On UHL and HUL

Călin Popescu

Abstract

Let R be a principal ideal domain of characteristic zero, containing $1/2$, and let $\varrho = \varrho(R) < \infty$ be the least non-invertible prime in R. Our main result is the following:

Let (L, d) be a connected differential non-negatively graded Lie algebra over R, whose underlying module is R-free of finite type. If $\mathrm{ad}^{\varrho-1}(x)(dx) = 0$, for homogeneous x in L_{even}, then the natural morphism $UFHL \to FHUL$ is an isomorphism of graded Hopf algebras; as usual, F stands for free part, H for homology, and U for universal enveloping algebra.

Related facts and examples are also considered.

This paper is a first part of a program of exploring the connections between UHL, the universal enveloping algebra of the homology of a differential graded Lie algebra L over a commutative ring containing $1/2$, and HUL, the homology of the universal enveloping algebra of L, via the natural morphism $UHL \to HUL$. For basic definitions, notation and results on the subject, we refer to standard references such as [2, 5, 9].

Our present goal is to prove the following:

1. Theorem. *Let R be a principal ideal domain of characteristic zero, containing $1/2$, and let $\varrho = \varrho(R) < \infty$ be the least non-invertible prime in R.*

Let further (L, d) be a differential graded Lie algebra over R, whose underlying module is R-free of finite type.

Let also L be r-reduced (i.e., L is trivial in dimensions less than r), with integer $r \geq 1$, and let $r' = 2[r/2 + 1]$ and non-negative integer $n < \varrho r' - 1$. Then:

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(1) *The natural morphism* $UFHL \to FHUL$ *is a monomorphism in all dimensions* (*thus, the Lie algebra* FHL *embeds naturally into* $FHUL$ *via* $UFHL$) *and an isomorphism in dimensions less than* $\varrho r' - 1$, *compatible with the Hopf algebra structures*; F *denotes the free part functor.*

(2) *The natural morphism* $HL \to HUL$ *is a split monomorphism of graded* R*-modules in dimensions less than* $\varrho r' - 2$, *compatible with the Lie algebra structures.*

(3) HUL *is* R*-free in dimensions less than* n *iff* HL *is* R*-free in dimensions less than* n; *in this case, the natural morphism* $UHL \to HUL$ *is an isomorphism in dimensions less than* n, *compatible with the Hopf algebra structures.*

(4) UL *is* n*-acyclic iff* L *is* n*-acyclic, in which case, the natural morphism* $\tilde{H}L \to \tilde{H}UL$, *between reduced homologies, is an isomorphism in dimensions less than or equal to* $\min\{2n-1, \varrho r' - 2\}$.

If, in addition, $\mathrm{ad}^{\varrho-1}(x)(dx) = 0$, *for homogeneous* x *in* L_{even}, *then*:

(5) *The natural morphism* $UFHL \to FHUL$ *is an isomorphism of graded Hopf algebras.*

(6) *The natural morphism* $HL \to HUL$ *is a split monomorphism of graded* R*-modules, compatible with the Lie algebra structures.*

(7) HUL *is* R*-free iff* HL *is* R*-free and* $d = 0$ *on* L_{even}, *in which case, the natural morphism* $UHL \to HUL$ *is an isomorphism of graded Hopf algebras.*

(8) *The natural morphism* $UHL \to HUL$ *is a split monomorphism of* R*-modules in dimensions less than* $2n+1$, *whenever* HL *is* R*-free in dimensions less than* n.

Before proceeding to prove the theorem, let us make some remarks upon the ingredients.

2. Remarks. (1) Since the ground ring is a principal ideal domain, and the modules under consideration are all of finite type, the free part functor F makes sense − recall that any such module M splits as $M = FM \oplus tM$, where t is the torsion functor, both summands being of finite type, as well [3, 4].

(2) Since the differential is compatible with the Lie brackets, the homology HL has a natural structure of a Lie algebra over R, and since torsion is a Lie ideal, by factoring it out, FHL can naturally be endowed with Lie brackets, as well, so $UFHL$ makes sense.

(3) Since torsion is an ideal in any algebra, by factoring it out of HUL, $FHUL$ acquires a natural multiplication, which is easily seen compatible with the comultiplication

$$FHUL \to FH(UL \otimes_R UL) \xleftarrow{\cong} FHUL \otimes_R FHUL,$$

the isomorphism $FHUL \otimes_R FHUL \xrightarrow{\cong} FH(UL \otimes_R UL)$ being given by the Künneth theorem. The unit and counit being obvious, $FHUL$ has indeed a natural structure of a Hopf algebra.

(4) The natural morphism $UFHL \to FHUL$ is obtained from the natural morphism of differential graded Lie algebras $L \to UL$ in several stages: this latter induces a morphism of graded Lie algebras in homology, $HL \to HUL$, torsion is sent into torsion, so, factoring it out both sides yields another morphism of graded Lie algebras, $FHL \to FHUL$, which produces the required morphism by universality.

(5) Statements (5) and (7) in the theorem may fail if the characteristic of R is an odd prime: indeed, given a field R of such characteristic, the pattern described in [6] yields connected differential non-negatively graded R-Lie algebras L of finite type, with arbitrarily prescribed nilpotency and non-isomorphic R-vector spaces UHL and HUL. As regards the other statements in the theorem, they hold partially under less restrictive hypotheses [1, 8].

