D/C *be an* H*-coextension and let* (ψ, α) *be a crossed cosystem associated to* D/C*. Then the following statements are equivalent:*
1) D/C *is* H*-twisted*
2) D/C *is the equivalence class of a crossed cosystem* (φ, β) *for which* $\varphi(x) = 1_H \otimes x$ *for any* $x \in C$*.*
3) There exists $v : C \to H$*, k-linear, convolution invertible, with* $\varepsilon_H \circ v = \varepsilon_C$ *such that*

$$\psi(c) = \sum v(c_1)v^{-1}(c_3) \otimes c_2 \tag{9}$$

for any $c \in C$ *(this means that* ψ *is* C*-inner with respect to* $id : C \to C$*).*

Proof: 1) $\Rightarrow$ 2) Follows immediately from Remark 3.14 and Lemma 3.1
2) $\Rightarrow$ 3) We have $(\psi, \alpha) \sim (\varphi, \beta)$, with $\varphi(x) = 1_H \otimes x$ for any $x \in C$. So, there exists $v : C \to H$, k-linear, convolution invertible, with $\varepsilon_H \circ v = \varepsilon_C$ such that, if we denote $\psi(c) = \sum c^1 \otimes c^2$, $\varphi(c) = \sum c_{-1} \otimes c_0$, we have the relations (7) and (8) which appeared in the proof of Proposition 3.12.
Since $\varphi(c) = 1_H \otimes c = \sum c_{-1} \otimes c_0$, (7) becomes
$\psi(c) = \sum v(c_1)(c_2)_{-1}v^{-1}(c_3) \otimes (c_2)_0 = \sum v(c_1)v^{-1}(c_3) \otimes c_2$
and this is just the relation (9).

3) $\Rightarrow$ 1) Let $v : C \to H$ be a k-linear map, convolution invertible, with $\varepsilon_H \circ v = \varepsilon_C$, such that $\psi(c) = \sum v(c_1)v^{-1}(c_3) \otimes c_2$. We consider the map $\gamma : C \rtimes_\alpha H \to H$ which appeared in the proof of Proposition 3.12, that is $\gamma(c \otimes h) = v^{-1}(c)h$. We proved there that γ is a cosection, and it is easy to see that the proof remains valid here. Now, we show that

$$\sum \gamma((c \otimes h)_1) \otimes \pi((c \otimes h)_2) = \sum \gamma((c \otimes h)_2) \otimes \pi((c \otimes h)_1)$$

where $\pi : C \rtimes_\alpha H \to C$, $\pi(c \otimes h) = \varepsilon(h)c$. We have:

$\sum \gamma((c \otimes h)_1) \otimes \pi((c \otimes h)_2) =$

$= \sum \gamma(c_1 \otimes (c_2)^1\alpha_1(c_3)h_1) \otimes \pi((c_2)^2 \otimes \alpha_2(c_3)h_2)$

$= \sum v^{-1}(c_1)(c_2)^1\alpha_1(c_3)h_1 \otimes \varepsilon(\alpha_2(c_3))\varepsilon(h_2)(c_2)^2$

$= \sum v^{-1}(c_1)(c_2)^1 h \otimes (c_2)^2$

$= \sum v^{-1}(c_1)v(c_2)v^{-1}(c_4)h \otimes c_3$

(using (9))

$= \sum v^{-1}(c_2)h \otimes c_1$

$\sum \gamma((c \otimes h)_2) \otimes \pi((c \otimes h)_1)$

$= \sum v^{-1}((c_2)^2)\alpha_2(c_3)h_2 \otimes \varepsilon((c_2)^1)\varepsilon(\alpha_1(c_3))\varepsilon(h_1)c_1$

$= \sum v^{-1}(c_2)h \otimes c_1$

The conclusion is that $C \rtimes_\alpha H$ is H-twisted, and since $C \rtimes_\alpha H/C$ is isomorphic to D, we obtain that D/C is H-twisted, q.e.d.

4 The case when H is commutative

From now on, H will be a commutative Hopf algebra.

Let $\pi : D \to C$ be a cleft coextension, $\phi : D \to H$ a cosection and (ψ, α) the associated crossed cosystem. Define $f : D \to H \otimes C$ by

$$f(c) = \sum \phi^{-1}(c_1)\phi(c_3) \otimes \pi(c_2)$$

We shall prove that $Ker\pi \subseteq Kerf$. Let $c \in D$, $h \in H^+$; it is enough to show that $f(ch) = 0$. We have $\pi(ch) = \varepsilon(h)\pi(c)$, ϕ is a right H-module homomorphism and (see [4]) $\phi^{-1}(ch) = S(h)\phi^{-1}(c)$, so

$f(ch) = \sum \phi^{-1}(c_1h_1)\phi(c_3h_2) \otimes \pi(c_2)$

$= \sum S(h_1)\phi^{-1}(c_1)\phi(c_3)h_2 \otimes \pi(c_2)$

$= \sum S(h_1)h_2\phi^{-1}(c_1)\phi(c_3) \otimes \pi(c_2)$

(because H is commutative)

$= \sum \varepsilon(h)\phi^{-1}(c_1)\phi(c_3) \otimes \pi(c_2) = 0$, q.e.d.

