Conical, ruled and deficiency one translation planes

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Abstract

Flocks of deficiency one of quadratic cones and hyperbolic quadrics in $PG(3, K)$, for K a finite field, correspond to translation planes admitting certain collineation groups that fix Baer subplanes pointwise. In this article, this theory is extended to the general situation where K is an arbitrary field. Flocks of quadratic cones and of hyperbolic quadrics in $PG(3, q)$ correspond to spreads in $PG(3, q)$ which are unions of q or $q + 1$ reguli respectively. A more general theory of the analogous conical and ruled spreads is developed for spreads in $PG(3, K)$ and K an arbitary skewfield.

1 Introduction.

In this article, some generalizations of the connections with flocks of quadric sets in $PG(3, q)$ are developed. For a complete history of the problems and theory of geometries related to flocks as well as certain fundamental definitions, the reader is referred to the survey article by Johnson and Payne [26].

Let F be a flock of a quadratic cone in $PG(3, q)$. It is now well known that there is a corresponding spread in $PG(3, q)$ which is a union of q reguli sharing a common line. Furthermore, there is a corresponding generalized quadrangle of order (q^2, q) obtained via an associated q -clan. When F is a flock of a hyperbolic quadric in $PG(3, q)$, there is an associated spread in $PG(3, q)$ which is a union of $q + 1$ reguli sharing two common lines.

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In addition to these connections with flocks and translation planes, there are translation planes with spreads in $PG(3,q)$ corresponding to partial flocks of deficiency one of quadratic cones and to partial flocks of deficiency one of hyperbolic flocks (see Johnson [16]). The associated translation planes of order q^2 admit collineation groups of order q and $q-1$ respectively which fix subplanes of order q pointwise.

More generally, it is possible to consider spreads in $PG(3,K)$, for K an arbitrary field, corresponding to both flocks of quadratic cones and of hyperbolic quadrics (see Jha-Johnson [12], De Clerck-Van Mandeghem [6] and Johnson [23]).

In this article, we consider the more general situation for spreads over arbitrary skewfields. In this case, the direct connection with flocks and generalized quadrangles is lost but some interesting translation planes emerge which are derivable but derive planes some of which have infinite dimension over their kernels.

The structure of any derivable net is known by the results of Johnson [17]. In particular, given a derivable net N, there is a skewfield K and a 4-dimensional left vector space V over K such that points of N are (x,y) for all x,y 2-vectors over K and the lines of N are vector translates of the $Z(K)$-subspaces given by the equations $x=0, y=x\delta\ \forall\delta\in K$ where $x\delta=(x_1,x_2)\delta=(x_1\delta,x_2\delta)$ where $x_i\in K$ for $i=1,2$. Any derivable partial spread within $PG(3,K)$ in this way is said to be a pseudo-regulus net.

In particular, we consider spreads in $PG(3,K)$ which are covered by pseudo-reguli that share a given line that we call conical spreads and spreads in $PG(3,K)$ which are covered by pseudo-reguli that share two given lines which we call ruled spreads and formulate the corresponding theory.

There are some technical problems forming unions of pseudo-reguli that can occur due to the possible non-commutativity of multiplication of K so we consider what are called normal sets of pseudo-reguli and we direct the reader to the relevant section for the definition. Our main results on conical and ruled spreads (translation planes) are as follows:

Theorem 1. (1) *Let π be a translation plane with spread in $PG(3,K)$, for K a skewfield, which is a union of a normal set of pseudo-reguli that share exactly one line L. We shall call π a conical translation plane under these conditions.*

Then there is an elation group E with axis L of π which acts regularly on lines $\neq L$ incident with the zero vector of each pseudo-regulus net.

(2) *Let π be a translation plane with spread in $PG(3,K)$ which is a union of a normal set of pseudo-reguli that share exactly two lines L and M. We shall call π a ruled translation plane under these conditions.*

Then there is a homology group H of π with axis and coaxis L and M which acts regularly on lines $\neq L$ or M incident with the zero vector of each pseudo-regulus net.

Theorem 2. (1) *If π is a conical translation plane then the spread for π may be represented in the form $x=0, y=x\begin{bmatrix} u+g(t) & f(t) \\ t & u \end{bmatrix}\ \forall\ t,u\in K$ and for g,f functions on K.*

(2) *Consider f and g are functions on K such that $x^2t+xg(t)-f(t)=\phi_x(t)$. Then $\phi_x(t)$ is bijective $\forall\ x\in K \Longleftrightarrow$ the functions define a conical spread of the form* (1).

(3) *If π is a ruled translation plane then the spread for π may be represented in the form* $x = 0, y = 0, y = x \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}, y = x \begin{bmatrix} g(t)u & f(t)u \\ tu & u \end{bmatrix}$ $\forall v, t, u$ *for* $ut \neq 0 \in K$ *where* g, f *functions on* K.

(4) *If π is a conical translation plane with line L and also a ruled translation plane with lines L and M such that two normal 1 -dimensional left K-subspaces lie on L then the spread for π may be represented in the form*

$x = 0, y = x \begin{bmatrix} u + gt & ft \\ t & u \end{bmatrix}$ $\forall t, u \in K$ *where g and f are constants in K.*

The above results are applied to determine a set of translation planes which we call skew-Hall planes. The planes obtained are derived from certain translation planes which are both conical and ruled. However, these planes may not have spreads within $PG(3, K)$ as the kernel of each such plane is always $Z(K)$. The reader is directed to section 5 for a description of these planes.

A partial flock of deficiency one of a quadratic cone P_C consists of $q - 1$ conics and a partial flock of deficiency one of a hyperbolic quadric H_C consists of q conics. Payne -Thas [31] have shown that it is always possible to extend a partial flock of a quadratic cone of $q - 1$ conics. However, there are non-extendable partial flocks of a hyperbolic quadric of q conics. In particular, there are deficiency one partial flocks in $PG(3, 4)$, $PG(3, 5)$, and $PG(3, 9)$ (see Johnson [19], Biliotti-Johnson [3], and Johnson-Pomareda [27] respectively).

In this article, we extend the theory of partial flocks of quadratic cones and of hyperbolic quadrics of deficiency one in $PG(3, K)$ for K an arbitrary field. There are corresponding translation planes admitting certain Baer groups. Furthermore, in the infinite case, it is an open question whether such partial flocks either of quadratic cones or of hyperbolic quadrics are extendible to flocks as there are no examples of either to the contrary.

Our main results on partial flocks of deficiency one are:

Theorem 3. (1) *The set of partial flocks of deficiency one of a quadratic cone in $PG(3, K)$, for K a field, is equivalent to the set of translation planes with spreads in $PG(3, K)$ that admit a point-Baer elation group B which acts transitively on nonfixed 1-dimensional K-subspaces of components of the fixed point subplane.*

(2) *A partial flock of deficiency one of a quadratic cone in $PG(3, K)$, for K a field, may be extended to a flock $\Longleftrightarrow$ in the corresponding translation plane which admits a point-Baer elation group B as in (1), the net defined by the point-Baer affine plane FixB defines a regulus in $PG(3, K)$.*

Theorem 4. (1) *The set of partial flocks of deficiency one of a hyperbolic quadric in $PG(3, K)$, for K a field, is equivalent to the set of translation planes with spreads in $PG(3, K)$ which admit a point-Baer homology group B which is transitive on the nonfixed 1-dimensional K-subspaces on any component of FixB.*

(2) *A partial flock of deficiency one of a hyperbolic quadric may be extended to a flock if and only if the net defined by FixB of the corresponding translation plane which admits a point-Baer homology group B as in (1) defines a regulus in $PG(3, K)$.*

Actually, the above theorems may be stated more generally when K is an arbitrary skewfield. The reader is referred to section 4 for the generalization.