(6) Recall that $\mathrm{ad}(u)(v)$ is a convenient notation for the Lie bracket $[u, v]$ of two elements u and v in a Lie algebra. The condition $\mathrm{ad}^{\varrho-1}(x)(dx) = 0$, for homogeneous x in L_{even}, is satisfied if, for instance, L is ν-nilpotent, $\nu \leq \varrho$, or $d = 0$ on L_{even}. The condition is essential in establishing statement (5) in the theorem, as the following example shows: Given positive odd prime ϱ, let $R = \mathbb{Z}[1/(\varrho-1)!]$, so $\varrho(R) = \varrho$, take

$$L_{2i} = \begin{cases} Ru, & i = 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$L_{2i+1} = \begin{cases} Ru_i, & i = 0, \ldots, \varrho - 1, \\ 0, & \text{otherwise}, \end{cases}$$

set $du = u_0$ and $du_i = 0$, $i = 0, \ldots, \varrho - 1$, and let, finally, $[u, u_i\,] = u_{i+1}$, $i = 0, \ldots, \varrho-2$, be the non-trivial relevant Lie brackets for L. Thus, except for $\mathrm{ad}^{\varrho-1}(x)(dx) = 0$, homogeneous x in L_{even}, the conditions in the second half of the theorem are all fulfilled; L_{even} is concentrated in degree 2, $L_2 = Ru$, and $\mathrm{ad}^{\sigma}(u)(du) = \mathrm{ad}^{\sigma}(u)(u_0) = u_{\sigma}$, $\sigma = 0, \ldots, \varrho - 1$, so $\mathrm{ad}^{\varrho-1}(u)(du)$ is *not* zero. The Lie algebra L is $(\varrho + 1)$-nilpotent, its homology HL is abelian, with R-free underlying module of finite type, concentrated in odd degrees $3, 5, \ldots, 2\varrho - 1$: $HL_{2\sigma+1} = Ru_{\sigma}$, $\sigma = 1, \ldots, \varrho - 1$, so

$$UFHL = UHL = \Lambda_R[u_1, \ldots, u_{\varrho-1}],$$

the exterior algebra over R, with generators $u_1, \ldots, u_{\varrho-1}$; for brevity, no distinction has been made between the cycles u_{σ}, $\sigma = 1, \ldots, \varrho-1$, and their respective homology classes. On the other hand, it can be shown that

$$FHUL = \Lambda_R[u_1, \ldots, u_{\varrho-2}, u^*_{\varrho-1}],$$

where

$$u^*_{\varrho-1} = u^{\varrho-1}u_0 + \sum_{\sigma=2}^{\varrho-1}(-1)^{\sigma-1}((\varrho-1)\ \ldots(\varrho-\sigma+1)/\sigma!)u^{\varrho-\sigma}u_{\sigma-1}.$$

This latter is a cycle in UL, for $u_{\varrho-1}$ is, and [2]

$$d(u^{\varrho}) = \varrho u^{*}_{\varrho-1} + u_{\varrho-1},$$

so

$$u_{\varrho-1} = -\varrho u^{*}_{\varrho-1}$$

in HUL, and since ϱ is not invertible in R, the natural morphism $UFHL \to FHUL$ is merely a (non-split R-) monomorphism, *not* an isomorphism. This example also shows that the upper bound $\varrho r' - 1$ is the best possible in 1(1), and that the natural arrows $FHL \to FHUL$, $UFHL \to FHUL$ and $HL_{\geq \varrho r'-2} \to HUL_{\geq \varrho r'-2}$ are not necessarily R-split; see also Examples 20 and 22, and the Remarks 21. As for the other upper bounds, Examples 19 and 20 and the Remarks 21 show that most of them are the best possible, as well. ■

The remainder of the paper is almost entirely devoted to the proof of the theorem.

The first step consists in choosing a suitable R-basis for L: since R is a principal ideal domain and the underlying module of L is R-free of finite type, by the normal form theorem on matrices with entries in a principal ideal domain [3,4], a *normal* R-basis $X \cup X' \cup Y$, disjoint union of graded sets, can be chosen for L in such a way that X and X' be related by a set-theoretic bijection $' : X \to X'$, with $dx = a(x)x'$, $a(x) \in R\backslash\{0\}$, whence $dx' = 0$, whatever x in X, and $dy = 0$, whatever y in Y (the fact that $d^2 = 0$ is essential). Fix this normal basis once and for all.

3. Example. Let $R = \mathbb{Z}[1/2]$, let L be concentrated in degrees 1, 2, 3 and 4, with L_1, L_2, L_3 and L_4 the respective R-free modules on the bases (s_0, s_1, s_2), (t_0, t_1, t_2), $(u_0, u_1, u_2, v_0, v_1, v_2)$ and (w_0, w_1, w_2), set

$$t = (t_0 + t_1 + t_2)/2, \quad u = (u_0 + u_1 + u_2)/8,$$

$$v = (v_0 + v_1 + v_2)/8, \quad w = (w_0 + w_1 + w_2)/2,$$

and equip L with the differential d, given by

$$ds_i = 0, \qquad dt_i = 0,$$

$$du_i = 3(2t_i - t), \quad dv_i = 3t_i,$$

$$dw_i = 3(u - 3v + v_{i+2}),$$

and the Lie brackets

$$
\begin{array}{rclrcl}
[s_i, s_i] & = & 2(t - t_{i+2}), & [s_i, s_{i+1}] & = & t, \\
[s_i, t_i] & = & (v_i - v_{i+2})/2, & [s_i, u_i] & = & \frac{3}{2}w_i - w, \\
[s_i, t_{i+1}] & = & (v_{i+2} - v_{i+1})/2, & [s_i, u_{i+1}] & = & w - \frac{3}{2}w_i, \\
[s_i, t_{i+2}] & = & (v_i - v_{i+1})/2, & [s_i, u_{i+2}] & = & (w_{i+2} - w_{i+1})/2,
\end{array}
$$

$$[s_i, v_j] = (w_{i+2} - w_{i+1})/2,$$

throughout indices being taken modulo 3; of course, the remaining Lie brackets are all trivial, unless they are deducible from the above. In this way, L becomes a connected differential graded R-Lie algebra of nilpotency 4. Bringing the matrices corresponding to the non-trivial components of d to the normal form, yields the normal (graded) basis $X \cup X' \cup Y$, with

$$X_3 = \{x_{30}, x_{31}, x_{32}\}, \quad \text{whence } X'_2 = (X_3)' = \{x'_{30}, x'_{31}, x'_{32}\},$$

$$X_4 = \{x_{40}, x_{41}, x_{42}\}, \text{ whence } X'_3 = (X_4)' = \{x'_{40}, x'_{41}, x'_{42}\},$$

$$Y_1 = \{y_0, y_1, y_2\},$$

$$dx_{3,i} = 3x'_{3,i}, \text{ whence } dx'_{3,i} = 0,$$

$$dx_{4,i} = 3x'_{4,i}, \text{ whence } dx'_{4,i} = 0,$$

$$dy_i = 0,$$

$$[y_i, y_{i+1}] = x'_{3,i+2},$$

$$[y_i, x'_{3,i}] = x'_{4,i+1} - x'_{4,i+2} \quad \text{and} \quad [y_i, x_{3,i}] = x_{4,i+2} - x_{4,i+1}.$$

■

With respect to the normal basis, in the splitting for HL, the free part is given by $FHL = \oplus_{y \in Y} Ry$, and the torsion by $tHL = \oplus_{x \in X}(R/a(x))x'$; for brevity, no distinction has been made between the cycles y and x', and their respective homology classes.