Hence, we have proved the following

Lemma 4.1. *There exists a k-linear map $F : C \to H \otimes C$, with $F(\pi(c)) = \sum \phi^{-1}(c_1)\phi(c_3) \otimes \pi(c_2)$ for any $c \in D$.*

Now, if C is a coalgebra, $Z(C)$ the cocentre of C, let $1^d : C \to Z(C)$ be the canonical (surjective) coalgebra homomorphism.
Hence $\sum 1^d(c_1) \otimes c_2 = \sum 1^d(c_2) \otimes c_1$ for any $c \in C$.

Lemma 4.2. *In the above situation, we have:*

$$\sum \phi(c_1)\phi^{-1}(c_3) \otimes 1^d(\pi(c_2)) \otimes \pi(c_4) = \sum \phi(c_2)\phi^{-1}(c_4) \otimes 1^d(\pi(c_3)) \otimes \pi(c_1)$$

for any $c \in D$.

Proof: Let $\varphi \in C^*$, and define $f_\varphi : D \to H$, $f_\varphi(c) = \sum \phi^{-1}(c_1)\phi(c_3)\varphi(\pi(c_2))$. It follows that $f_\varphi * \phi^{-1}(c) = \sum \phi^{-1}(c_1)\varphi(\pi(c_2))$ for any $c \in D$.
We have the map $F : C \to H \otimes C$, with $F(\pi(c)) = \sum \phi^{-1}(c_1)\phi(c_3) \otimes \pi(c_2)$ for any $c \in D$, so in this way we obtain a k-linear map $g_\varphi : C \to H$ with $f_\varphi(c) = g_\varphi(\pi(c))$ for any $c \in D$. Hence

$\sum \phi(c_1)\phi^{-1}(c_3)\varphi(\pi(c_4)) \otimes 1^d(\pi(c_2))$

$= \sum \phi(c_1)g_\varphi(\pi(c_3))\phi^{-1}(c_4) \otimes 1^d(\pi(c_2))$

$= \sum \phi(c_1)g_\varphi(\pi(c_2))\phi^{-1}(c_4) \otimes 1^d(\pi(c_3))$

(because $\sum 1^d(x_1) \otimes x_2 = \sum 1^d(x_2) \otimes x_1$ for any $x \in C$)

$= \sum \phi(c_1)\phi^{-1}(c_2)\phi(c_4)\varphi(\pi(c_3))\phi^{-1}(c_6) \otimes 1^d(\pi(c_5))$

$= \sum \phi(c_2)\phi^{-1}(c_4)\varphi(\pi(c_1)) \otimes 1^d(\pi(c_3))$

Since this equality is valid for any $\varphi \in C^*$, we obtain

$\sum \phi(c_1)\phi^{-1}(c_3) \otimes 1^d(\pi(c_2)) \otimes \pi(c_4) = \sum \phi(c_2)\phi^{-1}(c_4) \otimes 1^d(\pi(c_3)) \otimes \pi(c_1)$.

Proposition 4.3. *In the above situation, if we denote $\psi : C \to H \otimes C$, $\psi(\pi(c)) = \sum \phi(c_1)\phi^{-1}(c_3) \otimes \pi(c_2)$ for any $c \in D$, then there exists a k-linear map $\overline{\psi} : Z(C) \to H \otimes Z(C)$ with $\overline{\psi}(1^d(c)) = \sum \phi(c_1)\phi^{-1}(c_3) \otimes 1^d(\pi(c_2))$ for any $c \in C$.*

Proof: By [13], p.544, $Z(C) = e_{-C^e}(C)$, where $C^e = C^{cop} \otimes C$. By Proposition (2.2) of [13] the canonical map

$$\theta : C \to e_{-C^e}(C) \otimes C$$

is given by

$$\theta(c) = \sum 1^d(c_1) \otimes c_2$$

(for the definition of $e_{-C^e}(C)$ and the canonical map, we refer to [12]). By [12], 1.4, if W is a k-linear space and $\alpha : C \to W \otimes C$ is a C^e-right comodule homomorphism, then there exists a unique k-linear map $u : e_{-C^e}(C) \to W$ such that $\alpha = (u \otimes I) \circ \theta$. We shall take here $W = H \otimes Z(C)$; then, for $c \in C$, we denote $\psi(c) = \sum c^1 \otimes c^2$ and we take $\alpha : C \to [H \otimes Z(C)] \otimes C$,

$$\alpha(c) = \sum (c_1)^1 \otimes 1^d((c_1)^2) \otimes c_2$$

The C^e-right comodule structure of C is given by

$$\rho_C : C \to C \otimes C^e, \quad \rho_C(c) = \sum c_2 \otimes (c_1 \otimes c_3)$$