2 Conical and ruled translation planes.

In order for the reader to appreciate our definition of pseudo-regulus, we describe the characterization of a derivable net due to Johnson [17]. The following description may also be found in Johnson [24] and for a more complete history of derivation, the reader is referred to Johnson [21]

Definition 1. *A derivable net $N = (P, L, B, C, I)$ is an incidence structure with a set P of points, a set L of lines, a set B of Baer subplanes of the net, a set C of parallel classes of lines and a set I which is called the incidence set such that the following properties hold:*

(i) *Every point is incident with exactly one line from each parallel class, each parallel class is a cover of the points and each line of L is incident with exactly one of the classes of C.*

(ii) *Two distinct points are incident with at most one line of L.*

(iii) *If we refer to the set C as the set of infinite points then the subplanes of B are affine planes with infinite points exactly those of the set C.*

(iv) *Given any two distinct points a and b of P which are incident with a line of L, there is a Baer subplane $\pi_{a,b}$ of B containing (incident with) a and b.*

Remark 1. *Given a derivable net $N = (P, L, B, C, I)$, we may define a parallelism relation on the set B of subplanes. Two subplanes ρ and τ of B are defined to be parallel $\Longleftrightarrow$ they are disjoint on points. Define $N^* = (P, B, L, C^*, I)$ $= (P^*, L^*, B^*, C^*, I^*)$ as the incidence structure where $P^* = P$ is the set of points, $L^* = B$ is the set of lines, $B^* = L$ is the set of Baer subplanes, C^* is the parallelism on the set of lines L^* defined on the set of Baer subplanes B and I^* is the incidence set I.*

Then N^* is also a derivable net which is called the derived net. The transfer from N to N^* is called the derivation of N.

Theorem 5. *(Johnson [17]). Given a derivable net $N = (P, L, C, B, I)$, there is a 3-dimensional projective space Π and a fixed line R of Π such that the set P of points, the set L of lines, the set C of parallel classes, the set B of Baer subplanes of the net N are the set of lines of Π skew to R, the set of points of $\Pi - R$, the set of planes of Π that contain R, and the set of planes of Π which do not contain R respectively, where incidence I is the natural incidence of Π.*

The question was raised in Johnson [18] whether the process of derivation is geometric. In the present context, we might ask how the two combinatorial structures relate within the three dimensional projective spaces corresponding to a derivable net and its derived net. We shall provide a slight generalization of these geometric connections.

2.1 The annihilator mapping.

Let V_4 denote the left vector space over a skewfield $K \equiv (K, +, \cdot)$ such that the lattice of left vector subspaces defines the three-dimensional projective space Π corresponding to the derivable net N. Let V_4^* denote the dual space of V_4. We note that V_4^* is a right vector space over K by defining scalar multiplication as follows: If f is in V_4^* define $f\alpha$ by $f\alpha(x) = f(x)\alpha$ for x in V_4. V_4^* is a left $K^{opp} \equiv (K, +, \circ)$-space where $ab = a \cdot b = b \circ a$.

Now consider the annihilator mapping $\perp$ defined from V_4 onto V_4^* where if W is a left K-subspace of V_4 then $W^\perp = \langle f \,\epsilon V_4^* \mid f(w) = 0 \;\forall\; w \text{ in } W \rangle$ is a left K^{opp}-subspace of V_4^*.

Let Π and Π^* be the projective spaces defined by the lattices of left vector K-subspaces of V_4 and left K^{opp}-subspaces of V_4^* respectively.

Note that the annihilator mapping is dimension inverting and when considered on the associated 3-dimensional projective spaces Π and Π^* maps lines of Π onto lines of Π^*. Furthermore, points and planes of Π are interchanged with planes and points of Π^* respectively as $W \subseteq Z$ for W, Z left K-subspaces of $V_4 \Longleftrightarrow Z^\perp \subseteq W^\perp$ for $Z^\perp, W^\perp$ left K^{opp}-subspaces of V_4^*.

Let R denote the line of Π corresponding to the construction of the derivable net N. Let $R^* = R^\perp$. Then, clearly points of $\Pi - R$ map to planes of Π^* which do not contain R^*. Similarly, planes containing R are mapped to points of Π^* incident with R^* and points of R are mapped to planes of Π^* containing R^*.

Hence, the annihilator map defines a derivable net N^* which corresponds naturally to the 4-dimensional left K^{opp}-vector space V_4^* . If we identify the lines of V_4 and V_4^* via the annihilator mapping then we have made an identification between the points of N and N^*.

So, we have given a geometric connection between a derivable net and its derived net.

We have interpreted a derivable net within a three-dimensional projective space Π isomorphic to $PG(3, K)$ and its derived net similarly interpreted within a three-dimensional projective space Π^* isomorphic to $PG(3, K^{opp})$. In the following, we shall give an algebraic connection and also determine a three-dimensional projective space Σ isomorphic to $PG(3, K)$ corresponding to a derivable net. However, in this context, we may consider the lines of the net as translates of certain lines of Σ where as this was not the interpretation in Π.

2.2 Pseudo-reguli.

Note that it now follows from the above theorem that the collineation group of any derivable net is isomorphic to the full collineation group $P\Gamma L(4, K)_R$ which leaves the line R invariant of the 3 -dimensional projective space Π . Furthermore, with the use of this chacterization result and realization of the group of the derivable net, it is possible to give a complete structure theory for derivable nets.

Theorem 6. *(Johnson [17]). A net is derivable $\Longleftrightarrow$ there is a skewfield K and a left 4-dimensional vector space V over K such that the points of the net are the vectors of V and the lines of the net are translates of the following set of $Z(K)$-subspaces:*

The points are $(x_1, x_2, y_1, y_2) \equiv (x, y)\ \forall x_i\ ,\ y_i \in K,\ i = 1, 2,$ and the lines are translates of the $Z(K)$-subspaces $x = 0, y = \delta x$ where $\delta x = \delta(x_1, x_2) = (\delta x_1, \delta x_2)\ \forall \delta \in K$. The Baer subplanes are translates of the sets $\pi_{a,b} = \{(\alpha a, \alpha b, \beta a, \beta b) \mid \alpha, \beta \epsilon K\}$ and $(a, b) \in K \oplus K - \{(0, 0)\}$.

In Johnson [20], it was noted that given a derivable net N algebraically represented as in the above theorem with reference to a skewfield K then the derived net N^* may be algebraically represented as in the above theorem with reference to the skewfield K^{opp} where multiplication $\circ$ in K^{opp} is defined by $a \circ b = ba$ where juxtaposition denotes multiplication in K.

We note that, in the present notation, if the vector space V of points is a left vector space over the skewfield K then the lines of the derivable net incident with the zero vector are not necessarily always 2-dimensional left K-vector subspaces although they are natural 2-dimensional right K -vector subspaces. However, the Baer subplanes incident with the zero vector are left 2-dimensional left K-vector spaces.

For the derived net, the situation is reversed. The lines of the derived net incident with the zero vector are not always 2-dimensional left K^{opp} -vector subspaces but they are natural 2-dimensional right K^{opp} -vector subspaces as they are always 2-dimensional left K-subspaces. Similarly, the Baer subplanes incident with the zero vector of the derived net are 2-dimensional left K^{opp}-vector spaces as they are the lines of the original net incident with the zero vector which are 2-dimensional left K-vector subspaces.

Since we would like to represent the lines of our derivable net within the lattice of left subspaces of a 4-dimensional left vector space, we dualize everything and note the following:

Remark 2. *Let N denote a derivable net with lines incident with the zero vector represented in the form $y = x\delta$ for all δ in a skewfield J where the associated vector space V is a 4-dimensional left J-space and the lines indicated are $2-$dimensional left J -subspaces.*

So, the lines of N incident with the zero vector become lines in the projective space Σ isomorphic to $PG(3, J)$ defined as the lattice of left vector J-subspaces.

In terms of a given basis, V may also be defined as a 4-dimensional right or left J^{opp}-vector space V^* and the lines of N^* incident with the zero vector become lines in the projective space Σ^* isomorphic to $PG(3, J^{opp})$ defined as the lattice of left vector J^{opp} -subspaces.

Choose a left K-basis $B = \{e_i \text{ for } i \text{ in } \lambda\}$. For a vector $\Sigma x_i e_i$, x_i in K for $i = 1, 2, 3, 4,$ a left space over K^{opp} may be defined as follows: $u \circ \Sigma x_i e_i = \Sigma x_i u e_i = \Sigma(u \circ x_i)e_i$. Recall that $ac = c \circ a$ defines multiplication "$\circ$" in K^{opp} relative to K. So, there are many ways to form a projective space $PG(3, K^{opp})$ if K is a non-commutative skewfield.