4. Example. With reference to the previous example, FHL is concentrated in degree 1: $FHL_1 = \oplus_{i=0,1,2} Ry_i$; while tHL in degrees 2 and 3: $tHL_2 = \oplus_{i=0,1,2}(R/3)x'_{3,i}$ and $tHL_3 = \oplus_{i=0,1,2}(R/3)x'_{4,i}$. Also, note that, as Lie algebras, HL has nilpotency 4, while FHL is abelian (*i.e.*, has trivial Lie brackets). ■

Next, letting $x_{i,0}, \ldots, x_{i,m_i-1}$ denote the elements of X_i, $i \geq 2$ being the common degree of the elements $x_{i,k}$, and $y_{j,0}, \ldots, y_{j,n_j-1}$ those of Y_j, $j \geq 1$ being the common degree of the elements $y_{j,k}$, order the normal basis linearly by the relation $<$ defined below:

$$\begin{gathered}
x_{2,0} < x'_{2,0} < \ldots < x_{2,m_2-1} < x'_{2,m_2-1} < \\
x_{3,0} < x'_{3,0} < \ldots < x_{3,m_3-1} < x'_{3,m_3-1} < \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots < \\
x_{p,0} < x'_{p,0} < \ldots < x_{p,m_p-1} < x'_{p,m_p-1} < \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots < \\
y_{1,0} < \ldots < y_{1,n_1-1} < \\
y_{2,0} < \ldots < y_{2,n_2-1} < \\
\ldots\ldots\ldots\ldots\ldots\ldots < \\
y_{q,0} < \ldots < y_{q,n_q-1} < \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{gathered}$$

Of course, a copy of the restriction of $<$ to Y is naturally supposed to order this latter, when regarded as a basis for FHL. These orderings are both fixed once and for all, as well; once for all subsequent occurences, any phrase referring to an order for the basis letters actually refers to the order $<$ just defined.

5. Remarks. (1) Since there seems to be no possibility of confusion, we write $<$ indiscriminately for the ordering on either $\mathbb{N}$ or $X \cup X' \cup Y$.

(2) The restriction of $<$ to any of X_i or Y_j is actually subject to no other constraint than being linear: any such order can serve as a germ; while on $X'_i = (X_{i+1})'$ it is a copy of that on X_{i+1}, the resulting order on the block $X_{i+1} \cup X'_i$ being nothing but a braiding of the two, starting with the first member of X_{i+1}. These blocks are then ordered increasingly on i, the whole being followed by the increasing sequence of the Y_j's. ∎

The two examples below illustrate the way of ordering a normal basis.

6. Example. With the data in Example 3, the ordering on that normal basis is

$$x_{30} < x'_{30} < x_{31} < x'_{31} < x_{32} < x'_{32} <$$

$$x_{40} < x'_{40} < x_{41} < x'_{41} < x_{42} < x'_{42} <$$

$$y_0 < y_1 < y_2.$$

∎

For more convenience, our examples deal with normalized situations from now on; we will also avoid double indices, as much as possible.

7. Example. Consider the normalized situation in which: $L_0 = 0$, $L_1 = 0$, $L_{2i} = Rx'_i$ and $L_{2i+1} = Rx_i \oplus Ry_i$, $i \geq 1$; $dx_i = a_i x'_i$, $a_i \in R \backslash \{0\ \}$, whence $dx'_i = 0$, and $dy_i = 0$, $i \geq 1$; and, finally, $[x_i, x_j] = [y_i, y_j] = x'_{i+j+1}$, $i, j \geq 1$, are the non-trivial relevant Lie brackets. The verifications offer no difficulty at all, and the ordering on the given normal basis for L is thus

$$x_1 < x'_1 < x_2 < x'_2 < \ldots < x_m < x'_m < \ldots$$
$$< y_1 < y_2 < \ldots < y_n < \ldots$$

∎

By the Poincaré-Birkhoff-Witt theorem [2,5,9], the underlying modules of UL and $UFHL$ are both R-free of finite type: the former on the basis consisting of the *standard words*

$$x_0^{k_0} {x'}_0^{k'_0} \ldots x_{m-1}^{k_{m-1}} {x'}_{m-1}^{k'_{m-1}} y_0^{\ell_0} \ldots y_{n-1}^{\ell_{n-1}},$$

with $x_0 < \ldots < x_{m-1}$ in X and $y_0 < \ldots < y_{n-1}$ in Y (the elements x'_i are automatically at the right places), the latter on the basis whose standard words are

$$y_0^{\ell_0} \ldots y_{n-1}^{\ell_{n-1}},$$

with $y_0 < \ldots < y_{n-1}$ in Y, in both cases 0 and 1 being the sole possible values for the exponent of a basis letter of odd degree.

The following example illustrates the situation in the simplest non-abelian case.

