Extending the Thas-Walker construction

Rolf Riesinger

Abstract

A spread $\mathcal{S}$ of a Pappian projective 3-space admits a *regulization* Σ, if Σ is a collection of reguli contained in $\mathcal{S}$ and if each element of $\mathcal{S}$, except at most two lines, is contained either in exactly one regulus of Σ or in all reguli of Σ. Replacement of each regulus of Σ by its complementary regulus (exceptional lines remain unchanged) yields the *complementary congruence* $\mathcal{S}^{\mathrm{c}}_{\Sigma}$ *of* $\mathcal{S}$ *with respect to* Σ. If $\mathcal{S}^{\mathrm{c}}_{\Sigma}$ belongs to a single linear complex of lines, then Σ is called a *unisymplecticly complemented* regulization. For spreads with unisymplecticly complemented regulization we give a construction which can be seen as an extension of the well-known Thas-Walker construction of spreads admitting net generating regulizations.

1 Introduction

Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a Pappian projective 3-space with point set $\mathcal{P}$ and line set $\mathcal{L}$. We are going to investigate spreads composed of reguli and at most two exceptional lines. Therefore we standardize by defining: A *proper regulus* $\mathcal{R}$ is the set of lines meeting three mutually skew lines; the directrices of $\mathcal{R}$ form the complementary (opposite) regulus $\mathcal{R}^c$; if $x \in \mathcal{L}$, then $\{x\}$ is called an *improper regulus*; $\{x\}^c := \{x\}$.

Definition 1. *Let* $\mathcal{S}$ *be a spread of* Π *and let* Σ *be a collection of (proper or improper) reguli contained in* $\mathcal{S}$. *We call* Σ *a* regulization *of* $\mathcal{S}$, *if the following hold:*

(RZ1) *Each line of* $\mathcal{S}$ *belongs either to exactly one regulus of* Σ *or to all reguli of* Σ.

(RZ2) *There are at most two improper reguli in* Σ.

Received by the editors June 1997.
Communicated by J. Thas.
1991 *Mathematics Subject Classification* : 51A40.
Key words and phrases : spread, flock.

The set $\cup(\mathcal{R}^c|\mathcal{R}\in\Sigma) =: \mathcal{S}^c_\Sigma$ *is named* complementary congruence of $\mathcal{S}$ with respect to Σ. *If* $\mathcal{S}^c_\Sigma$ *belongs to a linear complex of lines, then we say that* Σ *is a* symplecticly complemented *regulization. If* $\mathcal{S}^c_\Sigma$ *belongs to a single linear complex of lines, then* Σ *is called a* unisymplecticly complemented *regulization, otherwise* multisymplecticly complemented. *If* $\mathcal{S}^c_\Sigma$ *is a non-degenerate linear congruence of lines, shortly a net (of lines), then we call* Σ *a* net generating *regulization, in particular, a* hyperbolic *or* parabolic *or* elliptic *regulization depending on the type of the* complementary net $\mathcal{S}^c_\Sigma$. *We say that* Σ *is a* preparabolic *regulization, if there exists a parabolic net* $\mathcal{Z}$ *with axis* z *such that* $\mathcal{S}^c_\Sigma = \mathcal{Z}\setminus\{z\}$.

For spreads with net generating regulizations and references to this subject, see [7] and [8]. Clearly, each net generating and each preparabolic regulization is multisymplecticly complemented. For the real projective 3-space $\mathrm{PG}(3,\mathbb{R})$ an example of a non-regular spread admitting a unisymplecticly complemented regulization is given in [7, (4.1,6)].

Let λ be the well-known Klein mapping of $\mathcal{L}$ onto the Klein quadric H_5 which is embedded into a projective 5-space Π_5 with point set $\mathcal{P}_5$; cf. e.g. [5]. If $\mathcal{R}$ is a proper or improper regulus, then $\lambda(\mathcal{R})$ is an irreducible conic or a point. For obvious reasons, we speak of *proper* or *improper conics*. If $\mathcal{S}$ is a spread of Π with the net generating regulization Ψ, then $\{\lambda(\mathcal{R}^c)|\mathcal{R}\in\Psi\}$ is a flock of the quadric $\lambda(\mathcal{S}^c_\Psi)\subset H_5$; cf. [7, Prop. 3.1] and [7, Def. 3.1].

Recall the Thas-Walker construction [7, Prop. 3.3]: If $\mathcal{F}$ is a flock of a quadric Q with $Q\subset H_5$, then $\cup((\lambda^{-1}(k))^c|k\in\mathcal{F})$ is a spread of Π with the net generating regulization $\{(\lambda^{-1}(k))^c|k\in\mathcal{F}\}$. This construction was discovered independently by M. Walker [11] and J. A. Thas (unpublished).

In Section 3 we start with a spread $\mathcal{S}$ of Π admitting a unisymplecticly complemented regulization Ω and investigate the set $\{\lambda(\mathcal{R}^c)|\mathcal{R}\in\Omega\} =: \mathcal{E}$ of conics. By statement (S3) of Section 2, $\mathcal{S}^c_\Omega$ belongs to a general linear complex $\mathcal{G}$ of lines. Each conic of $\mathcal{E}$ is contained in the quadric $\lambda(\mathcal{G})\subset H_5$. We sum up the properties of $\lambda(\mathcal{G})$ in

Definition 2. *A hyperquadric* L_4 *of a Pappian projective* 4*-space is called* Lie quadric*, if* L_4 *has no vertex and if* L_4 *contains a line. A generatrix of* L_4 *is a line* g *with* $g\subset L_4$.