Definition 2. *Let S be any set of mutually skew lines of $PG(3, K)$. A vector-transversal L to S is a line of some $PG(3, K^{opp})$ of $Z(K)$-projective points with the property that L as a left $Z(K)$-subspace has a nontrivial vector intersection with each line of S as a left $Z(K)$-subspace such that the direct sum of any two such* *intersections is L. A point-transversal to S is a line of $PG(3,K)$ which is also a vector-transversal.*

A pseudo-regulus $R = R_{\{U,V,W\}}^{\{L,M,N\}}$ in $PG(3,K)$ is a set of lines (as left 2-dimensional K -vector spaces) containing $\{L,M,N\}$ with a set of points $\{U,V,W\}$ of L such that any line T which intersects L in either $U,V,$ or W and also intersects M and N intersects each line of R and T is contained in the set of these intersections. So, any such line T becomes a point-transversal.

Definition 3. *The corresponding net defined by a pseudo-regulus is called a pseudo-regulus net. Often, we shall use the same notation for the pseudo-regulus and the corresponding net.*

Theorem 7. *Choose any three mutually skew lines L,M,N of $PG(3,K)$ and let U,V,W be any three distinct points on L.*

(1) *Then there exists a unique pseudo-regulus $R_{\{L,M,N\}}^{\{U,V,W\}}$ in $PG(3,K)$ which contains L,M,N and which has point-transversals intersecting L in U,V, and W.*

Furthermore, there is a unique basis such that L,M,N may be represented in the form $y=x, x=0, y=0$, respectively where $L = \langle U\rangle \oplus \langle V\rangle$ and $W = \langle U+V\rangle$.

The pseudo-regulus then has the form $x=0, y=x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ for all u in K.

(2) *Choose any set $\{L^*,M^*,N^*\}$ of three mutually skew lines of the pseudo-regulus $R_{\{L,M,N\}}^{\{U,V,W\}}$. Then there exist points U^*, V^*, W^* on L^* such that $R_{\{L^*,M^*,N^*\}}^{\{U^*,V^*,W^*\}} = R_{\{L,M,N\}}^{\{U,V,W\}}$.*

(3) *Any pseudo-regulus net is a derivable net.*

Proof: Consider the associated 4-dimensional left K-vector space V. Choose a basis for L as $\langle U,V\rangle$. Then $W = \alpha U + \beta V$ for α,β in K. Choose $U^* = \alpha U$ and $V^* = \beta V$. Hence, projectively, we may assume without loss of generality that $W = U+V$. Represent $V_4 = M \oplus N$, then there exist unique elements m_u, m_v of M and n_u, n_v of N such that $U = m_u + n_u$ and $V = n_v + n_u$. Then $W = U+V = (m_u+n_u)+(m_v+n_v) = ((m_u+m_v) = m_w) + ((n_u+n_v) = n_w)$. It is immediate that $\{m_u, m_v\}$ is a basis for M and $\{n_u, n_v\}$ is a basis for N.

Now form $T_U = \langle m_u, n_u\rangle$, $T_V = \langle m_v, n_v\rangle$, $T_W = \langle m_w, n_w\rangle$. Clearly, $T_U \cap L = \langle U\rangle$, $T_V \cap L = \langle V\rangle$ and $T_W \cap L = \langle W\rangle$. Choose a basis $\{m_u, m_v, n_u, n_v\}$ for V_4. In terms of this basis, $V_4 = \{(x_1,x_2,y_1,y_2) \mid x_i, y_i \in K \text{ for } i=1,2\}$. Let $x = (x_1,x_2)$ and $y = (y_1,y_2)$. Then L, M, N are $y=x, y=0, x=0$ respectively.

Furthermore, $T_U = \{(x_1,0,y_1,0) \mid x_i \in K \text{ for } i=1,2\}$, $T_V = \{(0,y_1,0,y_2) \mid y_i \in K \text{ for } i=1,2\}$ and $T_W = \{(x_1,x_1,y_1,y_1) \mid x_1,y_1 \in K\}$. It follows that any component of the pseudo-regulus is of the form $y = x\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for a,b,c,d in K and the intersection with T_U, T_V and T_W shows that $a=d=u$ and $b=c=0$. Let the pseudo-regulus R be represented in the form $x=0, y=0, y=x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ for $u \in \lambda \subseteq K$. Now in order that T_U is contained in $\{T_U \cap Z \mid z \in R\}$, we have $\{(x_1,0,x_1u,0) \mid x_1 \in K\} = T_U \ \forall\ u \in \lambda$. It clearly follows that this forces $\lambda = K$ so that the pseudo-regulus has the required form. It follows immediately that any pseudo-regulus is a derivable partial spread.

The derivable net R defined by $x=0, y=x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ $\forall\, u \in K$ has Baer subplanes $\rho_{a,b} = \{(a\alpha, b\alpha, a\beta, b\beta)\ \forall\ \alpha, \beta \in K\}$ and $(a,b) \neq (0,0)$ and by such a choice of basis for the vector space, we see that there are at least three 2-dimensional K^{opp}-subspaces which are vector-transversals which are also point-transversals (lines of $PG(3,K)$), namely $\rho_{0,1}$, $\rho_{1,0}$ and $\rho_{1,1}$. Note that if $Z(K)$ is isomorphic to $GF(2)$, there are exactly three lines of $PG(3,K)$ which are point-transversals to this net.

Now assume that there is another pseudo-regulus with these conditions. Assume that exist three point-transversals $\gamma_0, \gamma_1, \gamma_2$ (2-dimensional left K-subspaces) such that $\gamma_o \cap L = U, \gamma_1 \cap L = V$ and $\gamma_2 \cap L = W$. Assume that with respect to a basis $\{e_1, e_2, e_3, e_4\}$, L is $y = x$ and with respect to a basis $\{f_1, f_2, f_3, f_4\}$ we have $\langle f_3 \rangle = U$, $\langle f_4 \rangle = V$, and $\langle f_3 + f_4 \rangle = W$. Then it follows that $x_1e_1 + x_2e_2 + x_1e_3 + x_2e_4$ and $y_1f_1 + y_2f_2 + y_1e_3 + y_2e_4$ are both in L so that we must have $x_1(e_1 - f_1) + x_2(e_2 - f_2)$ $\forall\ x_i \in K$, $i = 1, 2$ is in $M \cap N$ which implies that $e_1 = f_1$ and $e_2 = f_2$.

In other words, any two sets of point-transversals of three elements to $\{L, M, N\}$ which intersect L in the same set of points are identical.

So, a second pseudo-regulus R' sharing the same three point-transversals must have the same basic form $x = 0, y = x\begin{bmatrix} u & 0 \\ 0 & g(u) \end{bmatrix}$ where g is some function on K such that $g(0) = 0$ and $g(1) = 1$. However, $\rho_{1,1} \cap (y = x\begin{bmatrix} u & 0 \\ 0 & g(u) \end{bmatrix}) = \{(\alpha, \alpha, \alpha u, \alpha g(u)\}$ which is incident with $\rho_{1,1} \iff g(u) = u\ \forall\ u \in K$. Hence, $R' = R$. This proves (1).

We have noted that there are at least three lines of $PG(3,K)$ which are point-transversals to the pseudo-regulus $R^{\{U,V,W\}}_{\{L,M,N\}}$. By the structure theory for derivable nets determined in Johnson[17], there exists a collineation group of the pseudo-regulus net which is triply transitive of the components of the pseudo-regulus and fixes each Baer subplane incident with the zero vector and hence every vector-transversal . Thus, there exists a collineation σ of the net which carries $\{L, M, N\}$ onto $\{L^*, M^*, N^*\}$ orderwise. Choose U^*, V^*, W^* as $U\sigma, V\sigma, W\sigma$ respectively. Then the above construction is merely a basis change so that the two pseudo-regulus nets are identical.

Remark 3. *Any pseudo-regulus in $PG(3,K)$ has a set of transversal lines in $1-1$ correspondence with a set of cardinality $Z(K) + 1$.*

We note that since the vector-transversals define Baer subplanes of the net incident with the zero vector, it is not necessarily true that every Baer subplane incident with the zero vector intersects each component in a 1 -dimensional left K-subspace (point of $PG(3,K)$).

In fact, the point-transversals are determined by any one intersection.