(see [13], p.538). The C^e-right comodule structure of $H \otimes Z(C) \otimes C$ is given by

$$\rho : H \otimes Z(C) \otimes C \to (H \otimes Z(C) \otimes C) \otimes C^e$$

$$\rho(h \otimes 1^d(c) \otimes d) = \sum h \otimes 1^d(c) \otimes d_2 \otimes d_1 \otimes d_3$$

We shall prove that α is a C^e-right comodule homomorphism; to see this, it is enough to show that $\rho \circ \alpha = (\alpha \otimes I) \circ \rho_C$ and then, by computation, it is enough to prove that

$$\sum (c_1)^1 \otimes 1^d((c_1)^2) \otimes c_2 = \sum (c_2)^1 \otimes 1^d((c_2)^2) \otimes c_1$$

for any $c \in C$, or equivalently

$$\sum \pi(c_1)^1 \otimes 1^d(\pi(c_1)^2) \otimes \pi(c_2) = \sum \pi(c_2)^1 \otimes 1^d(\pi(c_2)^2) \otimes \pi(c_1)$$

for any $c \in D$. But, for any $c \in D$, $\psi(\pi(c)) = \sum \phi(c_1)\phi^{-1}(c_3) \otimes \pi(c_2)$, hence the required equality follows using Lemma 4.2.
Therefore, there exists a unique k-linear map $u : Z(C) \to H \otimes Z(C)$ with $\alpha = (u \otimes I) \circ \theta$.
We have $(u \otimes I)(\theta(c)) = \sum u(1^d(c_1)) \otimes c_2$ for any $c \in C$. By applying $I \otimes \varepsilon$ we obtain $u(1^d(c)) = \sum \phi(c_1)\phi^{-1}(c_3) \otimes 1^d(\pi(c_2))$ and now we can define $\overline{\psi} = u$.

Proposition 4.4. *In the above situation, $\overline{\psi}$ defines a H-left comodule structure on $Z(C)$, and with this structure $Z(C)$ becomes a (cocommutative) H-comodule coalgebra.*

Proof: We shall prove first that $\overline{\psi}$ is a comodule structure; to see this, it is enough to prove that

$$\sum (1^d(c))^1 \otimes (1^d(c)^2)^1 \otimes (1^d(c)^2)^2 = \sum (1^d(c)^1)_1 \otimes (1^d(c)^1)_2 \otimes 1^d(c)^2$$

for any $c \in C$. We have:

$\sum (1^d(c))^1 \otimes (1^d(c)^2)^1 \otimes (1^d(c)^2)^2 = \sum c^1 \otimes 1^d(c^2)^1 \otimes 1^d(c^2)^2 =$

$\sum c^1 \otimes (c^2)^1 \otimes 1^d((c^2)^2)$

(because $\sum 1^d(c)^1 \otimes 1^d(c)^2 = \overline{\psi}(1^d(c)) = \sum c^1 \otimes 1^d(c^2)$)

For $c \in C$, the condition (TC) is

$\sum (c_1)^1 \alpha_1(c_2) \otimes ((c_1)^2)^1 \alpha_2(c_2) \otimes ((c_1)^2)^2 =$

$\sum \alpha_1(c_1)((c_2)^1)_1 \otimes \alpha_2(c_1)((c_2)^1)_2 \otimes (c_2)^2$

Now, taking $\varphi \in C^*$ and applying φ on the last position in the previous equality, we obtain two functions defined on C with values in $H \otimes H$; multiplying by convolution to the left with α^{-1}, we obtain, finally:

$\sum c^1 \otimes (c^2)^1 \otimes (c^2)^2 = \sum \alpha_1(c_1)((c_2)^1)_1 \alpha_1^{-1}(c_3) \otimes \alpha_2(c_1)((c_2)^1)_2 \alpha_2^{-1}(c_3) \otimes (c_2)^2$

Then

$\sum c^1 \otimes (c^2)^1 \otimes 1^d((c^2)^2)$

$= \sum \alpha_1(c_1)(1^d(c_2)^1)_1\alpha_1^{-1}(c_3) \otimes \alpha_2(c_1)(1^d(c_2)^1)_2\alpha_2^{-1}(c_3) \otimes 1^d(c_2)^2$