In the Proof of Proposition 1 we shall find that $\mathcal{E}$ is a "flockoid" of the Lie quadric $\lambda(\mathcal{G})$; we define the concept "flockoid", as follows

Definition 3. *A collection* $\mathcal{D}$ *of conics contained in a Lie quadric* L_4 *of a Pappian projective* 4*-space is called a* flockoid *of* L_4*, if the following two conditions hold:*

(FD1) *For each generatrix* g *of* L_4 *there exists exactly one conic* $k\in\mathcal{D}$ *with* $g\cap k\neq\emptyset$.

(FD2) *There are at most two improper conics in* $\mathcal{D}$.

The extended Thas-Walker construction starts with a flockoid $\mathcal{D}$ of a Lie quadric $L_4\subset H_5$. Then $\cup((\lambda^{-1}(k))^c|k\in\mathcal{D})$ is a spread of Π admitting the regulization $\{(\lambda^{-1}(k))^c|k\in\mathcal{D}\}$ which is either unisymplecticly complemented or elliptic; cf. Proposition 2. Each flock of an elliptic quadric Q_e can be interpreted as flockoid of a Lie quadric L_4 containing Q_e; cf. Remark 9. Note, a flock of a quadric Q covers

Q, but a flockoid of a Lie quadric L_4 is no covering of L_4. By $\mathbb{K}$ we denote the (commutative) coordinatizing field of Π, i.e., $\Pi = \mathrm{PG}(3, \mathbb{K})$. We combine Remark 7 and the Propositions 1 and 2 and get

Theorem 1. *To each spread of* $\mathrm{PG}(3, \mathbb{K})$ *with a unisymplecticly complemented or an elliptic regulization there corresponds a flockoid of a Lie quadric contained in the Klein quadric of* $\Pi_5 = \mathrm{PG}(5, \mathbb{K})$*, and vice versa.*

In Section 4 we state further properties of the extended Thas-Walker construction. The present paper will be continued by [9] wherein we apply the Thas-Walker construction to get topological spreads with unisymplecticly complemented regulization.

2 Preliminaries

If $\mathcal{S}$ is a spread of Π and Σ an arbitrary regulization of $\mathcal{S}$, then each point of Π is incident with at least one line of $\mathcal{S}_\Sigma^c$ and $\mathcal{S}_\Sigma^c$ contains at least one proper regulus. Thus $\mathcal{S}_\Sigma^c$ cannot be part of a degenerate linear congruence $\mathcal{C}$ of lines since such a $\mathcal{C}$ consists of all lines meeting two intersecting lines. Consequently,

(S1) *Each multisymplecticly complemented regulization is either net generating or preparabolic, and vice versa.*

If $\mathcal{S}_\Sigma^c$ belongs to a special linear complex of lines, then Σ is hyperbolic, parabolic or preparabolic by virtue of [7, Remark 2.7]. As an immediate consequence we obtain the following two statements.

(S2) *Let* $\mathcal{S}$ *be a spread of* Π *and let* Ω *be a symplecticly complemented regulization of* $\mathcal{S}$*. Then there exists at least one general linear complex* $\mathcal{G}$ *of lines with* $\mathcal{S}_\Omega^c \subset \mathcal{G}$*.*

(S3) *Let* $\mathcal{S}$ *be a spread of* Π *and let* Ω *be a unisymplecticly complemented regulization of* $\mathcal{S}$*. Then the linear complex* $\mathcal{H}$ *of lines with* $\mathcal{S}_\Omega^c \subset \mathcal{H}$ *is general.*

If Π_n is an arbitrary n-dimensional projective space, then the set of all subspaces of Π_n is a lattice with respect to the operations $\cap$ and $\vee$; we write $\mathrm{Lat}(\Pi_n)$ for this lattice and $\mathcal{P}_n$ for the point set of Π_n. By [7, Theorem 2.8] (compare also [3, Corollary 5.7]), a spread with net generating regulization is also a dual spread; we generalize this result in

Theorem 2. *Let* $\mathcal{S}$ *be a spread of* Π *and let* Φ *be a covering of* $\mathcal{S}$ *by (proper or improper) reguli. If* $\cup(\mathcal{R}^c | \mathcal{R} \in \Phi)$ *is contained in a general linear complex* $\mathcal{G}$ *of lines, then* $\mathcal{S}$ *is also a dual spread.*

Proof. The null polarity γ associated with $\mathcal{G}$ is an antiautomorphism of $\mathrm{Lat}(\Pi)$ fixing $\mathcal{G}$ elementwise. If $\mathcal{X}$ is an arbitrary regulus of Φ, then $\mathcal{X}^c \subset \mathcal{G}$ implies $\gamma(\mathcal{X}^c) = \mathcal{X}^c$. Consequently, $\gamma(\mathcal{X}) = \mathcal{X}$ for all $\mathcal{X} \in \Phi$. Therefore $\gamma(\mathcal{S}) = \mathcal{S}$ since $\mathcal{S}$ is covered by the reguli of Φ. As $\mathcal{S}$ is a spread, so $\gamma(\mathcal{S})$ is a dual spread. ∎

Corollary 1. *If a spread* $\mathcal{S}$ *of* Π *admits a symplecticly complemented regulization, then* $\mathcal{S}$ *is also a dual spread.*

A spread $\mathcal{S}$ of Π is called *symplectic*, if $\mathcal{S}$ belongs to a linear complex of lines.