Corollary 8. *Let R be any pseudo-regulus in $PG(3,K)$. If a vector-transversal intersects some line of R in a point (a 1-dimensional left K-space) then the vector-transversal is a point-transversal (line).*

Proof: We may represent R in the standard form $x = 0, y = x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ $\forall\, u \in K$. The vector-transversals are exactly the Baer subplanes $\rho_{a,b}$. Suppose $\rho_{a,b}$ intersects

$x = 0$ in a 1-dimensional left K-space. Then, it follows that the intersection is $\{(0, 0, a\alpha, b\alpha)\ \forall\ \alpha \text{ in } K\}$. However, this is a 1-dimensional left K-space if and only if a and b are in $Z(K)$ which implies that the intersection with $y = x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ which is $\{(a\alpha, b\alpha, a\alpha u, b\alpha u)\ \forall\ \alpha \in K\}$ is a 1-dimensional left K-space. Similarly, if any such intersection $\{(a\alpha, b\alpha, a\alpha u, b\alpha u)\ \forall\ \alpha \in K\}$ is a 1-dimensional left K-space then a and b are in $Z(K)$ so that all intersections are 1-dimensional left K-spaces.

This also proves the following well-known corollary.

Corollary 9. *If K is a field then there is a unique regulus containing any three mutually skew lines L, M, and N.*

Proof: Choose any three points on L and construct the corresponding regulus net. Any vector-transversal is a point-transversal and corresponds to a Baer subplane of the net. As any three points on L correspond to unique Baer subplanes of this regulus net, it follows that the choice of three points on L is arbitrary.

To illustrate that the previous corollary is not necessarily valid for pseudo-reguli, suppose K is a skewfield such that $Z(K) \simeq GF(2)$ but K is not isomorphic to $GF(2)$. Then, the above result shows that for any three distinct points U, V, W on a line L of $PG(3, K)$, there is a unique pseudo-regulus $R^{\{U,V,W\}}_{\{L,M,N\}}$ containing $\{L, M, N\}$. Furthermore, there are exactly three Baer-transversals which are point-transversals to this pseudo-regulus. So, take any three distinct points U^*, V^*, W^* on L such that $\{U, V, W\} \neq \{U^*, V^*, W^*\}$. Then, it is not possible that $R^{\{U,V,W\}}_{\{L,M,N\}}$ is equal to $R^{\{U^*,V^*,W^*\}}_{\{L,M,N\}}$.

Remark 4. *There exist skewfields K such that there are infinitely many pseudo-reguli that share any three mutually skew lines in $PG(3, K)$.*

Now suppose that D is a pseudo-regulus net represented in standard form. Suppose that a subplane $\rho_{a,b}$ intersects $x = 0$ in $\{(0, 0, a\beta, b\beta)\ \forall \beta \in K\}$. Assume that a and b are not both in $Z(K)$. Choose the vector $(0, 0, a, b)$ and let $\langle(0, 0, a, b)\rangle = U$ denote the left 1-dimensional K-subspace generated by $(0, 0, a, b)$. Assume that $\{x = 0, M, N\}$ is a set of skew lines in $PG(3, K)$ which is not in D. Choose any other points V, W of $x = 0$ and form $R^{\{U,V,W\}}_{\{x=0,M,N\}} = R$. Then R and D are pseudo-reguli which share at least one line but do not have the property that the vector-transversals to the two pseudo-reguli partition $x = 0$ in the same set of sublines. We shall be interested in situations where there is such a partition and to this end, we formulate the following definition.

Definition 4. *Let D_1 and D_2 be any two pseudo-regulus nets whose union defines a partial spread in $PG(3, K)$ that share either one or more components. Assume that on one of the common components L there exists two points of $PG(3, K)$ (1 -dimensional left K-subspaces) which are in point-transversals to D_1 and D_2.*

Any two pseudo-reguli sharing one or more two lines whose nets satisfy the above property shall be said to be normalizing. Furthermore, any set of pseudo-reguli sharing one or more lines each pair of which satisfies the above property with respect to the same two points shall be said to be normal set. Any such point shall be said to be a normal point.

Proposition 10. (1) *Two normalizing pseudo-reguli in $PG(3,K)$ share one or two lines.*

(2) *If two normalizing pseudo-reguli share exactly one line then there is an elation group with axis the common line which acts regularly on the remaining lines of each pseudo-regulus. Furthermore, the vector-transversals to each pseudo-regulus net induce the same partition on this common line.*

(3) *If two normalizing pseudo-reguli share at least two lines then there is a homology group with axis and coaxis the two common lines which acts regularly on the remaining lines of each pseudo-regulus. Furthermore, on one of the common lines, the vector-transversals to the various pseudo-reguli induce the same partition on this line.*

Proof: Assume the hypothesis of (2). By appropriate choice of coordinates, any given pseudo-regulus may be brought into the form $x = 0, y = x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ $\forall\ u \in K$. Let D_1 have this standard form. We assume that $x = 0$ is the common line. The Baer subplanes incident with the zero vector are $\pi_{a,b} = \{(a\alpha, b\alpha, a\beta, b\beta)\ \forall\ \alpha, \beta \in K\}$. Note the Baer subplanes which are transversal lines are $\pi_{a,b}$ where both a and b are in $Z(K)$. Furthermore, we may assume that the two 1-dimensional left K-subspaces on $x = 0$ which belong to transversal lines of each of the two pseudo-reguli have the general form $\langle(0,0,1,0)\rangle$ and $\langle(0,0,0,1)\rangle$. Note that the Baer subplanes of D_1 containing the indicated 1-dimensional left K-subspaces are $\pi_{1,0}$ and $\pi_{0,1}$ respectively.

Assume that $y = xT_i$ for i in λ and $x = 0$ are the lines of the second pseudo-regulus net D_2.

Select two distinct values c, d of λ and change bases by the mapping $(x, y) \to (x, -xT_c + y)$.

This mapping fixes $x = 0$ pointwise and carries $y = xT_c$ onto $y = 0$. Now change bases again by the mapping $(x, y) \to (x(T_d - T_c), y)$. It follows, after the basis change, that the components of the second pseudo-regulus net are $x = 0, y = x(T_d - T_c)^{-1}(T_i - T_c)$. In particular, $x = 0, y = x, y = 0$ are components of the second net D_2. Since $x = 0$ is fixed pointwise by both of the above basis changes, it follows that there are two transversal lines L, M to D_2 intersecting $x = 0$ in $\langle(0,0,0,1)\rangle$ and $\langle(0,0,1,0)\rangle$ respectively such that $L \cap (y = 0) = \langle(m,n,0,0)\rangle$ and $M \cap (y = 0) = \langle(s,t,0,0)\rangle$. Since L and M both intersect $y = x$ in a $1-$dimensional left K-subspace, it follows that $m = 0 = t$. Hence, L and M are $\pi_{0,1}$ and $\pi_{1,0}$ respectively. It then easily follows since D_2 is a derivable net with partial spread in $PG(3,K)$ that the components of the net have the general form $x = 0, y = x\begin{bmatrix} u^\sigma & 0 \\ 0 & u \end{bmatrix}$ $\forall u \in K$ and σ an automorphism of K. However, there is at least a third transversal line to the derivable net. Since the Baer subplanes of the net in question now have the form $\rho_{a,b} = \{(a\alpha^\sigma, b\alpha, a\beta^\sigma b, \beta)\}$ then there exist a, b such that $ab \neq 0$ and a and b are in $Z(K)$ so it must be that $\sigma = 1$. Hence, the set of images $x = 0, y = 0, y = x(T_d - T_c)^{-1}(T_i - T_c)$ is equal to $x = 0, y = 0, y = x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ $\forall\ u \in K$. Thus, $T_i = T_c + (T_d - T_c)uI\ \forall\ u \in K$. To prove part (2), it suffices to show that $T_d - T_c = v_oI$ for some v_o in K. Since $x = 0$ is fixed pointwise, it would then also follow that the vector-transversals to both pseudo-regulus nets share the same points on $x = 0$.

Hence, we have a partial spread $x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall\, u \in K$ and $y = x(T_c + (T_d - T_c)vI) \;\forall\, v \in K$.

Let $T_d = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $T_c = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$. It then follows that all matrix differences are nonsingular or zero so that we must have

$$\begin{bmatrix} a_2 + (a_1 - a_2)v - u & b_2 + (b_1 - b_2)v \\ c_2 + (c_1 - c_2)v & d_2 + (d_1 - d_2)v - u \end{bmatrix}$$

is nonsingular $\forall\, u, v \in K$. Assume that $(b_1 - b_2) \neq 0$. Then choose v so that $b_2 + (b_1 - b_2)v = 0$. Then there exists a $u \in K$ such that $a_2 + (a_1 - a_2)v - u = 0$. This is a contradiction so $b_1 = b_2$ and similarly $c_1 = c_2$.