8. Example. For L concentrated in degrees $1, 2$ and 3, with $L_1 = Rx'$, $L_2 = Rx$ and $L_3 = Ry$; $dx = x'$, so $dx' = 0$, and $dy = 0$; and $[x, x'] = y$ as single non-trivial relevant Lie brackets, the ordering on the given normal basis for L is $x < x' < y$, so UL is R-free on the standard words

$$x^k, \quad x^k y, \quad x^k x', \quad x^k x' y,$$

with integer $k \geq 0$ (of course, $x^0 = 1$). On the other hand, HL is an R-free abelian Lie algebra concentrated in degree 3, $HL_3 = Ry$, so $UHL = \Lambda_R[y]$. ∎

The unique expression of a word in either universal enveloping algebra as a linear combination of standard words in a basis will simply be called the *standard expression* of that word (with respect to that basis).

9. Example. With the data in the previous example, for integer $k \geq 1$, the standard expression of the word $x'x^k$ in UL is: $x'x^k = x^k x' - kx^{k-1}y$. ∎

The next two technical lemmas deal with standard expressions in UL. As usual, if x is a homogeneous element in a graded object, $\deg x$ denotes the degree of x, while $|x|$ is the modulo 2 reduction of $\deg x$.

10. Lemma. *If $x \in X$ and $z, z_0, \ldots, z_{m-1} \in X \cup X' \cup Y$, $m \geq 1$, are basis letters satisfying $x' < z$, $\deg x' \leq \deg z$, $x' < z_0 \leq \ldots \leq z_{m-1}$ and $z_{i-1} < z_i < z_{i+1}$ whenever $|z_i| = 1$, then x' precedes any basis letter occuring in the standard expression in UL of a word of the form*

$$z_0 \ldots z_i z z_{i+1} \ldots z_{m-1},$$

$i = -1, \ldots, m-1$ (*of course*, $z_{-1} = z_m = 1$).

Proof. By induction on m. ■

We illustrate the situation in the lemma by means of the following example.

11. Example. With the data in Example 7, let $x = x_2$, $z = y_2$, $z_0 = y_3$, $z_1 = y_4$ and note that $x' = x_2'$ precedes any basis letter occuring in the standard expression of $y_3 y_4 y_2$:

$$y_3 y_4 y_2 = -x_6' y_4 + x_7' y_3 + y_2 y_3 y_4.$$

This also shows that the lemma might fail if the condition $\deg x' \leq \deg z$ were removed: replace, for instance, x_2 by x_7 in the preceding. ■

12. Lemma. *The standard expression of the differential of a standard word in UL involves no basis letter preceding the first letter of that standard word.*

Proof. By wordlength induction along with Lemma 10 and the well-known formula [2]

$$d(x^k) = a(x) \sum_{j=1}^{k} (-1)^{j-1} (j, k-j) x^{k-j} \mathrm{ad}^{j-1}(x)(x'),$$

for x in X_{even}; as usual, $(j, k-j)$ denotes the coefficient of t^j in the expansion of $(1+t)^k$ in $R[t]$. ■

The previous formula for $d(x^k)$, $|x| = 0$, illustrates itself the situation in the lemma. However, the following example might throw a better light on it.

13. Example. Let R contain 1/6, *e.g.*, $R = \mathbb{Z}[1/6]$; let L be concentrated in degrees 1, 2, 3 and 5: $L_1 = Rx_1' \oplus Rx_2'$, $L_2 = Rx_1 \oplus Rx_2$, $L_3 = Ry_1$ and $L_5 = Ry_2$; set $dx_i = x_i'$, so $dx_i' = 0$, and $dy_i = 0$, $i = 1, 2$; and let, finally, $[x_i, x_j'] = y_1$, $i, j = 1, 2$, and $[x_i, y_1] = y_2$, $i = 1, 2$, be the non-trivial relevant Lie brackets for L. In this way, L becomes a connected differential graded Lie algebra, whose underlying module is R-free on the ordered normal basis $x_1 < x_1' < x_2 < x_2' < y_1 < y_2$. Noting that L is 4-nilpotent, the standard expression of $d(x_2^k)$,

$$d(x_2^k) = k x_2^{k-1} x_2' - \frac{1}{2} k(k-1) x_2^{k-2} y_1 + \frac{1}{6} k(k-1)(k-2) x_2^{k-3} y_2,$$

does not involve any basis letter preceding x_2; nor does

$$d(x_2^k y_1) = k x_2^{k-1} x_2' y_1 - \frac{1}{6} k(k-1)(k-2) x_2^{k-3} y_1 y_2,$$

for instance. Observe that, from the view point of Lemma 12, the other examples yield trivial situations, for either the differential is zero in even degrees, or X_{even} is a singleton set consisting of the first member of the ordered normal basis (Example 8). ■

Recall that $\varrho \geq 3$ is the least non-invertible prime in R, so $(\varrho - 1)!$ is a unit in R.

The next step consists in considering “standard germs” in UL.

14. Definition. For x in X and integer $k \geq 1$, define the *standard germs*

$$\xi_k(x) = \begin{cases} x^k, & \text{if } |x| = 0, \\ x x'^{k-1}, & \text{if } |x| = 1, \end{cases}$$

and

$$\zeta_k(x) = \begin{cases} x^{k-1}x' + \sum_{j=2}^{\varrho-1} c_{kj} x^{k-j} \mathrm{ad}^{j-1}(x)(x'), & \text{if } |x| = 0, \\ \\ x'^k, & \text{if } |x| = 1, \end{cases}$$

where $c_{kj} = (-1)^{j-1}(k-1)\dots(k-j+1)/j!$, $j = 2, \dots, \varrho-1$ (compare to the formula for $d(x^k)$, $|x| = 0$, in the proof of Lemma 12); for more convenience, let also

$$\epsilon_k(x) = \begin{cases} k, & \text{if } |x| = 0, \\ 1, & \text{if } |x| = 1, \end{cases}$$

and note that

$$d\xi_k(x) = \epsilon_k(x) a(x) \zeta_k(x) \quad \text{and} \quad d\zeta_k(x) = 0,$$

for any x in X_{odd} and any positive integer k, and for any x in X_{even} and positive integer $k < \varrho$. These latter equalities might no longer hold for x in X_{even} and integer $k \geq \varrho$: indeed, with a mere change of notation, the example in 2(6) shows that

$$d\xi_k(u) = k\zeta_k(u) + (\varrho, k - \varrho) u^{k-\varrho} u_{\varrho-1},$$

for integer $k \geq \varrho$, and therefore $\zeta_k(u)$ is no longer a cycle, for integer $k > \varrho$. However, we can easily recover the previous situation by simply requiring that $\mathrm{ad}^{\varrho-1}(x)(dx) = 0$, for any x in X_{even}, as actually done in the second half of the theorem. ■