(because $\sum 1^d(x)^1 \otimes 1^d(x)^2 = \sum x^1 \otimes 1^d(x^2)$)

$=\sum \alpha_1(c_1)(1^d(c_3)^1)_1\alpha_1^{-1}(c_2) \otimes \alpha_2(c_1)(1^d(c_3)^1)_2\alpha_2^{-1}(c_2) \otimes 1^d(c_3)^2$

(because $\sum 1^d(x_1) \otimes x_2 = \sum 1^d(x_2) \otimes x_1$)

$=\sum(1^d(c)^1)_1 \otimes (1^d(c)^1)_2 \otimes 1^d(c)^2$

where the last equality follows because H is commutative.
Now, the fact that $Z(C)$ is a H-comodule coalgebra follows immediately, using the relations:

$\sum(1^d(c))_1 \otimes (1^d(c))_2 = \sum 1^d(c_1) \otimes 1^d(c_2)$ and

$\sum 1^d(c)^1 \otimes 1^d(c)^2 = \sum c^1 \otimes 1^d(c^2)$

for any $c \in C$.

Lemma 4.5. *In the above situation, if ϕ' is another cosection, then the coaction of H on $Z(C)$ induced by ϕ' (it is a strong coaction) is just $\overline{\psi}$, i.e. the coaction induced by ϕ.*

Proof: Let (φ, β) be the crossed cosystem induced by ϕ'. From Lemma 3.1 we know that $(\psi, \alpha) \sim (\varphi, \beta)$, so there exists $v : C \to H$, k-linear and convolution invertible such that $\varphi(c) = \sum v(c_1)(c_2)^1v^{-1}(c_3) \otimes (c_2)^2$ for any $c \in C$, where $\psi(c) = \sum c^1 \otimes c^2$. Therefore it is enough to prove that

$$\sum v(c_1)(c_2)^1v^{-1}(c_3) \otimes 1^d((c_2)^2) = \sum c^1 \otimes 1^d(c^2)$$

for any $c \in C$, and this follows immediately, using the relations

$\sum 1^d(c)^1 \otimes 1^d(c)^2 = \sum c^1 \otimes 1^d(c^2)$

$\sum 1^d(c_1) \otimes c_2 = \sum 1^d(c_2) \otimes c_1$

and the fact that H is commutative.

Remark 4.6. *By Proposition 3.9 and the proof of Lemma 4.5 it follows that if D'/C is a cleft coextension isomorphic to D/C, then the coaction of H on $Z(C)$ induced by D'/C equals the one induced by D/C. Hence, an isomorphism class of cleft coextensions [D/C] gives a unique left H-comodule coalgebra structure on $Z(C)$.*

Now, let H be a commutative Hopf algebra, B a cocommutative left H-comodule coalgebra with structure map $\rho : B \to H \otimes B$, $\rho(b) = \sum b^1 \otimes b^2$. In [6] the cohomology groups $Coalg - H^n(B, H)$ were defined; they are dual to the cohomology groups introduced by Sweedler in [11]. In the sequel, we use only $Coalg - H^2(B, H)$. If $v : B \to H$ is k-linear and convolution invertible, define a (k-linear and convolution invertible) map $D^1(v) : B \to H \otimes H$, by

$$D^1(v)(b) = \sum(b_1)^1v(b_2)v^{-1}(b_3)_1 \otimes v((b_1)^2)v^{-1}(b_3)_2$$

Then $Coalg - H^2(B, H) = Z^2(B, H)/B^2(B, H)$, where
$Z^2(B, H) =$

$$= \{\alpha : B \to H \otimes H,\ k-linear,\ convolution\ invertible,\ with\ (CU)\ and\ (C)\}$$

$B^2(B,H) =$

$$= \{D^1(v)/v : B \to H,\ k - linear,\ convolution\ invertible,\ with\ \varepsilon_H \circ v = \varepsilon_B\}.$$

Proposition 4.7. *Let H be a commutative Hopf algebra , D/C a cleft coextension, $\phi : D \to H$ a cosection and (ψ, α) the corresponding crossed cosystem. If $\Gamma : Z(C) \to H \otimes H$, let $\gamma : C \to H \otimes H$, $\gamma(c) = \Gamma(1^d(c))$. Then:*

1) If $\Gamma \in Z^2(Z(C), H)$, then $(\psi, \alpha * \gamma)$ is a crossed cosystem for H over C.

2) Conversely, if $\alpha' : C \to H \otimes H$ is k-linear and convolution invertible, and (ψ, α') is a crossed cosystem for H over C (with the same ψ), then there exists $\Gamma \in Z^2(Z(C), H)$ such that $\alpha' = \alpha * \gamma$.