Corollary 2. *A symplectic spread $\mathcal{S}$ of Π is also a dual spread.*

Proof. Let $\mathcal{H}$ be a linear complex with $\mathcal{S} \subset \mathcal{H}$. By [7, Remark 4.1.3], $\mathcal{H}$ is general. Hence $\mathcal{S}$ and the collection $\Phi_0 := \{\{x\}|x \in \mathcal{S}\}$ of improper reguli satisfy the assumptions of Theorem 2. ∎

In connection with the Klein mapping λ we often use Plücker coordinates. We may assume that $\Pi = \mathrm{PG}(3, \mathbb{K})$ and $\Pi_5 = \mathrm{PG}(5, \mathbb{K})$ are the projective spaces on $\mathbb{K}^4$ and $\mathbb{K}^4 \wedge \mathbb{K}^4$, respectively, and that λ maps $\mathbf{c}\mathbb{K} \vee \mathbf{d}\mathbb{K} \in \mathcal{L}$ onto $(\mathbf{c} \wedge \mathbf{d})\mathbb{K} \in \mathcal{P}_5$. The standard basis $\mathbf{B}$ of $\mathbb{K}^4$ yields the ordered basis $(\mathbf{p}_0, \dots, \mathbf{p}_5) =: \mathbf{B}_5$ of $\mathbb{K}^4 \wedge \mathbb{K}^4$ with

$$\mathbf{p}_0 := \mathbf{b}_0 \wedge \mathbf{b}_1,\ \mathbf{p}_1 := \mathbf{b}_0 \wedge \mathbf{b}_2,\ \mathbf{p}_2 := \mathbf{b}_0 \wedge \mathbf{b}_3,\ \mathbf{p}_3 := \mathbf{b}_2 \wedge \mathbf{b}_3,$$
$$\mathbf{p}_4 := \mathbf{b}_3 \wedge \mathbf{b}_1,\ \mathbf{p}_5 := \mathbf{b}_1 \wedge \mathbf{b}_2.$$

Thus

$$H_5 = \{\mathbf{p}\mathbb{K} \in \mathcal{P}_5 |\ \mathbf{p} = \sum_{k=0}^{5} \mathbf{p}_k p_k \text{ and } p_0p_3 + p_1p_4 + p_2p_5 = 0\}. \tag{1}$$

Next we give some properties of Lie quadrics.

Remark 1. Let L_4 be a Lie quadric of $\Pi_4 = \mathrm{PG}(4, \mathbb{K})$. We may assume that Π_4 is the projective space on $\mathbb{K}^5$. By [10, (7.40), (7.41), (7.49)] there exists a basis $(\mathbf{a}_0, \dots, \mathbf{a}_4)$ of $\mathbb{K}^5$ such that

$$L_4 = \{\mathbf{x}\mathbb{K} \in \mathcal{P}_4 |\ \mathbf{x} = \sum_{k=0}^{4} \mathbf{a}_k x_k \text{ and } \quad x_0x_3 + x_1x_4 - x_2^2 = 0\}. \tag{2}$$

This shows that in Π_4 there exists an essentially unique Lie quadric.

Remark 2. Throughout this paper, the polarities associated with a Lie quadric L_4 and with a Klein quadric H_5 are denoted by π_4 and π_5, respectively. From (1) we deduce that π_5 always is an antiautomorphism of $\mathrm{Lat}(\Pi_5)$. Yet, π_4 is an antiautomorphism of $\mathrm{Lat}(\Pi_4)$ if, and only if, $\mathrm{Char}\,\mathbb{K} \neq 2$.

Remark 3. Let H_5 be the Klein quadric of $\mathrm{PG}(5, \mathbb{K})$ and let U be a hyperplane of $\mathrm{PG}(5, \mathbb{K})$ which is not tangent to H_5. Then $H_5 \cap U$ is a Lie quadric.

Remark 4. From Remark 1 and 3 we deduce that each Lie quadric of $\mathrm{PG}(4, \mathbb{K})$ is embeddable into the Klein quadric of $\mathrm{PG}(5, \mathbb{K})$.

Remark 5. Let L_4 be a Lie quadric of an arbitrary Pappian projective 4-space Π_4. A simple application of Witt's theorem (cf. e.g. [2, p.376]) shows that the group $\mathrm{Aut}\, L_4 := \{\xi \in \mathrm{P\Gamma L}(\Pi_4) | \xi(L_4) = L_4\}$ operates transitively both on the points of L_4 and on the set of all generatrices of L_4.

Lemma 1. *Let L_4 be a Lie quadric of an arbitrary Pappian projective 4-space.*
(i) *If $P \in L_4$, then the intersection of L_4 and the tangent hyperplane $\pi_4(P)$ of L_4 at P is a quadratic cone ("tangent cone of L_4 at P").*
(ii) *If g is a generatrix of L_4, then $L_4 \cap \pi_4(g) = g$.*

(iii) *If the intersection of a plane α and L_4 consists of a single point, say P, then $\alpha \subset \pi_4(P)$ and the tangent cone of L_4 at P has no generatrix in α.*
(iv) *There exists a plane α with $\#(\alpha \cap L_4) = 1$ if, and only if, there exist $p, q \in \mathbb{K}$ such that $x^2 + qx \neq p$ for all $x \in \mathbb{K}$.*

We leave the proof of Lemma 1 to the reader.

Remark 6. Let L_4 and $\widetilde{L}_4$ be Lie quadrics contained in the Klein quadric H_5 of $\mathrm{PG}(5, \mathbb{K})$. By Remark 1 and the theorem of Witt, there exists a collineation κ of $\mathrm{PG}(5, \mathbb{K})$ with $\kappa(L_4) = \widetilde{L}_4$ and $\kappa(H_5) = H_5$.