15. Example. With reference again to Example 8, note that L is 3-nilpotent, so

$$\xi_k(x) = x^k, \quad \zeta_k(x) = x^{k-1}x' - \frac{1}{2}(k-1)x^{k-2}y$$

and $d\xi_k(x) = k\zeta_k(x)$; had we defined $dx = ax'$, $a \in R\backslash\{0\}$, in Example 8, we would now have obtained $d\xi_k(x) = ka\zeta_k(x)$. Clearly, $d\zeta_k(x) = 0$ in either case, whatever the index k is. ■

Except for the $\zeta_k(x)$, with $|x| = 0$ and $k > 1$, the other germs are all front blocks (or head blocks) in standard words; as regards the former, it rather contains such a block, namely, $x^{k-1}x'$, with a *unit* coefficient. This remark is the key to the following lemma.

16. Lemma. *The elements $\xi_k(x)u$ and $\zeta_k(x)v$, with x in X, integer $k \geq 1$, and u and v standard words in UL in basis letters following x', form, along with the standard words in letters in Y, an R-basis for the underlying module of UL.*

In the light of Lemma 10, the proof should offer no major difficulty and is hence omitted.

In other words (sic!), Lemma 16 states that an R-basis for UL can be derived from the standard basis under consideration, by replacing in this latter the words of the form $x^kx'w$, with x in X_{even}, integer $k \geq 0$ and standard w in letters following x', by $\zeta_{k+1}(x)w$. For instance, in the standard basis in Example 8, the words x^kx' and $x^kx'y$, integer $k \geq 0$, can respectively be replaced by $\zeta_{k+1}(x)$ and $\zeta_{k+1}(x)y$, to derive the R-basis

$$\{x^k,\, x^ky,\, \zeta_{k+1}(x),\;\; \zeta_{k+1}(x)y\}_{k\geq 0} =$$

$$\{\xi_k(x),\, \xi_k(x)y,\, \zeta_k(x), \zeta_k(x)y\}_{k\geq 1} \cup \{1, y\};$$

recall that $\xi_k(x) = x^k$ and $\zeta_k(x) = x^{k-1}x' - (1/2)(k-1)x^{k-2}y$ (Example 15). Similarly, with the data in Example 13, the passage from the standard word $x_1^{k_1}x_1'x_2^{k_2}$ to the corresponding $\zeta_{k_1+1}(x_1)x_2^{k_2}$, k_1, $k_2 \geq 1$, is easily seen to be given by

$$\begin{aligned} x_1^{k_1}x_1'x_2^{k_2} \;\; &= \;\; \zeta_{k_1+1}(x_1)x_2^{k_2} + \tfrac{1}{2}k_1x_1^{k_1-1}x_2^{k_2}y_1 - \\ &\qquad \tfrac{1}{2}k_1k_2x_1^{k_1-1}x_2^{k_2-1}y_2 - \tfrac{1}{6}k_1(k_1-1)x_1^{k_1-2}x_2^{k_2}y_2. \end{aligned}$$

As will be seen in the sequel, Lemma 16 yields an R-basis for UL, which behaves well under differentiation.

The pieces of the proof toward which we have been heading are now all available and nothing remains but fit them together for the *coup de grâce.*

17. Proof of the Theorem. Leaving, for the time being, the injectivity of the natural morphism $UFHL \to FHUL$ aside, let us first deal with the remainder of the theorem.

According to Lemma 16, any member w of UL has a unique expression of the form

$$w = \sum_{x \in X, k \geq 1} (\xi_k(x) u_k(x) + \zeta_k(x) v_k(x)) + \bar{w},$$

where: the sum is obviously finite; for each x in X and each integer $k \geq 1$, $u_k(x)$ and $v_k(x)$ are R-linear combinations of standard words in basis letters following x'; and, finally, $\bar{w}$ is an R-linear combination of standard words in letters in Y.

For w in $UL_{<\varrho r'-1}$, a degree argument and a simple computation yield

$$dw = \sum_{x,k} ((-1)^{|x|} \xi_k(x) du_k(x) + \zeta_k(x)(\epsilon_k(x) a(x) u_k(x) - (-1)^{|x|} dv_k(x))),$$

with appropriate x and k, and Lemma 12 assures us that the standard expressions of $du_k(x)$ and $dv_k(x)$ involve no basis letter $\leq x'$, whatever appropriate x and k are.

Consequently, with reference again to Lemma 16, $dw = 0$ iff

$$du_k(x) = 0 \;\text{ and }\; dv_k(x) = (-1)^{|x|} \epsilon_k(x) a(x) u_k(x),$$

for all appropriate x and k.

Noting finally that, by the preceding,

$$d(\xi_k(x) v_k(x)) = \epsilon_k(x) a(x) (\xi_k(x) u_k(x) + \zeta_k(x) v_k(x)),$$

that no R-linear combination of standard words in y's, y in $Y_{<\varrho r'-1}$, is a boundary, and recalling that R is a domain of characteristic zero, the second half of statement (1) in the theorem follows at once. Incidentally, observe that the last equality shows precisely where the argument breaks down in non-zero characteristic. Note also that we have actually obtained a recursive method of determining the homology of $UL_{<\varrho r'-1}$.

The ingredients being already available, the proofs of the remaining statements consist in completing and/or adapting the previous reasoning to the context by means of one obvious argument or another (*e.g.*, a degree argument) and are hence omitted.