Note that if $c_2 = 0$ then choosing $u = a_2$ shows the matrix difference to be singular. Hence, $c_2 d_2 \neq 0$. Also, since $c_1 = c_2$ and $T_d - T_c$ is nonsingular, it follows that $a_1 - a_2 \neq 0$.

Then, for each u in K, we may choose v in K so that $a_2 + (a_1 - a_2)v - u = 1$. This leaves us with the matrix

$$\begin{bmatrix} 1 & b_2 \\ c_2 & 1 - a_2 + d_2 + ((d_1 - d_2) - (a_1 - a_2))v \end{bmatrix}.$$

Then, in order that the matrix be nonsingular, we must have

$$b_2 - c_2^{-1}(1 - a_2 + d_2 + ((d_1 - d_2) - (a_1 - a_2))v) \neq 0 \forall v \in K.$$

Clearly, this forces $(d_1 - d_2) - (a_1 - a_2) = 0$.

Hence, $T_d - T_c = \begin{bmatrix} a_1 - a_2 & 0 \\ 0 & a_1 - a_2 \end{bmatrix}$. Thus, there is an associated elation group E of the form

$$\left\langle \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall\, u \in K \right\rangle.$$

Assume the conditions of (3). Let D_1 be represented in the form

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in K$$

and choose the two common components to be $x = 0, y = 0$. Let D_2 have components $y = xT_i$ and $x = 0, y = 0$. Further, assume the two $1-$dimensional left K subspaces lie on $x = 0$ and are $< (0,0,1,0) >$ and $< (0,0,0,1) >$. Choose a new basis by $(x, y) \to (xT_d, y)$. Then there are Baer subplanes in the image of D_2 which are 2-dimensional left K-subspaces that share $x = 0, y = 0, y = x$ and such that the subplanes intersect $x = 0$ in $\langle (0,0,1,0) \rangle$, and $\langle (0,0,0,1) \rangle$. It follows, similarly as in the previous argument, that the two Baer subplanes have the form

$\langle (1,0,0,0),(0,0,1,0)\rangle$ and $\langle (0,1,0,0),(0,0,0,1)\rangle$. Hence, as before, D_2 is now represented as $x=0, y=0, y=x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ $\forall\ u \in K-\{0\}$. Thus, $T_i = T_d uI$. Hence, there is an associated homology group H of the form

$$\left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix} \forall\ K-\{0\} \right\rangle.$$

Since $y=0$ is pointwise fixed by the basis change above, this proves (3). Furthermore, if D_2 shares at least three components with D_1 then this will force $T_d = v_o I$ so that the two pseudo-regulus nets are identical. This proves (1).

Thus, we have the following result:

Theorem 11. (1) *Let π be a translation plane with spread in $PG(3,K)$, for K a skewfield, which is a union of a normal set of pseudo-reguli that share exactly one line L. Then there is an elation group E with axis L of π which acts regularly on lines $\neq L$ of each pseudo-regulus.*

(2) *Let π be a translation plane with spread in $PG(3,K)$ which is a union of a normal set of pseudo-reguli that share exactly two lines L and M. Then there is a homology group H of π with axis and coaxis L and M which acts regularly on lines $\neq L$ or M of each pseudo-regulus .*

In a following section, we discuss what are called "skew-Hall" planes which are defined by deriving translation planes which, although are not necessarily always Desarguesian, are related to translation planes admitting families of reguli in their spreads. Also, because of the connection with flocks of quadratic cones and flocks of hyperbolic quadrics in $PG(3,K)$, for K a field, we formulate the following definitions.

Definition 5. *Let π be a translation plane with spread in $PG(3,K)$, for K a skewfield.*

If the spread for π is a normal set of pseudo-reguli sharing exactly one line L, we shall call π a conical translation plane.

If the spread for π is a normal set of pseudo-reguli sharing exactly two lines M and N, we shall call π a ruled translation plane.

Theorem 12. (1) *If π is a conical translation plane then the spread for π may be represented in the form $x=0, y=x\begin{bmatrix} u+g(t) & f(t) \\ t & u \end{bmatrix}$ $\forall\ t,u \in K$ and for g,f functions on K.*

(2) *f and g are functions on K such that $x^2t + xg(t) - f(t) = \phi_x(t)$ is bijective $\forall\ x$ in $K \iff$ the functions define a spread of the form* (1).

(3) *If π is a ruled translation plane then the spread for π may be represented in the form*

$$x=0, y=0, y=x\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}, y=x\begin{bmatrix} g(t)u & f(t)u \\ tu & u \end{bmatrix} \forall t,v,u,ut \neq 0 \in K$$

where g, f functions on K.

(4) *If π is a conical translation plane with line L and also a ruled translation plane with lines L and M such that two normal points (1-dimensional left K-subspaces) lie on L then the spread for π may be represented in the form*

$$x = 0, y = x \begin{bmatrix} u + gt & ft \\ t & u \end{bmatrix} \forall t, u \in K$$

where g and f are constants in K.

Proof: If π is a conical translation plane, choose coordinates so that the common component is $x = 0$, and one of the pseudo-reguli has the form $x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ $\forall\ u \in K$. We shall refer to this as the *standard form.* It follows from the above results that each remaining pseudo-regulus has the form $x = 0, y = x(S + uI\ )$ for a set $\{S\}$ of 2×2 K-matrices. Recall that the set of components must be of the general form

$$x = 0, y = x \begin{bmatrix} G(t,u) & F(t,u) \\ t & u \end{bmatrix} \forall t, u \in K$$

and for functions G and F from $K \times K$ to K. Since the group E exists as a collineation group, the result (1) now directly follows.

If π is a ruled translation plane, choose coordinates so that the two common components are $x = 0, y = 0$ and that the two 1-dimensional left K -subspaces referred to in the statement lie on $x = 0$. Choose one pseudo-regulus to have the standard form. Since, the plane now admits the group H listed previously, the result (3) now follows immediately.

If π is a ruled translation plane of the type listed in statement (4) , use the form of (1) and apply the group H to obtain the conclusion that

$$g(t)v = g(tv) \text{ and } f(t)v = f(tv)\ \forall t, v \neq 0 \in K.$$

Hence, letting $g(1) = g$ and $f(1) = f$, (4) is now clear.

Assume the conditions of (2). We have that

$$x = 0, y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u \end{bmatrix} \forall t, u \in K$$

and for g, f functions on K defines a spread $\Longleftrightarrow$, for each nonzero vector (a, b, c, d) such that not both a and b zero, there exists a unique pair (u, t) such that

$$\begin{aligned} a(u + g(t)) + bt &= c \text{ and} \\ af(t) + bu &= d. \end{aligned}$$

If a is zero then there is a unique such pair, namely $(b^{-1}d, b^{-1}c)$. If $b = 0$ then since $f(t)$ is bijective, the unique pair is $(f^{-1}(a^{-1}d), a^{-1}c - g(f^{-1}(a^{-1}d))$. If $ab \neq 0$ then

$$\begin{aligned} b^{-1}af(t) - g(t) - a^{-1}bt &= b^{-1}d - a^{-1}c \Longleftrightarrow \\ z^2t + zg(t) - f(t) &= a^{-1}d - a^{-1}ba^{-1}c \text{ for } z = a^{-1}b. \end{aligned}$$

Hence, if ϕ_z is bijective then, for any given d,c, there is a unique t in K that satisfies the above equation and defining

$$u = -a^{-1}b - a^{-1}bt - g(t) = b^{-1}d + b^{-1}af(t),$$

there is a unique pair (u, t) which satisfies the first system of equations.

3 Point-Baer subplanes in spreads of $PG(3, K)$.

We first recall some results of Jha and Johnson [15].

Definition 6. *Let N denote an affine net containing an affine subplane π_o. We shall say that π_o is a point-Baer subplane if and only if every point of the projective extension of N is incident with a line of the projective extension of π_o.*

Similarly, a line-Baer subplane is a subplane such that every line of the projective extension of N is incident with a point of the projective extension of π_o.