Lemma 2. *Let L_4 be a Lie quadric which belongs to the Klein quadric H_5 of $\Pi_5 = \mathrm{PG}(5, \mathbb{K})$. If a plane α of $\operatorname{span} L_4$ intersects L_4 in a single point, say P, then $\pi_5(\alpha) \cap H_5 = \{P\}$.*

Proof. We may assume that $\Pi_5 = \mathrm{PG}(5, \mathbb{K})$ is the projective space on $\mathbb{K}^4 \wedge \mathbb{K}^4$. In $\mathbb{K}^4 \wedge \mathbb{K}^4$ we change coordinates according to

$$p_j = p'_j \quad (j = 0, ..., 4), \qquad p_5 = -p'_2 + p'_5 \tag{3}$$

and denote the corresponding basis by $(\mathbf{p}'_0, ..., \mathbf{p}'_5)$. From (1) follows

$$H_5 = \{\mathbf{p}\mathbb{K} \in \mathcal{P}_5 | \quad \mathbf{p} = \sum_{k=0}^{5} \mathbf{p}'_k p'_k \quad \text{and} \quad p'_0 p'_3 + p'_1 p'_4 - {p'_2}^2 + p'_2 p'_5 = 0\}. \tag{4}$$

The hyperplane η with $p'_5 = 0$ is not tangent to H_5. By Remark 5 and 6, we may assume that L_4 is the intersection of H_5 and η, and that $P = \mathbf{p}'_0\mathbb{K}$. There must be $a_1, a_2 \in \mathbb{K}$ such that $p'_5 = p'_3 = a_1 p'_1 + a_2 p'_2 + p'_4 = 0$ describes α and such that $x^2 + a_2 x + a_1 \neq 0$ for all $x \in \mathbb{K}$. The plane $\pi_5(\alpha)$ is spanned by $(\mathbf{p}'_2 + \mathbf{p}'_5 2)\mathbb{K} =: P_1$, $\mathbf{p}'_0\mathbb{K}$, and $(\mathbf{p}'_1 + \mathbf{p}'_4 a_1 + \mathbf{p}'_5 a_2)\mathbb{K} =: P_2$. Because of $\mathbf{p}'_0\mathbb{K} \in \alpha \Rightarrow \pi_5(\alpha) \subset \pi_5(\mathbf{p}'_0\mathbb{K})$, the determination of $\pi_5(\alpha) \cap H_5$ is equivalent to finding $(P_1 \vee P_2) \cap H_5$ and, consequently, equivalent solving the equation $x^2 + a_2 x + a_1 = 0$. ■

3 The extended Thas-Walker construction

This Section generalizes [7, Section 3]. In the subsequent, the star of lines with vertex A is denoted by $\mathcal{L}[A] := \{x \in \mathcal{L} | A \in x\}$; let α be a plane, then the set of lines $\mathcal{L}[\alpha] := \{x \in \mathcal{L} | x \subset \alpha\}$ is called a ruled plane. If $A \in \alpha$, then $\mathcal{L}[A, \alpha] := \mathcal{L}[A] \cap \mathcal{L}[\alpha]$ is a pencil of lines.

Proposition 1. *Let $\mathcal{S}$ be a spread of Π and let Ω be a unisymplecticly complemented regulization of $\mathcal{S}$. Then $\{\lambda(\mathcal{R}^c) | \mathcal{R} \in \Omega\} =: \mathcal{D}$ is a flockoid of a uniquely determined Lie quadric $L_4 \subset H_5$.*

Proof. Clearly, (RZ2) implies (FD2).

We consider $i(\Omega) := \#(\cap(\mathcal{X} | \mathcal{X} \in \Omega)) \in \{0, 1, 2\}$, cf. [7, (2,1) and Remark 2.4]. First we show $i(\Omega) = 0$. Assume, to the contrary, $i(\Omega) \in \{1, 2\}$ then, by [7, Remarks 2.5 and 2.6], Ω is a parabolic or preparabolic ($i(\Omega) = 1$) or a hyperbolic

($i(\Omega) = 2$) regulization. From statement (S1) of Section 2 follows that Ω is a multi-symplecticly complemented regulization, a contradiction to the hypothesis.

By statement (S3), the linear complex $\mathcal{G}$ of lines with $\mathcal{S}^c_\Omega \subset \mathcal{G}$ is general, hence the conics of $\mathcal{D}$ are contained in the Lie quadric $\lambda(\mathcal{G}) \subset H_5$. By γ we denote the null polarity associated with $\mathcal{G}$. Let g be an arbitrary generatrix of $\lambda(\mathcal{G})$, then $\lambda^{-1}(g)$ is a pencil $\mathcal{L}[A, \gamma(A)]$ of lines. If $\lambda(\mathcal{R}^c)$ is a conic of $\mathcal{D}$ with $g \cap \lambda(\mathcal{R}^c) \neq \emptyset$, then the regulus $\mathcal{R}^c$ contains a line of $\mathcal{L}[A, \gamma(A)]$ and, consequently, $\mathcal{R}^c$ has a unique directrix $d \in \mathcal{R} \subset \mathcal{S}$ incident with $\gamma(A)$. By Corollary 1, $\mathcal{L}[\gamma(A)]$ and $\mathcal{S}$ have a single line $s_0 = d$ in common. Because of $i(\Omega) = 0$ and (RZ1), in Ω there exists exactly one regulus $\mathcal{R}_d$ with $d \in \mathcal{R}_d$. Conversely, $d \in \mathcal{R}_d$ and $d \subset \gamma(A)$ imply that there is exactly one line $h \in \mathcal{R}^c_d$ incident with $\gamma(A)$, and from $\mathcal{R}^c_d \subset \mathcal{S}^c_\Omega \subset \mathcal{G}$ we deduce $h \in \mathcal{L}[A, \gamma(A)]$ and $\lambda(h) \in g \cap \lambda(\mathcal{R}^c_d)$ with $\lambda(\mathcal{R}^c_d) \in \mathcal{D}$ because of $\mathcal{R}_d \in \Omega$. Thus $\mathcal{D}$ is a flockoid of the Lie quadric $\lambda(\mathcal{G})$. ■

Remark 7. *Let $\mathcal{S}$ be a spread of Π and let Ω be an elliptic regulization of $\mathcal{S}$. Then there exists a Lie quadric L_4 of H_5 such that $\{\lambda(\mathcal{R}^c) | \mathcal{R} \in \Omega\} =: \mathcal{D}$ is a flockoid of L_4.*

Proof. (a) There exists a general linear complex $\mathcal{G}$ of lines which contains the elliptic net $\mathcal{S}^c_\Omega$. The Lie quadric $\lambda(\mathcal{G})$ contains the elliptic quadric $\lambda(\mathcal{S}^c_\Omega)$ and $\operatorname{span} \lambda(\mathcal{S}^c_\Omega)$ is a hyperplane of the 4-space $\operatorname{span} \lambda(\mathcal{G})$. By [7, Proposition 3.1], $\mathcal{D}$ is a flock of $\lambda(\mathcal{S}^c_\Omega)$.