Finally, to prove the first half of 1(1), fit the natural morphism $UFHL \to FHUL$ into a commutative diagram

$$\begin{array}{ccc}
UFHL & \longrightarrow & FHUL \\
\| & & \| \\
R \otimes_R UFHL & & R \otimes_R FHUL \\
\downarrow & & \downarrow \\
Q \otimes_R UFHL & \rightarrow & Q \otimes_R FHUL
\end{array}$$

of obvious canonical morphisms of R-algebras, by extending the scalars to the quotient field Q of R, and note that the left vertical arrow is monic, for $UFHL$ is R-free by the Poincaré-Birkhoff-Witt theorem. On the other hand, regarding $Q \otimes_R UFHL \to Q \otimes_R FHUL$ as a morphism of Q-algebras, we can fit it, via standard identifications or Künneth isomorphisms, into a commutative diagram

$$\begin{array}{ccc}
Q \otimes_R UFHL & \longrightarrow & Q \otimes_R FHUL \\
\uparrow\cong & & \uparrow\cong \\
U(Q \otimes_R FHL) & & Q \otimes_R HUL \\
\uparrow\cong & & \downarrow\cong \\
U(Q \otimes_R HL) & & H(Q \otimes_R UL) \\
\downarrow\cong & & \uparrow\cong \\
UH(Q \otimes_R L) & \longrightarrow & HU(Q \otimes_R L)
\end{array}$$

of obvious morphisms of Q-algebras, in which $UH(Q \otimes_R L) \to HU(Q \otimes_R L)$ is an isomorphism, by a result of Quillen [7] (see also our Remark 18(3)). It then follows that $Q \otimes_R UFHL \to Q \otimes_R FHUL$ is an isomorphism, so, with reference to the situation in the first diagram, $UFHL \to FHUL$ is indeed a monomorphism. ∎

18. Remarks. (1) To prove the theorem in the special case where $d = 0$ on L_{even}, a spectral sequence argument works as well; it suffices to compare the spectral sequences associated with the Lie filtrations of $UFHL$ and UL, respectively (the former being obviously equipped with a trivial differential).

(2) If we go through the preceding, we see that the fact that R is a principal ideal domain has essentially been used to describe the objects under consideration in a very simple way, a key step being the construction of the normal basis $X \cup X' \cup Y$. If the existence of such a basis were assumed by hypothesis (this holds for vector spaces of arbitrary dimension in each degree), it should now be clear that similar results still hold with R just a domain of characteristic zero; in addition, L would no longer be asked of finite type, and $UFHL$ and $FHUL$ should respectively be replaced by $U(HL/tHL)$ and $HUL/tHUL$ everywhere (the ordering for the normal basis would similarly be derived from germs whose existence is guaranteed by the well-ordering principle). Thus, in the normalized situations described in the examples, the theorem works with any domain of characteristic zero as a ground ring.

(3) It should also be clear that the theorem is valid for $\varrho = \infty$, as well; that is, for $\mathbb{Q} \subseteq R$, in which case we agree that $\mathrm{ad}^{\varrho-1} = \mathrm{ad}^{\infty} = 0$, and the condition that d be trivial on L_{even} is to be removed in statement (7) in the theorem. Thus, when R is a field of characteristic zero, we recover Quillen's result [7] on UHL and HUL (by the preceding remark, L is not necessarily of finite type in this case).

(4) It seems, under the hypothesis in the second half of the theorem, that the natural morphism of algebras $UHL \to HUL$ is a monomorphism in all dimensions. However, we do not yet know whether this is indeed the case or not. Nor do we know whether the natural morphism $U(HL/tHL) \to HUL/tHUL$ is an isomorphism under less restrictive hypotheses (on R and/or L) than those in the theorem. Let us just recall that, given integer ν and prime ϱ, with $\nu > \varrho \geq 3$, there exist commutative rings R of characteristic zero, which are not domains, with $\varrho(R) = \varrho$, and

connected differential non-negatively graded Lie algebras L of nilpotency ν, with R-free underlying module and R-free homology, both of finite type, but non-injective natural morphism $UHL \to HUL$ [6]. ∎

Let us now show how the previous general pattern applies to the situation described in Example 8, a situation which can actually be dealt with by direct calculation, as well.

19. Example. As we have already seen, with the data in Example 8, L is 3-nilpotent and the standard basis yields, via Lemma 16, the basis

$$\{\xi_k(x),\, \xi_k(x)y,\, \zeta_k(x),\, \zeta_k(x)y\}_{k\geq 1} \cup \{1, y\}$$

for UL, with $\xi_k(x) = x^k$ and $\zeta_k(x) = x^{k-1}x' - (1/2)(k-1)x^{k-2}y$ (Example 15). Thus, a member w of UL has a unique expression of the form

$$w = \sum_{k\ \geq 1} (\xi_k(x)(\alpha_k y + \beta_k) + \zeta_k(x)(\gamma_k y + \delta_k)) + \alpha y + \beta,$$

with $\alpha_k, \beta_k, \gamma_k, \delta_k, \alpha$ and β in R. Recalling that $d\xi_k(x) = k\zeta_k(x)$ and $d\zeta_k(x) = 0$, $k \geq 1$, we infer

$$dw = \sum_{k\geq 1} k\zeta_k(x)(\alpha_k y + \beta_k),$$

which shows immediately what the boundaries and the cycles are. Consequently, the homology of UL is given by

$$HUL_0 = R,\ HUL_1 = 0,\ HUL_2 = 0,\ HUL_3 = Ry,$$

$$\begin{aligned} HUL_{2k+2} &= (R/k)\zeta_k(x)y \\ &= (R/k)x^{k-1}x'y, \quad \text{and} \\ HUL_{2k+3} &= (R/(k+2))\zeta_{k+2}(x) \\ &= (R/(k+2))(x^{k+1}x' - \tfrac{1}{2}(k+1)x^k y), \end{aligned}$$