3) If $\Gamma, \Gamma' \in Z^2(Z(C), H)$, then $(\psi, \alpha * \gamma) \sim (\psi, \alpha * \gamma')$ if and only if Γ and Γ' are cohomologous, i.e. there exists $v : Z(C) \to H$, k-linear, convolution invertible, with $\varepsilon_H \circ v = \varepsilon_{Z(C)}$, such that $\Gamma^{-1} * \Gamma' = D^1(v)$.

4) The map $\Gamma \mapsto (\psi, \alpha * \gamma)$ induces a bijection between $Coalg - H^2(Z(C), H)$ and the set of the equivalence classes of all those crossed cosystems for H over C which have ψ as weak coaction.

Proof: 1) Follows after a tedious (but straightforward) computation.
2) Define $\gamma : C \to H \otimes H, \gamma = \alpha^{-1} * \alpha'$. It is enough to show that there exists $\Gamma \in Z^2(Z(C), H)$ such that $\Gamma(1^d(c)) = \gamma(c)$ for any $c \in C$.

We had $\psi : C \to H \otimes C$, $\psi(\pi(c)) = \sum \phi(c_1)\phi^{-1}(c_3) \otimes \pi(c_2)$ for any $c \in D$ and

$F : C \to H \otimes C$, $F(\pi(c)) = \sum \phi^{-1}(c_1)\phi(c_3) \otimes \pi(c_2)$ for any $c \in D$. If $c \in C$ we

denote $\psi(c) = \sum c^1 \otimes c^2$ and $F(c) = \sum c_{-1} \otimes c_0$. Then, if $c \in D$ and $d = \pi(c)$, we have:

$\sum d^1(d^2)_{-1} \otimes (d^2)_0 = \sum \phi(c_1)\phi^{-1}(c_5)\phi^{-1}(c_2)\phi(c_4) \otimes \pi(c_3) =$

$1 \otimes \pi(c) = 1 \otimes d$

(because H is commutative).

Hence

$\sum x^1(x^2)_{-1} \otimes (x^2)_0 = 1 \otimes x \qquad (*)$

for any $x \in C$.

We have seen before that, since (ψ, α) is a crossed cosystem, we have

$\sum c^1 \otimes (c^2)^1 \otimes (c^2)^2 = \sum \alpha_1(c_1)((c_2)^1)_1\alpha_1^{-1}(c_3) \otimes \alpha_2(c_1)((c_2)^1)_2\alpha_2^{-1}(c_3) \otimes (c_2)^2$

But (ψ, α') is also a crossed cosystem, then

$\sum \alpha_1(c_1)((c_2)^1)_1\alpha_1^{-1}(c_3) \otimes \alpha_2(c_1)((c_2)^1)_2\alpha_2^{-1}(c_3) \otimes (c_2)^2 =$

$= \sum \alpha'_1(c_1)((c_2)^1)_1\alpha_1'^{-1}(c_3) \otimes \alpha'_2(c_1)((c_2)^1)_2\alpha_2'^{-1}(c_3) \otimes (c_2)^2 \qquad (**)$

for any $c \in C$.

Now, let $\varphi \in C^*$; we shall prove that $\sum \varphi(c_1)\gamma(c_2) = \sum \varphi(c_2)\gamma(c_1)$ for any $c \in C$.

$\sum \varphi(c_1)\gamma(c_2) = \sum \varphi(c_1)\alpha_1^{-1}(c_2)\alpha'_1(c_3) \otimes \alpha_2^{-1}(c_2)\alpha'_2(c_3)$

$$= \sum \alpha_1^{-1}(c_1)\alpha_1(c_2)((c_3)^1)_1(((c_3)^2)_{-1})_1\varphi(((c_3)^2)_0)\alpha_1^{-1}(c_4)\alpha_1'(c_5)\otimes$$

$$\otimes\alpha_2^{-1}(c_1)\alpha_2(c_2)((c_3)^1)_2(((c_3)^2)_{-1})_2\alpha_2^{-1}(c_4)\alpha_2'(c_5)$$

(applying (*) for $(c_1)^1((c_1)^2)_1$ instead of x)

$$= \sum \alpha_1^{-1}(c_1)\alpha_1(c_2)((c_3)^1)_1\alpha_1^{-1}(c_4)(((c_3)^2)_{-1})_1\varphi(((c_3)^2)_0)\alpha_1'(c_5)\otimes$$

$$\otimes\alpha_2^{-1}(c_1)\alpha_2(c_2)((c_3)^1)_2\alpha_2^{-1}(c_4)(((c_3)^2)_{-1})_2\alpha_2'(c_5)$$