(b) An arbitrary generatrix g of $\lambda(\mathcal{G})$ has exactly one common point G with $\operatorname{span} \lambda(\mathcal{S}^c_\Omega)$ and $G \in \lambda(\mathcal{S}^c_\Omega)$. In the flock $\mathcal{D}$ there exists a unique conic k containing G. Thus (FD1) is valid for $\mathcal{D}$ and $\lambda(\mathcal{G})$. ■

Remark 8. Remark 7 does not hold true for a hyperbolic, parabolic or preparabolic regulization Ω. Part (a) of the above Proof can be done, mutatis mutandis. Part (b) splits into two cases. If the generatrix g does not belong to the hyperbolic quadric resp. quadratic cone $\lambda(\mathcal{S}^c_\Omega)$, then, as above, there is a unique conic $k \in \mathcal{D}$ with $g \cap k \neq \emptyset$. If the generatrix g belongs to $\lambda(\mathcal{S}^c_\Omega)$, then $g \cap k \neq \emptyset$ holds for all conics $k \in \mathcal{D}$; such a generatrix of the Lie quadric $\lambda(\mathcal{G})$ could be called a *transversal* of $\mathcal{D}$.

Remark 9. By [8, 2.1], each elliptic quadric Q_e of $\mathrm{PGL}(3, \mathbb{K})$ is embeddable into the Klein quadric H_5 of $\mathrm{PGL}(5, \mathbb{K})$, shortly $Q_e \subset H_5$. There exists a 4-space V of $\mathrm{PGL}(5, \mathbb{K})$ containing $\operatorname{span} Q_e$ and being not tangent to H_5. Now $V \cap H_5$ is a Lie quadric with $V \cap H_5 \supset Q_e$, consequently, *each elliptic quadric Q_e of $\mathrm{PGL}(3, \mathbb{K})$ is embeddable into the Lie quadric L_4 of $\mathrm{PGL}(4, \mathbb{K})$. If $\mathcal{F}$ is a flock of Q_e with $Q_e \subset L_4$, then $\mathcal{F}$ is a flockoid of L_4* (see part (b) of the above Proof).

Before formulating and proving the converse of Proposition 1 and Remark 7 in Proposition 2 we state some Lemmas about flockoids. The following two Lemmas are immediate consequences of (FD1) and the properties of a plane section of a quadric.

Lemma 3. *Let $\mathcal{D}$ be a flockoid of the Lie quadric L_4.*
(i) *Then different conics of $\mathcal{D}$ are disjoint.*
(ii) *If $\{P_1\}$ and $\{P_2\}$ are different improper conics of $\mathcal{D}$, then $P_1 \vee P_2 \not\subset L_4$.*
(iii) *If g is a generatrix of L_4 and $k \in \mathcal{D}$ satisfies $k \cap g \neq \emptyset$, then $g \not\subset \operatorname{span} k$ and $\#(k \cap g) = 1$.*

Lemma 4. *Let $\mathcal{D}$ be a flockoid of the Lie quadric L_4 and let k_1 be a proper conic of $\mathcal{D}$. If $k_2 \in \mathcal{D} \setminus \{k_1\}$, then there exists no tangent cone C_3 of L_4 with $k_1 \cup k_2 \subset C_3$.*

Proposition 2. *If $\mathcal{D}$ is a flockoid of the Lie quadric L_4 with $L_4 \subset H_5$, then*

$$\cup\Big((\lambda^{-1}(k))^c | k \in \mathcal{D}\Big) =: T_E(\mathcal{D}) \tag{5}$$

is a spread of Π admitting the regulization

$$\{(\lambda^{-1}(k))^c | k \in \mathcal{D}\} =: T_R(\mathcal{D}) \tag{6}$$

and $T_R(\mathcal{D})$ is either unisymplecticly complemented or elliptic.

Proof. Let X be an arbitrary point of Π. In $T_E(\mathcal{D})$ there exists a line incident with X if, and only if, there is a conic $k_X \in \mathcal{D}$ such that X is on a line h of the regulus $\lambda^{-1}(k_X)$. But $\lambda^{-1}(k_X) \subset \lambda^{-1}(L_4)$ implies $h \in \mathcal{L}[X, \gamma(X)]$, wherein γ denotes the null polarity associated with $\lambda^{-1}(L_4)$. By (FD1) there is a unique $k_X \in \mathcal{D}$ with $k_X \cap \lambda(\mathcal{L}[X, \gamma(X)]) \neq \emptyset$. Hence there is a unique regulus $(\lambda^{-1}(k_X))^c \subseteq T_E(\mathcal{D})$ which contains a line through X. Consequently, $T_E(\mathcal{D})$ is a spread.