with integer $k \geq 1$, which is precisely the result obtained by a direct calculation appealing to descriptions in terms of the standard basis. Noting that $HUL/tHUL = FHUL = \Lambda_R[y]$, that HUL is R-free in dimensions less than $2\,\varrho - 1$, but $HUL_{2\varrho-1} = (R/\varrho)\zeta_\varrho(x)$ is torsion (recall that R is a domain of characteristic zero and $\varrho \geq 3$ is the least non-invertible prime in R), that UL is 3-acyclic, that $r = 1$, so $r' = 2$, and recalling that HL is an R-free abelian Lie algebra concentrated in degree 3, $HL_3 = Ry$, so $UHL = \Lambda_R[y]$ (Example 8, again), the statements in the theorem are readily verified; and if, in addition, we let $R = \mathbb{Z}[1/2]$, we get $\varrho = 3$ and see that the upper bound in 1(4) is actually the best possible. Finally, let us note that this example also shows how different the situation may be in non-zero characteristic: were $R = \mathbb{Z}/3\mathbb{Z}$, we would find the same $UHL = \Lambda_R[y]$, but a quite different HUL, namely,

$$\begin{aligned} HUL_{6k} &= Rx^{3k}, \\ HUL_{6k+5} &= R\zeta_{3k+3}(x) \oplus Rx^{3k+1}y, \\ HUL_{6k+8} &= R\zeta_{3k+3}(x)y \\ &= Rx^{3k+2}x'y, \end{aligned}$$

with integer $k \geq 0$, the other components being trivial. ∎

As for the other examples we have considered so far, it should now be clear that, in each case, the corresponding Hopf algebra $FHUL$ is isomorphic to the exterior algebra on the respective y's.

The following example shows that, without any further assumption, the upper bound $\varrho r' - 2$ in 1(2) is indeed the best possible.

20. Example. Take $R = \mathbb{Z}[1/2]$, so $\varrho = 3$; let L be concentrated in degrees $1, \ldots, 5$: $L_1 = Rx'_1$, $L_2 = Rx_1 \oplus Rx'_2$, $L_3 = Rx_2$, $L_4 = Rx'_3$ and $L_5 = Rx_3$; set $dx_1 = x'_1$, $dx_2 = x'_2$, $dx_3 = 3x'_3$ and, of course, $dx'_i = 0$, $i = 1, 2, 3$; and let, finally, $[x_1, x'_1] = x_2$, $[x_1, x_2] = x_3$, $[x_1, x'_2] = 2x'_3$, $[x'_1, x'_1] = x'_2$ and $[x'_1, x_2] = x'_3$ be the non-trivial relevant Lie brackets for L. The conditions in the first half of the theorem are thus satisfied with $r = 1$; so $r' = 2$ and $\varrho r' - 2 = 4$. Finally, the homology Lie algebra HL is abelian, concentrated in degree 4, $HL_4 = (R/3)x'_3$, and the natural arrow $HL \to HUL$ is trivial, for $x'_3 = d(x_1x_2 - x_1^2x'_1)$ in UL.

A related example is that of the free Lie algebra $L(x, dx)$ over $R = \mathbb{Z}[1/2]$, with x of degree 2 (again, $\varrho = 3, r = 1$, so $r' = 2$ and $\varrho r' - 2 = 4$): the homology $HL(x, dx)$ is trivial below dimension 4, while $HL(x, dx)_4 = (R/3)[dx, [x, dx]]$. On the other hand, $UL(x, dx) = T(x, dx)$, the tensor algebra over R, with generators x and dx, which is clearly acyclic [2], so the natural morphism $HL(x, dx) \to HUL(x, dx)$ is again trivial; incidentally, note that $[dx, [x, dx]] = d(x^2dx + (dx)x^2)$ in $T(x, dx)$. ∎

21. Remarks. (1) It might be worth noticing that, if HL is R-free in dimensions less than $\varrho r' - 1$, then so is HUL, but not conversely (the second situation in Example 20). Note also that the upper bound $\varrho r' - 1$ cannot be improved for HUL, even though HL is R-free (Example 19).

(2) Similarly, if L is $(\varrho r' - 1)$-acyclic, then so is UL, but not conversely (the second situation in Example 20, again). Once more, the upper bound $\varrho r' - 1$ cannot be improved for UL, even though L is acyclic. Here is an example: Take $R = \mathbb{Z}[1/2]$, so $\varrho = 3$; let L be concentrated in degrees $1, \ldots, 4$, with $L_{2i-1} = Rx'_i$ and $L_{2i} = Rx_i$, $i = 1, 2$; set $dx_i = x'_i$, so $dx'_i = 0$, $i = 1, 2$; and let $[x_1, x'_1] = x'_2$ be the sole non-trivial relevant Lie brackets for L. We thus get an acyclic 3-nilpotent (L, d) satisfying the conditions in the theorem with $r = 1$ (so $r' = 2$ and $\varrho r' - 1 = 5$). However, it is not hard to see that $HUL_5 = (R/3)\zeta_3(x_1)$, with $\zeta_3(x_1) = x_1^2x'_1 - x_1x'_2$.

(3) If, in the conditions in the first half of the theorem, $L = L(x_\alpha, dx_\alpha)$, the free Lie algebra on an appropriate generating set $\{x_\alpha, dx_\alpha\}$, then $UL = UL(x_\alpha, dx_\alpha) = T(x_\alpha, dx_\alpha)$ is acyclic [2], so, letting $n = \varrho r' - 2$ in 1(4), we recover a result of Anick

[1], namely: $L(x_\alpha, dx_\alpha)$ is $(\varrho r' - 2)$-acyclic. The second situation in Example 20 shows that the upper bound $\varrho r' - 2$ cannot be improved in the preceding. ∎

We end with a non-nilpotent extension of the situation in 2(6).