(because H is commutative)

$$= \sum \alpha_1^{-1}(c_1)\alpha_1'(c_2)((c_3)^1)_1\alpha_1'^{-1}(c_4)(((c_3)^2)_{-1})_1\varphi(((c_3)^2)_0)\alpha_1'(c_5)\otimes$$

$$\otimes\alpha_2^{-1}(c_1)\alpha_2'(c_2)((c_3)^1)_2\alpha_2'^{-1}(c_4)(((c_3)^2)_{-1})_2\alpha_2'(c_5)$$

(applying (**) for c_1 instead of c)

$= \sum \alpha_1^{-1}(c_1)\alpha_1'(c_2)[(c_3)^1((c_3)^2)_{-1}]_1\varphi(((c_3)^2)_0)\otimes$
$\alpha_2^{-1}(c_1)\alpha_2'(c_2)[(c_3)^1((c_3)^2)_{-1}]_2$

(because H is commutative)

$= \sum \alpha_1^{-1}(c_1)\alpha_1'(c_2)\varphi(c_3)\otimes\alpha_2^{-1}(c_1)\alpha_2'(c_2) = \sum \gamma(c_1)\varphi(c_2)$

(applying (*))

Therefore we have $\sum \gamma(c_1)\otimes c_2 = \sum \gamma(c_2)\otimes c_1$ for any $c \in C$.

We shall define $f : C \to H\otimes H\otimes H$, $f(c) = \sum \gamma(c_1)\otimes c_2$. C is a C^e-right comodule with structure map $\rho_C : C \to C\otimes C^e$, $\rho_C(c) = \sum c_2\otimes c_1\otimes c_3$ and $H\otimes H\otimes C$ is a C^e right comodule with structure map $\rho : H\otimes H\otimes C \to H\otimes H\otimes C\otimes C^e$, $\rho(h\otimes g\otimes c) = \sum h\otimes g\otimes c_2\otimes c_1\otimes c_3$.
Using the relation $\sum \gamma(c_1)\otimes c_2 = \sum \gamma(c_2)\otimes c_1$, it is easy to see that f is a right comodule homomorphism. Now, from [12] ,1.4, there exists a unique k-linear map $u : Z(C) \to H\otimes H$ such that $f = (u\otimes I)\circ\theta$, where $\theta : C \to Z(C)\otimes C$, $\theta(c) = \sum 1^d(c_1)\otimes c_2$. Hence $\gamma(c) = u(1^d(c))$ for any $c \in C$.
Define $\Gamma = u$. We shall prove that $\Gamma \in Z^2(Z(C), H)$. Since (ψ, α') is a crossed cosystem, it appears , by Proposition 3.9, from a cleft coextension, say D'/C, in fact from a cosection $\phi' : D' \to H$. So, using the same proof, there exists $\Gamma' : Z(C) \to H\otimes H$, k-linear, with $\Gamma'(1^d(c)) = \gamma'(c)$ for any $c \in C$, where $\gamma' = \alpha'^{-1} * \alpha$, and then obviously Γ' is the convolution inverse of Γ. It remains to prove that Γ satisfies (CU) and (C). Let $\Gamma(1^d(c)) = \sum \Gamma_1(1^d(c))\otimes\Gamma_2(1^d(c)) = \gamma(c) = \sum \gamma_1(c)\otimes\gamma_2(c)$.
The condition (CU) for Γ is trivial, because α and α' satisfy (CU). We shall prove now the condition (C).
Since $\sum 1^d(c_1)\otimes c_2 = \sum 1^d(c_2)\otimes c_1$ and H is commutative, we have that $\gamma = \alpha' * \alpha^{-1}$. Then:

$\sum \gamma_1(c_1)(\gamma_1(c_2))_1\otimes\gamma_2(c_1)(\gamma_1(c_2))_2\otimes\gamma_2(c_2) =$

$= \sum \alpha_1^{-1}(c_1)\alpha_1'(c_2)[\alpha_1'(c_3)\alpha_1^{-1}(c_4)]_1\otimes$

$$\otimes\alpha_2^{-1}(c_1)\alpha_2'(c_2)[\alpha_1'(c_3)\alpha_1^{-1}(c_4)]_2\otimes\alpha_2'(c_3)\alpha_2^{-1}(c_4)$$

(using $\alpha' * \alpha^{-1} = \alpha^{-1} * \alpha'$)

$$= \sum \alpha_1^{-1}(c_1)(c_2)^1\alpha_1'(c_3)\alpha_1^{-1}(c_4)_1 \otimes$$

$$\otimes \alpha_2^{-1}(c_1)\alpha_1'((c_2)^2)\alpha_2'(c_3)_1\alpha_1^{-1}(c_4)_2 \otimes \alpha_2'((c_2)^2)\alpha_2'(c_3)_2\alpha_2^{-1}(c_4)$$