Next we prove the validity of (RZ1) and (RZ2) for $T_R(\mathcal{D})$. Clearly, (FD2) $\Rightarrow$ (RZ2). Instead of (RZ1) we show even more:
(RZ1*) *Each line of $T_E(\mathcal{D})$ belongs to exactly one regulus of $T_R(\mathcal{D})$.*
Let $b \in T_E(\mathcal{D})$ be arbitrary. We assume

$$b \in (\lambda^{-1}(k_1))^c \cap (\lambda^{-1}(k_2))^c, \quad \{k_1, k_2\} \subseteq \mathcal{D}, \quad k_1 \neq k_2. \tag{7}$$

In the case that both $(\lambda^{-1}(k_1))^c$ and $(\lambda^{-1}(k_2))^c$ are improper reguli with $(\lambda^{-1}(k_i))^c = \{g_i\}$ and $g_i \in \mathcal{L}$, $i = 1, 2$, the lines g_1 and g_2 are skew and (7) yields the absurdity $b \in \{g_1\} \cap \{g_2\} = \emptyset$. Hence we may assume, without loss of generality, that $(\lambda^{-1}(k_1))^c$ is a proper regulus. Each line of $(\lambda^{-1}(k_1)) \cup (\lambda^{-1}(k_2))$ meets b. Thus $k_1 \cup k_2$ is contained in the tangent cone of L_4 at the point $\lambda(b)$, a contradiction to Lemma 4. Therefore $T_R(\mathcal{D})$ is a regulization and, because of $k \subset L_4$ for all $k \in \mathcal{D}$, $T_R(\mathcal{D})$ is symplecticly complemented.

As (RZ1*) holds for $T_R(\mathcal{D})$, so $i(T_R(\mathcal{D})) = 0$ and, by [7, Remarks 2.5 and 2.6], $T_R(\mathcal{D})$ is neither hyperbolic nor parabolic nor preparabolic. ∎

Now Theorem 1 is proved completely. The process of gaining a spread from a flockoid via formula (5) is called *extended Thas-Walker construction*. Using Proposition 1, Remark 7, and Proposition 2 we see: The construction of all spreads of $\mathrm{PG}(3, \mathbb{K})$ with unisymplecticly complemented or elliptic regulization is equivalent to the construction of all flockoids of the Lie quadric of $\mathrm{PG}(4, \mathbb{K})$.

4 Thas-Walker line sets

This Section is a generalization of [8, Section 2.2]. For the rest of this paper, we assume that the Lie quadric L_4 is contained in the Klein quadric H_5. We want a proper conic $k \subset L_4$ to be uniquely determined by the line $\pi_4(\operatorname{span} k)$, hence we assume $\operatorname{Char} \mathbb{K} \neq 2$ throughout Section 4. Thus $\operatorname{span} L_4 =: \overline{L_4}$ and the pole Z of $\overline{L_4}$ with respect to H_5 are complementary subspaces of Π_5, and the projection

$\Delta : \mathcal{P}_5 \setminus Z \to \overline{L_4},\ X \mapsto (X \vee Z) \cap \overline{L_4}$ is well-defined. A set T_ℓ of lines contained in $\overline{L_4}$ is called *Thas-Walker line set with respect to* L_4, if

$$D(T_\ell) := \{\pi_4(x) \cap L_4 | x \in T'_\ell\} \quad \text{with} \quad T'_\ell := \{x \in T_\ell | \pi_4(x) \cap L_4 \neq \emptyset\} \tag{8}$$

is a flockoid of L_4. By Lemma 1 (ii), a Thas-Walker line set with respect to L_4 must not contain a generatrix of L_4. If $\mathbb{K}$ is quadratically closed, then, by virtue of Lemma 1 (iv), formula (8) does not yield flockoids of L_4 which contain improper conics. We put

$$T^p_\ell := \{x \in T_\ell | \#(\pi_4(x) \cap L_4) > 1\}. \tag{9}$$

Remark 10. Let $\{P\} \subset L_4$ be an improper conic. In the case $\mathbb{K} = \mathbb{R}$ there are infinitely many lines a with $\pi_4(a) \cap L_4 = \{P\}$; see Lemma 1 (iii). In other words, if $T_{\ell 1}$ and $T_{\ell 2}$ are Thas-Walker line sets with respect to L_4, then $D(T_{\ell 1}) = D(T_{\ell 2})$ implies $T^p_{\ell 1} = T^p_{\ell 2}$, but not $T'_{\ell 1} = T'_{\ell 2}$.

Lemma 5. *Denote by* $\mathcal{G}[L_4]$ *the set of all generatrices of the Lie quadric* L_4 *and put* $\mathcal{G}^*[L_4] := \pi_4(\mathcal{G}[L_4])$. *A set* A *of lines is a Thas-Walker line set with respect to* L_4 *if, and only if, the following four conditions hold true:*

(TL1) $a \subset \operatorname{span} L_4 =: \overline{L_4}$ *for all* $a \in A$.

(TL2) $\#(A_e) \leq 2$ *with* $A_e := \{a \in A | a \cap \pi_4(a) \neq \emptyset\}$.

(TL3) *If* $a_e \in A_e$, *then* $\#(\pi_4(a_e) \cap L_4) = 1$.

(TL4) *For each plane* $\xi \in \mathcal{G}^*[L_4]$ *there exists exactly one line* $a \in A$ *with* $\xi \cap a \neq \emptyset$.

Proof. If the intersection of the line $a \in A$ and the plane $\pi_4(a)$ is empty, then $\pi_4(a) \cap L_4$ is either a proper conic or empty, and conversely. We define $D(A)$ according to (8). Now (TL2) and (TL3) imply that all elements of $D(A)$ are proper or improper conics and that $D(A)$ satisfies (FD2), and vice versa. Finally, (TL4)⇔(FD1). ∎

If $k \subset L_4$ is a proper conic, then

$$\Big(\lambda^{-1}(k)\Big)^c = \lambda^{-1}\Big(Z \vee \pi_4(\operatorname{span} k)\Big) \text{ and } (\Delta \circ \lambda)\Big(\big(\lambda^{-1}(k)\big)^c\Big) = \pi_4(\operatorname{span} k). \tag{10}$$