A Baer subplane is a subplane which is both point-Baer and line-Baer.

A planar collineation of an affine net is said to be a point-Baer collineation if its fixed plane is a point-Baer affine subplane.

Definition 7. *Under the assumptions of the previous definition, a point-Baer collineation σ of the net N is called a point-Baer perspectivity with axis Fix$\sigma = \pi_\sigma$ if and only if N admits a partition of its affine points by a collection C of σ-invariant point-Baer subplanes of N such that each plane of C different from π_σ meets π_σ in at most one point. C is called the center of the Baer perspectivity σ and the members of C are called the central planes of σ. If π_s lies in C then σ is said to be a point-Baer elation and is called a point-Baer homology otherwise.*

Remark 5. *If σ is a collineation of an affine plane then Fix σ is a Baer subplane if and only if Fixσ is either point-Baer or line-Baer.*

Proof: Jha and Johnson [14] Theorem 14.

Definition 8. *Let D denote a vector space net which contains a point-Baer affine plane π_o sharing the same parallel classes. Let K_o denote the kernel of π_o. A full K_o-point-Baer elation group is a collineation group of the net which fixes π_o pointwise which may be identified with*

$$\left\langle \begin{bmatrix} I & \gamma \\ 0 & I \end{bmatrix} \forall\gamma \in K_o \right\rangle.$$

A full K_o-point-Baer homology group is a group of the net which fixes π_o pointwise which may be identified with

$$\left\langle \begin{bmatrix} I & 0 \\ 0 & \gamma \end{bmatrix} \forall\, \gamma \in K_o - \{0\} \right\rangle.$$

Theorem 13. *(Jha and Johnson [15]). Let π be a translation plane with kernel K. Assume that σ is a point-Baer collineation and $Fix\sigma = \pi_o$. Let K_o denote the kernel of π_o. Let G denote a collineation group in the translation complement which fixes π_o pointwise.*

Then π_o is a K-subspace in any of the following situations:
(i) K is a field and $\mid G \mid > 2$.
(ii) The characteristic is not 2 and σ is a point-Baer elation.
(iii) G is a full K_o-point-Baer elation group and $|K_o| > 2$
(iv) L_o is a subskewfield of K_o and G is a full L_o-point-Baer homology group of order > 2.

Theorem 14. *(Jha and Johnson [15]). Let S be a spread in $PG(3, K)$ for K a skewfield. Let π denote the corresponding translation plane. Let σ be a point-Baer collineation where $Fix\sigma = \pi_o$ and K_o is the kernel of π_o. Let G denote a collineation group in the translation complement which fixes π_o pointwise.*

If G is a full K_o-point-Baer elation group of order > 2 or a full K_o -point-Baer homology group of order > 2 then the kernel of π is K, K_o is isomorphic to K and K is a field.

Specifically, we are interested in the nature of the point-Baer elation and point-Baer homology groups. Most of the following theorem follows from more general results of Jha and Johnson in [15].

Theorem 15. *Let π be a translation plane with spread in $PG(3, K)$, for K a skewfield. Note that the following are considered the kernel mappings:*

$$(x_1, x_2, y_1, y_2) \to (kx_1, kx_2, ky_1, ky_2) \text{ for } k \in K - \{0\}$$

whereas linear collineations may be represented by 4×4 matrices acting on the right and components have the general form $x = 0, y = xM$ where M is a 2×2 matrix acting on the right.

(1) Let π admit a nontrivial point-Baer elation group B which fixes the point-Baer subplane π_o pointwise. Let the kernel of π_o be K_o.

If B is a full K_o-point-Baer elation group of order > 2 then π_o is a K-space, K is a field isomorphic to K_o and B may be represented in the form

$$\left\langle \begin{bmatrix} 1 & \beta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall \beta \in K \right\rangle.$$

(2) If π admits a nontrivial point-Baer homology group C then let $FixC = \pi_o$ have kernel K_o. If C acts as the full K_o-point-Baer homology group of the net containing $FixC$ and has order > 2 then K is a field isomorphic to K_o and C may be represented in the form

$$\left\langle \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall \lambda \in K^* \right\rangle.$$

Note, in this case, there is another point-Baer subplane sharing its infinite points with FixC and invariant under C. This second point-Baer sublane is called coFixC.

(3) *If L is any 2-dimensional left K -subspace which is disjoint from FixB then the orbit of L under B union FixB is a pseudo-regulus net in $PG(3,K)$.*

(4) *If L is any 2-dimensional left K-subspace which is disjoint from FixC or coFixC then the orbit of L under C union FixC and coFixC is a pseudo-regulus net in $PG(3,K)$.*

Proof: (1) and (2) follow from Jha and Johnson [15].

Now assume the conditions of (3).

Change coordinates so that Fix$B$ is represented by $x = 0$. B now has the following form

$$\left\langle \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall u \in K \right\rangle.$$

The set of B -images of $y = 0$ is $y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall\, u \in K$. It follows immediately from Johnson [17] that this defines a derivable net. We may assume that a 2-dimensional subspace disjoint from $x = 0$ has the general form $y = xN$ where N is a 2×2 matrix (possibly singular). By a coordinate change of the form $\begin{bmatrix} I & -N \\ 0 & I \end{bmatrix}$, we may take the 2-dimensional subspace as $y = 0$ without changing the new form of the group B. This then proves (3). (See also Jha-Johnson [12]

Assume the conditions of (4). Change bases over the prime field so that the fixed subplanes have the form $x = 0$, and $y = 0$. The group now has the form

$$\left\langle \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall u \in K^* \right\rangle.$$

Any 2-dimensional K-subspace disjoint from $x = 0$ and $y = 0$ has the form $y = xN$ where N is a nonsingular 2×2 matrix. Change bases by $\begin{bmatrix} I & 0 \\ 0 & N^{-1} \end{bmatrix}$. This basis change leaves the new form of the group invariant and the images of the 2-dimensional subspace are $y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall\, u \in K^*$.

By the above remarks, we have the proof to (4).

4 Partial flocks of deficiency one.

We now apply the results of the previous section to $PG(3,K)$, where K is a field.

Definition 9. *Let C_o be a conic in a plane of $PG(3,K)$, for K a field. Let v_o be a point exterior to C_o and form the quadratic cone.*

A partial flock of the quadratic cone is a set of mutually disjoint conics which lie in the cone.

A partial flock of deficiency one is a partial flock such that for each line of the cone, the union of the conics cover all but exactly one of the nonvertex points.

Theorem 16. (1) *The set of partial flocks of a quadratic cone of deficiency one in $PG(3,K)$, for K a field, is equivalent to the set of translation planes with spreads in $PG(3,K)$ that admit a point-Baer elation group B which acts transitively on nonfixed 1-dimensional K-subspaces of components of the fixed point subplane.*

(2) *A partial flock of a quadratic cone of deficiency one in $PG(3,K)$, for K a field, may be extended to a flock $\Longleftrightarrow$ in the corresponding translation plane which admits a point-Baer elation group B as in* (1) *the net defined by the point-Baer affine plane FixB defines a regulus in $PG(3,K)$.*

Proof: Assume that such a translation plane π exists. We point out that when we use the term Baer subplane of a regulus net we are not asserting that the subplane indicated is a Baer subplane of any other net containing the regulus net as this is not always the case in the infinite case.

Since K is a field, the point-Baer subplane is always a 2-dimensional K -subspace. Hence, the group has the following representation:

$$\left\langle \begin{bmatrix} 1 & \beta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall \beta \in K \right\rangle .$$

The components of the net N containing π_o have the basic form

$$x = 0, y = x \begin{bmatrix} u & b(u) \\ 0 & u \end{bmatrix} \forall u \in K$$

and b a function on K.

Since the kernel may be given by the mappings

$$\begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix} \forall \beta \in K$$

then the components of the spread which are not in the net N have the following general form:

$$y = x \begin{bmatrix} G(t,u) & F(t,u) \\ t & u \end{bmatrix} \forall t \neq 0, u \in K$$

where G and F are functions of $K \times K$.