22. Example. Given odd prime ϱ, let $R = \mathbb{Z}[1/(\varrho - 1)!]$, so $\varrho(R) = \varrho$, take

$$L_{2i} = \begin{cases} Rx, & i = 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$L_{2i+1} = \begin{cases} Rx', & i = 0, \\ Ry_i, & i = 1, 2, \ldots, \\ 0, & \text{otherwise}, \end{cases}$$

set $dx = x'$, whence $dx' = 0$, and $dy_i = 0$, $i = 1, 2, \ldots$, and let, finally, $[x, x'] = y_1$ and $[x, y_i] = y_{i+1}$, $i = 1, 2, \ldots$, be the non-trivial relevant Lie brackets for L. Thus, $\mathrm{ad}^i(x)(x') = y_i$, $i = 1, 2, \ldots$, and we get a non-nilpotent object (L, d) satisfying the conditions in the first half of the theorem with $r = 1$ (so $r' = 2$). The order on the given normal basis is $x < x' < y_1 < y_2 < \ldots$. The homology HL is clearly an R-free abelian Lie algebra concentrated in odd degrees, $HL_{2i+1} = Ry_i$, $i = 1, 2, \ldots$, so $UFHL = UHL = \Lambda_R[y_1, y_2, \ldots]$. On the other hand, the relevant standard germs in UL are $\xi_k(x) = x^k$ and

$$\zeta_k(x) = x^{k-1}x' + \sum_{j=2}^{\varrho-1}(-1)^{j-1}((k-1)\ldots(k-j+1)/j!)x^{k-j}y_{j-1},$$

$k = 1, 2, \ldots$, and they satisfy $d\xi_k(x) = k\zeta_k(x)$ and $d\zeta_k(x) = 0$, for $k = 1, 2, \ldots, \varrho - 1$; however, these latter are no longer satisfied for $k \geq \varrho$:

$$d\xi_k(x) = k\zeta_k(x) + \sum_{j=\varrho}^{k}(-1)^{j-1}(j, k-j)x^{k-j}y_{j-1},$$

so $d\zeta_k(x) \neq 0$, for $k > \varrho$. An application of the previous general pattern yields

$$HUL_{<2\varrho-1} = \Lambda_R[y_1, y_2, \ldots]_{<2\varrho-1} = \Lambda_R[y_1, \ldots, y_{\varrho-2}]_{<2\varrho-1}$$

but

$$HUL_{2\varrho-1} = \Lambda_R[y_1, \ldots, y_{\varrho-2}]_{2\varrho-1} \oplus R\zeta_\varrho(x),$$

with $y_{\varrho-1} = -\varrho\zeta_\varrho(x)$ in homology, which plainly confirms the results in the first half of the theorem. Moreover, since ϱ is not invertible in R, the embeddings $FHL \to FHUL$ and $UFHL \to FHUL$ are no longer R-split, and $UFHL \to FHUL$ is no longer an isomorphism in dimensions beyond $2\varrho - 2$, so the upper bound $\varrho r' - 1$ in the theorem is indeed the best possible. Like its quotient in 2(6), this example also emphasizes the importance of the “nilpotency” condition in the second half of the theorem. ∎

23. Remark. The result in the second half of 1(1) can be slightly improved under the additional hypothesis that $\mathrm{ad}^{\varrho-1}(x)(dx)$ be a cycle with a torsion homology class in L, for each x in X_{even}. In this case, the pattern previously developed leads, by a degree argument, to the conclusion that the natural arrow $UFHL \to FHUL$ is actually an isomorphism in dimensions less than $\varrho r' + r$. Here is an example: To be more specific, let again $R = \mathbb{Z}[1/(\varrho - 1)!]$, with positive odd prime ϱ, so $\varrho(R) = \varrho$, let further $L_{2i-1} = Rx'_i$ and $L_{2i} = Rx_i$, $i = 1, 2, \ldots$, set $dx_i = x'_i$ and $dx'_i = 0$, $i = 1, 2, \ldots$, and define the non-trivial relevant Lie brackets for L by $[x_i, x'_j] = x'_{i+j}$, $i, j = 1, 2, \ldots$. Thus, the conditions in the first half of the theorem are satisfied with $r = 1$, so $r' = 2$, and, in addition, $[x_{i_0}, [\ldots, [x_{i_{m-1}}, x'_{i_m}] \ldots]] = x'_{i_0+\ldots+i_m} = dx_{i_0+\ldots+i_m}$, whatever the indices. Since L is acyclic, $UFHL = UHL = R$. We also find that $FHUL_0 = R$ and $FHUL_i = 0$, $i = 1, \ldots, 2\varrho$, so the natural arrow $UFHL \to FHUL$ is indeed an isomorphism in dimensions less than $2\varrho + 1$. As a matter of fact, it is an isomorphism in dimensions less than $4\varrho - 1$, and this is not accidental at all: with a slight modification in the definition of standard germs, it can be shown that, under the assumptions in the first part of the theorem, given σ in $\{0, \ldots, \varrho - 1\}$, the additional condition that $\mathrm{ad}^{\varrho-1}(x)(dx), \ldots, \mathrm{ad}^{\varrho+\sigma-1}(x)(dx)$ be all boundaries in L, whatever x in X_{even}, implies that the natural arrow $UFHL \to FHUL$ is an isomorphism in dimensions less than $(\varrho + \sigma + 1)r' - 1$; examples can be given to show that, without further assumptions, in dimension $(\varrho + \sigma + 1)r' - 1$ or beyond, $UFHL$ and $FHUL$ may no longer be isomorphic under the natural arrow. However, if $\mathrm{ad}^i(x)(dx) = d\omega_i(x)$ in L, $i = \varrho - 1, \ldots, 2\varrho + \sigma - 1$, and

$$\sum_{j=\varrho}^{k-\varrho} (j, k - j)\mathrm{ad}(\omega_{k-j-1}(x))(d\omega_{j-1}(x))$$

is a boundary in L, $k = 2\varrho, \ldots, 2\varrho + \sigma$, whatever x in X_{even}, then it can be shown that the natural arrow $UFHL \to FHUL$ is an isomorphism in dimensions less than $(2\varrho + \sigma + 1)r' - 1$, a fact readily confirmed by the above example; once again, examples can be given to show that, without further assumptions, in dimension $(2\varrho + \sigma + 1)r' - 1$ or beyond, $UFHL$ and $FHUL$ may no longer be isomorphic under the natural arrow. Similar considerations hold for the R-split injectivity of the natural morphism $HL \to HUL$. ■

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Université Catholique de Louvain
Institut de Mathématique
2, chemin du Cyclotron
B-1348 Louvain-La-Neuve
BELGIQUE