If $\alpha \subset L_4$ is a plane such that $\alpha \cap L_4$ is the improper conic $\{A\}$, then, by Lemma 2,

$$\Big(\lambda^{-1}(\{A\})\Big)^c = \lambda^{-1}\Big(Z \vee \pi_4(\alpha)\Big) \quad \text{and} \quad (\Delta \circ \lambda)\Big(\big(\lambda^{-1}(\{A\})\big)^c\Big) = \{A\}. \tag{11}$$

Thus we have the subsequent modification of the extended Thas-Walker construction:

Lemma 6. *Let* H_5 *be the Klein quadric of a classical projective* 5*-space. If* T_ℓ *is a Thas-Walker line set with respect to the Lie quadric* $L_4 \subset H_5$, *then*

$$\mathcal{T}_\ell := \cup\Big(\lambda^{-1}(x \vee Z) | x \in T_\ell\Big) \quad \text{with} \quad Z = \pi_5(\operatorname{span} L_4) \tag{12}$$

is a spread of Π *admitting the regulization*

$$\Theta_\ell := \{\lambda^{-1}(x \vee Z) | x \in T'_\ell\} \tag{13}$$

wherein T'_ℓ *is defined by* (8); Θ_ℓ *is either unisymplecticly complemented or elliptic.*

Remark 11. If T'_ℓ is contained in a 3-space σ, then $\mathcal{T}_\ell$ is a symplectic spread, since $\lambda(\mathcal{T}_\ell)$ belongs to the hyperplane $Z \vee \sigma$ of Π_5.

Remark 12. If all lines of T'_ℓ have a common point, then Θ_ℓ is an elliptic regulization.

Remark 13. If T^p_ℓ contains two skew lines, then Θ_ℓ is a unisymplecticly complemented regulization of $\mathcal{T}_\ell$.

The image of a proper conic m under any projection through a point $Z \in \operatorname{span} m =: \overline{m}$ onto a line of $\overline{m}$ (not through Z) will be called a *linear segment.* We say that $\Phi(T'_\ell) := \cup(t|t \in T'_\ell)$ is the *ruled surface determined by* T'_ℓ and that each line $t \in T'_\ell$ is a T'_ℓ*-generatrix of* $\Phi(T'_\ell)$.

Lemma 7. *Suppose that the conditions (and notations) of Lemma* 6 *hold. If each linear segment* s_x *with* $s_x \subset \Phi(T'_\ell)$ *is contained in a* T'_ℓ*-generatrix of* $\Phi(T'_\ell)$ *and if* $\Phi(T'_\ell)$ *contains no proper conic which is the* Δ*-image of a conic of* H_5*, then*
(1) *each proper regulus contained in* $\mathcal{T}_\ell$ *belongs to* Θ_ℓ*;*
(2) $\mathcal{T}_\ell$ *admits exactly one regulization, namely* Θ_ℓ*.*

Proof. Assume, to the contrary, that $\mathcal{R}$ is a proper regulus with $\mathcal{R} \subset \mathcal{T}_\ell$ and $\mathcal{R} \notin \Theta_\ell$. Put $\overline{r} := \operatorname{span} \lambda(\mathcal{R})$. If $Z \notin \overline{r}$, then $(\Delta \circ \lambda)(\mathcal{R}) \subset \Phi(T'_\ell)$ is a proper conic which is the Δ-image of the proper conic $\lambda(\mathcal{R}) \subset H_5$. If $Z \in \overline{r}$, then $(\Delta \circ \lambda)(\mathcal{R}) =: s_r$ is a linear segment with $s_r \subset \Phi(T'_\ell)$. From $\mathcal{R} \notin \Theta_\ell$ follows that s_r is not contained in a T'_ℓ-generatrix of $\Phi(T'_\ell)$. ■

Remark 14. Using the language of descriptive geometry we can say that L_4 is the contour (silhouette) of H_5 under Δ. Without proof we mention: If c is a proper conic of H_5 with $c \not\subset L_4$ and $Z \notin \overline{c} := \operatorname{span} c$, then $\Delta(c)$ is "doubly tangent to L_4", i.e., the determination of $L_4 \cap \Delta(c)$ is equivalent to the determination of the zeroes of a biquadratic polynomial which splits into two (not necessarily different) quadratic polynomials. An arbitrary biquadratic polynomial $Ax^4+Bx^3+Cx^2+Dx+E \in \mathbb{K}[x]$ splits into two quadratic polynomials if, and only if,

$$AD^2 - EB^2 = 0 \quad \text{and} \quad 8A^2D + B^3 - 4ABC = 0; \tag{14}$$

(extend [1, p.60] where $\mathbb{K} = \mathbb{R}$ is assumed). In geometric terms: If $\overline{L_4} \cap \overline{c} =: l_4$ is not tangent to L_4, then $\Delta(c)$ and L_4 determine the same involution of conjugate points in l_4 and the pole of l_4 with respect to $\Delta(c)$ is incident with $\pi_4(l_4)$; if l_4 is tangent to L_4 at the point H, then $\Delta(c)$ hyperosculates $L_4 \cap \operatorname{span} \Delta(c)$ at H. The converse is not always true: Let $b \subset \overline{L_4}$ be a proper conic which is tangent to L_4 at the different points D_1 and D_2. The quadratic cone $Z \vee b$ and the quadric $H_5 \cap \operatorname{span}(Z \vee b) =: h_5$ have common tangent planes at D_1 and D_2. If $h_5 \cap (Z \vee b) \neq \{D_1, D_2\}$, then $h_5 \cap (Z \vee b)$ consists of two (not necessarily different) conics. But for $\mathbb{K} = \mathbb{R}$ it is easy to give an example of a quadratic cone and a quadric such that their complete intersection consists of two different points.