(using condition (C) for α')

$$= \sum \alpha_1^{-1}(c_1)(c_2)^1\alpha_1(c_3)\gamma_1(c_4)\alpha_1^{-1}(c_5)_1 \otimes$$

$$\otimes \alpha_2^{-1}(c_1)\alpha_1(((c_2)^2)_1)\gamma_1(((c_2)^2)_2)\alpha_2(c_3)_1\gamma_2(c_4)_1\alpha_1^{-1}(c_5)_2 \otimes$$

$$\otimes \alpha_2(((c_2)^2)_1)\gamma_2(((c_2)^2)_2)\alpha_2(c_3)_2\gamma_2(c_4)_2\alpha_2^{-1}(c_5)$$

(using $\alpha' = \alpha * \gamma$)

$$= \sum \alpha_1^{-1}(c_1)(c_2)^1\alpha_1(c_3)\Gamma_1(1^d(c_4))\alpha_1^{-1}(c_5)_1 \otimes$$

$$\otimes \alpha_2^{-1}(c_1)\alpha_1(((c_2)^2)_1)\Gamma_1(1^d(((c_2)^2)_2))\alpha_2(c_3)_1\Gamma_2(1^d(c_4))_1\alpha_1^{-1}(c_5)_2 \otimes$$

$$\otimes \alpha_2(((c_2)^2)_1)\Gamma_2(1^d(((c_2)^2)_2))\alpha_2(c_3)_2\Gamma_2(1^d(c_4))_2\alpha_2^{-1}(c_5)$$

$$= \sum \alpha_1^{-1}(c_1)(c_2)^1(c_3)^1\alpha_1(c_4)\Gamma_1(1^d(c_5))\alpha_1^{-1}(c_6)_1 \otimes$$

$$\otimes \alpha_2^{-1}(c_1)\alpha_1((c_2)^2)\Gamma_1(1^d((c_3)^2))\alpha_2(c_4)_1\Gamma_2(1^d(c_5))_1\alpha_1^{-1}(c_6)_2 \otimes$$

$$\otimes \alpha_2((c_2)^2))\Gamma_2(1^d((c_3)^2))\alpha_2(c_4)_2\Gamma_2(1^d(c_5))_2\alpha_2^{-1}(c_6)$$

(using the definition of the weak coaction for c_2)

$$= \sum \alpha_1^{-1}(c_1)(1^d(c_2))^1(c_3)^1\alpha_1(c_4)\Gamma_1(1^d(c_5))\alpha_1^{-1}(c_6)_1 \otimes$$

$$\otimes \alpha_2^{-1}(c_1)\alpha_1((c_3)^2)\alpha_2(c_4)_1\Gamma_1(1^d(c_2)^2)\Gamma_2(1^d(c_5))_1\alpha_1^{-1}(c_6)_2 \otimes$$

$$\otimes \Gamma_2(1^d(c_2)^2)\alpha_2((c_3)^2))\alpha_2(c_4)_2\Gamma_2(1^d(c_5))_2\alpha_2^{-1}(c_6)$$

(because: by Proposition 4.3 we have $\sum 1^d(c)^1 \otimes 1^d(c)^2 = \sum c^1 \otimes 1^d(c^2)$; we apply this here for c_3. Then we have $\sum 1^d(c_3) \otimes c_2 = \sum 1^d(c_2) \otimes c_3$ and H is commutative)

$$= \sum \alpha_1^{-1}(c_1)(1^d(c_2))^1\alpha_1(c_3)\alpha_1(c_4)_1\Gamma_1(1^d(c_5))\alpha_1^{-1}(c_6)_1 \otimes$$

$$\otimes \alpha_2^{-1}(c_1)\Gamma_1(1^d(c_2)^2)\alpha_2(c_3)\alpha_1(c_4)_2\Gamma_2(1^d(c_5))_1\alpha_1^{-1}(c_6)_2 \otimes$$

$$\otimes \Gamma_2(1^d(c_2)^2)\alpha_2(c_4)\Gamma_2(1^d(c_5))_2\alpha_2^{-1}(c_6)$$

(applying (C) for α)

$$= \alpha_1^{-1}(c_1)\alpha_1(c_2)(c_3)^1\gamma_1(c_4)\alpha_1(c_5)_1\alpha_1^{-1}(c_6)_1 \otimes$$

$$\otimes \alpha_2^{-1}(c_1)\alpha_2(c_2)\Gamma_1(1^d(c_2)^2)\gamma_2(c_4)\alpha_1(c_5)_2\alpha_1^{-1}(c_6)_2 \otimes$$

$$\otimes \Gamma_2(1^d(c_2)^2)\gamma_2(c_5)_2\alpha_2(c_5)\alpha_2^{-1}(c_6)$$

(because $\sum 1^d(c_2) \otimes c_3 = \sum 1^d(c_3) \otimes c_2$ and $\sum 1^d(c_5) \otimes c_4 = \sum 1^d(c_4) \otimes c_5$)

$$= \sum((1^d(c)_1)^1\Gamma_1(1^d(c)_2) \otimes \Gamma_1((1^d(c)_1)^2)\Gamma_2(1^d(c)_2)_1 \otimes \Gamma_2((1^d(c)_1)^2)\Gamma_2(1^d(c)_2)_2$$

hence Γ satisfies (C).