Change coordinates by $(x_1, x_2, y_1, y_2) \to (x_1, y_1, x_2, y_2)$ so that the group has the following form:

$$\left\langle \begin{bmatrix} 1 & 0 & \beta & 0 \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall \beta \in K \right\rangle .$$

The components which are not in the net have the form:

$$y = x \begin{bmatrix} -G(t,u)t^{-1} & F(t,u) - G(t,u)t^{-1}u \\ t^{-1} & t^{-1}u \end{bmatrix} \forall t \neq 0, u \in K.$$

Hence, the components not in N have the general form

$$y = x \begin{bmatrix} G^*(t,u) & F^*(t,u) \\ t & u \end{bmatrix} \forall t \neq 0, u \in K$$

for functions G^* and F^* on $K \times K$.

Since the plane admits the group listed as above, it follows that $G^*(t,u) = g(t)+u$ and $G^*(t,u) = f(t)$ for functions g, f on K.

It is now clear that there is a set of regulus nets R_t with partial spread $x = 0$ (which is the new equation for π_o) and

$$y = x \begin{bmatrix} g(t)+u & f(t) \\ t & u \end{bmatrix} \forall u \in K \text{ and } t \text{ fixed in } K - \{0\}.$$

Note: We are not claiming that the matrices involved are nonsingular. However, we may select one of the matrices and change bases without changing $x = 0$ so that this matrix is zero.

The resulting partial spread has the form: R_t with partial spread $x = 0$ which is the new equation for π_o and

$$y = x \begin{bmatrix} g_1(t)+u & f_1(t) \\ t & u \end{bmatrix} \forall u \in K \text{ and } t \text{ fixed in } K - \{t_o\}, \text{ for } t_o \neq 0$$

where the matrices and differences are either zero or nonsingular.

Now given any 1-dimensional K-subspace X. If X is in the original net N then the B-orbit of X generates a 2-dimensional K-subspace which is a component of the net and which intersects π_o in a 1 -dimensional subspace. If X is not in the original net N then the B -orbit of X generates a 2-dimensional K-subspace which again intersects π_o in a 1-dimensional K-subspace.

Take the set of 2-dimensional K-subspaces each of which is invariant under B and each of which intersects π_o in the same 1-dimensional K-subspace P_o. Then, there is *exactly* one which defines a component of the net N. It then follows directly that any other such 2 -dimensional subspace is a Baer subplane of one of the regulus nets R_t.

Using the Klein quadric, it is not hard to see that corresponding to a partial flock of a quadratic cone in $PG(3,K)$, there is a partial spread in $PG(3,K)$ which is a union of a set of regulus partial spreads which share a common component. Furthermore, the points of the lines of the cone which are covered by the conics of the partial flock correspond to Baer subplanes of the corresponding regulus nets defined by the reguli in $PG(3,K)$. The correspondence is such that the nonvertex points on a given line of the cone correspond to the set of Baer subplanes which share a given 1-dimensional K-subspace on the common component of the partial spread.

We consider the following partial flock of a quadratic cone in $PG(3,K)$ defined by equation $x_o x_1 = x_2^2$ in the plane $x_3 = 0$ given by homogeneous coordinates (x_o, x_1, x_2, x_3) with vertex $(0,0,0,1)$:

The planes containing the conics of intersection are:

$\rho_t : x_o t - x_1 f(t) + x_2 g(t) + x_3 = 0.$

It follows easily that $\{\rho_t \ \forall \ t \neq 0 \}$ defines a partial flock of the quadratic cone (see for example Jha-Johnson [12]).

Since the points on the lines of the cone correspond to Baer subplanes incident with the zero vector of the regulus nets and each of these is a 2 -dimensional K-subspace which is invariant under B, and for each 1 -dimensional K -subspace on π_o, there is exactly one component containing this 1-dimensional K-subspace, it follows that there is exactly one point on each line of the cone which is not covered by the partial flock. This shows that the deficiency is one.

Now assume that there is a partial flock F of a quadratic cone of deficiency one in $PG(3,K)$. Again, note that there is a corresponding partial spread P_F in $PG(3,K)$ which is the union of a set of reguli that share a common component and admit a group B. The points on a line of the cone which are covered by the partial flock correspond to Baer subplanes of the regulus nets of the translation net corresponding to the partial spread. Hence, for each line of the cone, there is exactly one point which then corresponds to a 2-dimensional K-subspace which is B-invariant in the corresponding vector space (which is the ambient space of the translation plane) and which is not a Baer subplane of one of the regulus nets. The points on the line of the cone correspond to the set of all B-invariant 2 -dimensional K-spaces which intersect the common component L in a fixed 1-dimensional K-subspace.

Hence, for each 1-dimensional K-subspace Z of L, there is a unique 2-dimensional K-subspace π_Z containing Z which is B-invariant which does not lie as a Baer subplane in one of the regulus nets of the partial spread. So, there is a set $R = \{\pi_Z \ ; \ Z$ is a 1-dimensional K-subspace of $L\}$ of B-orbits which then must consist of mutually disjoint 2-dimensional K-subspaces. It also clearly follows that these subspaces are disjoint from the partial spread excluding the common component L.

Clearly, $R \cup (P_F - L)$ covers the set of all 1-dimensional K-subspaces and is a set of 2-dimensional K-subspaces. Hence, we have a translation plane with spread in $PG(3,K)$ and which admits B as a collineation group that fixes L pointwise. Since L is a 2-dimensional K-subspace, L becomes an affine plane of the new translation plane Σ. We note that L defines a line-Baer subplane of Σ which admits a collineation group B fixing it pointwise. We have also noted that this implies that the subplane is Baer and hence point-Baer. This completes the proof of part (1) of the theorem.

If the partial flock can be extended to a flock then there is a translation plane π^+ corresponding to the flock by Jha-Johnson [12] which admits B as an elation group. By (1), there is a translation plane π corresponding to the partial flock which admits B as a point-Baer group and which shares all components with π^+ which do not lie in the net N defined by the components of Fix$B$. Let N^+ denote the subnet of π^+ which replaces N. We notice that the components of N are generated by B -orbits. Moreover, it follows that N^+ is a regulus net so that B -orbits of 1-dimensional K-subspaces within N^+ generate the Baer subplanes of N^+ which are the components of N. Hence, N is the opposite regulus net of N^{+}.

Now assume that the plane π is derivable with net N. Then the net contains at least three point-Baer subplanes so it follows from the preceding section that each of the point-Baer subplanes is a K-subspace. This means the net N is a regulus net. By Johnson [24], derivation of this net determines a translation plane with spread in $PG(3,K)$ which admits an elation group B such that any component orbit union the axis of the group forms a regulus net. It follows from Jha-Johnson [12] that the derived translation plane corresponds to a flock of a quadratic cone which extends the original partial . This completes the proof of the theorem.

Remark 6. *If one assumes that there is a partial flock of a quadratic cone which has the property that the union of the conics cover all but at most one nonvertex point on each line then either the partial flock is either a flock or a partial flock of deficiency one.*

Proof: Assume that partial flock is not a flock. Then there is a partial spread P admitting an elation group E such that each component orbit union the axis of E is a regulus net. Furthermore, the assumption implies that there is a subset W of 1-dimensional K-subspaces of Fix$E$ such that for each element ω of W there is a unique 2-dimensional K -subspace which is E $-$invariant which is not covered by the partial spread. Remove W from Fix$E$ and adjoin the set S of these E-invariant 2-dimensional K-subspaces. It follows that $\{P - W\} \cup S$ is a spread in $PG(3,K)$. If W consists of exactly one 1-dimensional K -subspace Z then Fix$E$ remains a component of the new translation plane because Z is removed then readjoined with the 2-dimensional E-invariant subspace of S. If W has more than one element then Fix$E$ cannot be a component but is a 2-dimensional K-subspace so defines a line-Baer subplane of the translation plane. However, E is now a collineation group which fixes the line-Baer subplane Fix$E$ pointwise so that, by the previous remark, Fix$E$ is a Baer subplane and hence a point-Baer subplane.

In either case, the partial spread has been extended to a spread. In Jha-Johnson [13], it is shown that if one line of the cone is completely covered by the conics then the partial flock is a maximal partial flock. Hence, if there is at least one line of the cone such that all but one points are covered and for every line at most one point is uncovered then there is a unique point on each line of the cone which is uncovered. Hence, the partial spread has deficiency one.

Definition 10. *Let H be a ruled quadric of the form $x_1x_4 = x_2x_3$ in $PG(3,K)$ represented by homogeneous coordinates (x_1, x_2, x_3, x_4). A partial flock of H is a set of mutually disjoint conics which lie in H.*

A partial flock of deficiency one is a partial flock such that on any line of either ruling, there is exactly one point which is not covered by the conics of the flock.