Lemma 8. *Suppose that the conditions of Lemma* 7 *hold and that* T^p_ℓ *contains two skew lines* t_1,t_2*. Let* $\kappa \in \operatorname{Aut} \mathcal{T}_\ell \subset \mathrm{P\Gamma L}(\Pi)$ *and let* κ_λ *be the collineation of* Π_5 *induced by* κ *(i.e.,* $\lambda \circ \kappa = \kappa_\lambda \circ \lambda$*). Then*

$$(3) \quad \kappa_\lambda(Z) = Z \quad \text{and} \quad \kappa_\lambda(L_4) = L_4 \qquad (4) \quad \kappa_\lambda(T^p_\ell) = T^p_\ell.$$

(5) *If Θ_ℓ contains two different improper reguli $\{g_1\}$ and $\{g_2\}$, then $\{g_1\}$ and $\{g_2\}$ are fixed or interchanged by κ. The points $\lambda(g_1)$ and $\lambda(g_2)$ are fixed or interchanged by κ_λ.*

Proof. Now $(Z \vee t_j) \cap H_5 =: c_j^*$ are proper conics with $\lambda^{-1}(c_j^*) \in \Theta_\ell$ $(j = 1, 2)$. As t_1 and t_2 are skew, so

$$Z = \operatorname{span} c_1^* \cap \operatorname{span} c_2^*. \tag{15}$$

By Lemma 7 (1), $\kappa(\lambda^{-1}(c_j^*)) \in \Theta_\ell$, hence $Z \in \kappa_\lambda(\operatorname{span} c_j^*)$ for $j = 1, 2$. Consequently, $\kappa_\lambda(Z) = Z$ and $\kappa_\lambda(L_4) = L_4$.

If $t \in T_\ell^p$, then $\mathcal{R}_t := \lambda^{-1}(t \vee Z) \in \Theta_\ell$ is a proper regulus contained in $\mathcal{T}_\ell$ and hence, by Lemma 7 (1), $\kappa(\mathcal{R}_t) \in \Theta_\ell$. Thus $\kappa_\lambda(t) = \operatorname{span} \lambda(\kappa(\mathcal{R}_t)) \cap \overline{L_4} \in T_\ell^p$, i.e., (4) is valid.

By Remark 13, Θ_ℓ is a unisymplecticly complemented regulization and $i(\Theta_\ell) = 0$, because of [7, Remarks 2.5 and 2.6]. By Lemma 7 (1) and [7, Remark 2.8], there is no proper regulus $\mathcal{X} \subset \mathcal{T}_\ell$ with $\{g_k\} \subset \mathcal{X}$, thus there is no proper regulus $\mathcal{Y} \subset \mathcal{T}_\ell$ with $\kappa(\{g_k\}) \in \mathcal{Y}$ and, consequently, $\kappa(\{g_k\}) \in \{\kappa(\{g_1\}), \kappa(\{g_2\})\}$, $k = 1, 2$. ∎

Remark 15. By Remark 10, the statement $\kappa_\lambda(T'_\ell) = T'_\ell$ is not necessarily true.

Lemma 9. *Assume $\mathbb{K} = \mathbb{R}$ and let $\mathcal{T}_\ell$ be a spread constructed from a Thas-Walker line set T_ℓ via (12). Put $\overline{L_4} := \operatorname{span} L_4$ and*

$$\operatorname{Aut}(L_4, T_\ell^p) := \{\xi \in \operatorname{PGL}(\overline{L_4}) | \xi(L_4) = L_4 \text{ and } \xi(T_\ell^p) = T_\ell^p\}.$$

If each collineation $\kappa \in \operatorname{Aut} \mathcal{T}_\ell \subseteq \operatorname{PGL}(\Pi)$ induces a collineation κ_λ of Π_5 with $\kappa_\lambda(L_4) = L_4$ and $\kappa_\lambda(T_\ell^p) = T_\ell^p$, then

$$g : \operatorname{Aut} \mathcal{T}_\ell \to \operatorname{Aut}(L_4, T_\ell^p), \quad \eta \mapsto \eta_\lambda | \overline{L_4}$$

is an isomomorphism and $\operatorname{Aut} \mathcal{T}_\ell = \{\mathrm{id}_{\mathrm{Lat}(\Pi)}\} \Leftrightarrow \operatorname{Aut}(L_4, T_\ell^p) = \{\mathrm{id}_{\mathrm{Lat}(\overline{\mathrm{L_4}})}\}$.

Proof. The assumptions imply that g is a map from the group $\operatorname{Aut} \mathcal{T}_\ell$ into the group $\operatorname{Aut}(L_4, T_\ell^p)$. Clearly, g is homomorphic. Up to notational modifications, the proof of the surjectivity of g can be taken from the proof of [8, Lemma 2.2.4]; we point out that a quadratic form which describes the Lie quadric L_4 has signature $(+++--)$ or $(---++)$. Finally,

$$\xi_\lambda | \overline{L_4} = \mathrm{id}_{\mathrm{Lat}(\overline{\mathrm{L_4}})} \Leftrightarrow \xi_\lambda | L_4 = \mathrm{id}_{L_4} \Leftrightarrow \xi | \lambda^{-1}(L_4) = \mathrm{id}_{\lambda^{-1}(L_4)} \Leftrightarrow \xi = id_{\mathrm{Lat}(\Pi)}$$

implies $\ker g = \{\mathrm{id}_{\mathrm{Lat}(\Pi)}\}$. ∎

Remark 16. A spread $\mathcal{S}$ of Π with $\operatorname{Aut} \mathcal{S} = \{id_{\mathrm{Lat}(\Pi)}\}$ is called *rigid*. Explicitly given examples of rigid spreads are very rare; cf. [4] for the finite case and [6] for $\mathrm{PG}(3, \mathbb{R})$.

I would like to express my thanks to H. HAVLICEK (Vienna) for valuable suggestions in the preparation of this article.

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Rolf Riesinger
Patrizigasse 7/14
A–1210 Vienna
Austria