3) Suppose $(\psi, \alpha * \gamma) \sim (\psi, \alpha * \gamma')$. We shall prove that Γ and Γ' are cohomologous.

From the above equivalence, there exists $v : C \to H$, k-linear, convolution invertible, with $\varepsilon_H \circ v = \varepsilon_C$ such that (denote $\psi(c) = \sum c^1 \otimes c^2$):

$$\sum c^1 \otimes c^2 = \sum v(c_1)(c_2)^1 v^{-1}(c_3) \otimes (c_2)^2 \tag{10}$$

$$(\alpha * \gamma')(c) = \sum v(c_1)(c_2)^1(\alpha * \gamma)_1(c_3)v^{-1}(c_4)_1 \otimes v((c_2)^2)(\alpha * \gamma)_2(c_3)v^{-1}(c_4)_2 \tag{11}$$

First, we shall prove that $\sum v(c_1) \otimes c_2 = \sum v(c_2) \otimes c_1$ for any $c \in C$. Let $\varphi \in C^*$; we denoted $F(\pi(c)) = \sum c_{-1} \otimes c_0$, $F : C \to H \otimes C$. Then, for $c \in C$, we have

$\sum v(c_1)\varphi(c_2) = \sum v(c_1)(c_2)^1((c_2)^2)_{-1}\varphi(((c_2)^2)_0)v^{-1}(c_3)v(c_4)$

(because $\sum c^1(c^2)_{-1} \otimes (c^2)_0 = 1 \otimes c$)

$= \sum (c_1)^1((c_1)^2)_{-1}\varphi(((c_1)^2)_0)v(c_2)$

(applying the fact that H is commutative and (10))

$= \sum \varphi(c_1)v(c_2)$

Hence, $\sum v(c_1) \otimes c_2 = \sum v(c_2) \otimes c_1$.

Now, define $f : C \to H \otimes C$, $f(c) = \sum v(c_1) \otimes c_2$. The equality proved above says that f is a right C^e-comodule homomorphism. Therefore, there exists a unique k-linear map $u : Z(C) \to H$ such that $(u \otimes I) \circ \theta = f$, where θ is the canonical map. So, $u(1^d(c)) = v(c)$, and therefore u is convolution invertible.
If we denote $A : Z(C) \to H \otimes H$, $A(1^d(c)) = \alpha(c)$, then A is convolution invertible and from (11) we obtain immediately $A * \Gamma' = (A * \Gamma) * D^1(u)$, hence $\Gamma' = \Gamma * D^1(u)$ q.e.d.
Conversely, if Γ and Γ' are cohomologous, we can prove in a similar way that $(\psi, \alpha * \gamma) \sim (\psi, \alpha * \gamma')$.
4) Is a direct consequence of 1), 2) and 3).

Remark 4.8. *In the conditions of the above theorem, if there exists a map* $A : Z(C) \to H \otimes H$, *k-linear, such that* $A(1^d(c)) = \alpha(c)$ *for any* $c \in C$, *then* $A \in Z^2(Z(C), H \otimes H)$, *and (TC) implies that* C *is a left* H*-comodule coalgebra via* ψ. *The pair* (ψ, α_0) , *where* $\alpha_0(c) = \varepsilon(c)1_H \otimes 1_H$, *is also a crossed cosystem for* H *over* C. *Therefore, by 3) we obtain that* B/C *is* H*-smash if and only if* $A \in B^2(Z(C), H)$.

Remark 4.9. If H is a commutative Hopf algebra, then, for $B = k$, the cohomology groups $coalg - H^n(B, H)$ are also known under the name Harrison cohomology groups. It is known (see [2], Th.3.4) that the second Harrison cohomology group is isomorphic to the group of Galois coobjects with normal basis. Recall that a Galois coobject with normal basis is a right H-module coalgebra C satisfying the following properties:
1) the map $\delta : C \otimes H \to C \otimes C$, $\delta(c \otimes h) = \sum c_1 \otimes c_2 h$ is an isomorphism
2) H and C are isomorphic as right H-modules
(see [2], [3], [9]).
The group operation is the tensor product over H (see [2], Th.2.3). We can conclude that a cleft H-coextension of k is an H-Galois coobject with normal basis.

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