Theorem 17. (1) *The set of partial flocks of ruled quadrics in $PG(3,K)$, for K a field, of deficiency one is equivalent to the set of translation planes with spreads in $PG(3,K)$ which admit a point-Baer homology group B which is transitive on the nonfixed* 1*-dimensional K -subspaces on any component of FixB.*

(2) *A partial flock of deficiency one of a ruled quadric may be extended to a flock $\Longleftrightarrow$ the net defined by FixB of the corresponding translation plane is a regulus net.*

Proof: A partial flock of a ruled quadric in $PG(3,K)$ gives rise to a partial spread in $PG(3,K)$ which is the union of a set of reguli that share two lines. Furthermore, a flock corresponds to a translation plane whose spread has the same property. In addition, the translation plane admits an affine homology group B each of whose components orbits union the axis and coaxis is a regulus net. This is also true of any such corresponding partial spread.

In general, the Baer subplanes of the regulus nets correspond to points on the ruling lines. Again, we are not necessarily assuming that the subplanes are Baer in any net other than the regulus net in question. Furthermore, the set of points on a given ruling line correspond to the set of Baer subplanes that share a given 1-dimensional K-space on a given common component of the partial spread. The corresponding Baer subplanes are 2-dimensional K -subspaces generated by point orbits under B. Let the two sets of ruling lines be denoted by R_+ and R_-. Let J_+ be a line of R_+ . The points on J_+ correspond to the 2-dimensional K-subspaces which are generated by B-orbits of 1-dimensional K-subspaces which are not on the common components L and M and which intersect on one of the common components, say L, in a particular 1-dimensional K-subspace.

First, let π be a translation plane with spread in $PG(3,K)$ that admits a point-Baer homology group so there are at least two point-Baer subplanes incident with the zero vector which share the same infinite points. By the previous section, since K is a field, each point-Baer subplane is a K-subspace and the group B may be represented in the form

$$\left\langle \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \forall \lambda \in K^* \right\rangle .$$

Following Johnson [23], it follows that each component orbit disjoint from the two subplanes π_o and π_1 under B union π_o and π_1 defines a K-regulus in $PG(3,K)$.

Let N denote the net containing the two point-Baer subplanes. It then follows that corresponding to the partial spread $\{\pi - N\} \cup \{L,M\}$ is a partial flock since $\{\pi - N\} \cup \{L,M\}$ admits the appropriate "homology group" with axis π_o and coaxis π_1 if $\mathrm{Fix}B = \pi_o$ and $\{L,M\} = \{\pi_o,\pi_1\}$.

The 2-dimensional K-subspaces which are generated by point orbits under B intersect π_o and π_1 in 1-dimensional K-subspaces. Each such point-orbit of a point which is not on a component of N lies in a Baer subplane of a K-regulus net which corresponds to a conic. For a given 1-dimensional K-subspace of π_o, there is a unique B-orbit 2 -dimensional K-subspace which is not in one of the regulus nets. This unique B-orbit is the component of N containing the subspace in question. This component also nontrivially intersects π_1. Hence, it follows that for each line of either ruling of the ruled quadric, there is a unique point of the line which is not covered by conics corresponding to the regulus nets of $\{\pi - N\} \cup \{L,M\}$. Hence, the partial spread has deficiency one.

Now assume that a partial spread has deficiency one. Then, there is a partial spread of the form $P \cup \{L,M\}$ admitting a homology group B which may be represented in the form

$$\left\langle \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \forall \lambda \in K^* \right\rangle$$

and such that each component orbit of P union $\{L,\ M\}$ defines a regulus in $PG(3, K)$.

Similarly as above, the points of the ruling lines correspond to the 2 -dimensional K-subspaces which are B-invariant and not equal to L or M. The deficiency one assumption implies that for each 1-dimensional K -subspace of L, there is a unique 2-dimensional K-subspace invariant under B which is not in a regulus net of the partial spread. The union of the set S of these 2-dimensional subspaces cover both L and M and are mutually disjoint. Form $P \cup S$. Clearly, this partial spread completely covers all 1-dimensional K-subspaces and consists of 2 -dimensional K -subspaces so that a translation plane with spread in $PG(3, K)$ is obtained. Since the components are defined via B-orbits, B is a collineation group of the constructed translation plane. However, now L and M are 2-dimensional K-subspaces which then define line-Baer affine planes. Since

$$\left\langle \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \forall \lambda \in K^* \right\rangle$$

is a collineation group of the translation plane that fixes L pointwise and

$$\left\langle \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \forall \lambda \in K^* \right\rangle$$

defines the kernel homology group, it follows that

$$\left\langle \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \forall \lambda \in K^* \right\rangle$$

is also a collineation group of the translation plane that fixes M pointwise. We have noted that any point-Baer or line-Baer subplane which is pointwise fixed by a collineation is a Baer subplane. This completes the proof of part (1).

Now assume that there is a partial flock of deficiency one which may be extended to a flock. Let π denote the translation plane admitting the point-Baer group B and let Σ denote the translation plane corresponding to the flock. Let G denote the regulus-inducing homology group of Σ. We note that the subplanes of Σ which are the Baer subplanes of the regulus nets sharing two components L and M correspond to the 2-dimensional K -spaces which are G-invariant. Hence, all of the components of the net N of π are now various of these subplanes of Σ. Hence, $G = B$. In other words, N is a regulus net and π and Σ are derivates of each other by replacement of N.

Conversely, if N is a regulus net in π then, by Johnson [24], it must be that derivation by N produces a translation plane Σ and B becomes a homology group in Σ of the correct form to produce a flock of a quadratic cone. (There are derivable nets in infinite affine planes such that derivation of the net does not produce an affine plane.)

Remark 7. *If a partial flock has the property that the conics cover all but at most one point of each line of a ruling class then the partial flock is either a flock or a partial flock of deficiency one.*

Proof: The proof is very similar to the proof of the corresponding remark for partial flocks of quadratic cones and is left to the reader.

Definition 11. *The translation planes corresponding to partial flocks of deficiency one of either quadratic cones or hyperbolic quadrics shall be called deficiency one translation planes.*

4.1 Skewfields:

Actually, the above results are not stated in the most general way. We have noted that certain point-Baer elation or point-Baer homology groups force the kernel of an associated spread to be commutative. In particular,

Theorem 18. *Let π denote a translation plane with spread in $PG(3, K)$, for K a skewfield.*

Let π_o be a point-Baer subplane with kernel K_o. Let G denote a collineation group of π.

(1) *If G is a full K_o-point-Baer elation group of order > 2 then K is a field and there is a corresponding partial flock of deficiency one of a quadratic cone in $PG(3, K)$.*

(2) *If G is a full K_o-point-Baer homology group of order > 2 then K is a field and there is a corresponding partial flock of deficiency one of a hyperbolic quadric in $PG(3, K)$.*

5 Skew-Hall planes.

The finite Hall planes of order q^2 originally constructed by changing multiplication in a finite field so as to construct an associated quasifield are exactly those translation planes obtained by the derivation of a regulus net in a Desarguesian spread in $PG(3, GF(q))$. If K is a field which admits a quadratic extension $K[\theta]$ then there are analogous infinite Hall planes constructed via Pappian spreads in $PG(3, K)$. A Pappian plane defines flocks of both quadratic cones and ruled quadrics in $PG(3, K)$.

We have seen that flocks of quadratic cones or ruled quadrics are equivalent to translation planes with spreads in $PG(3, K)$ which are unions of reguli either sharing one or two lines respectively.

A regulus cannot exist in $PG(3,K)$, for K a skewfield, unless K is a field (see Grundhöfer [11]). However, pseudo-reguli do exist in $PG(3,K)$ as we have seen (also see Johnson [17]).

In this section, we consider translation planes with spreads in $PG(3,K)$ which are unions of pseudo-reguli sharing either one or two lines. We construct a class of translation planes which may be derived from a translation plane which has these properties. Because of the similarity to Hall planes, we call these skew-Hall planes. Note that the translation planes deriving the skew-Hall planes are both conical and ruled.

Theorem 19. *Let K be a skewfield and let π denote a 4-dimensional left K -vector space.*

(1) *Then the set of left 2-dimensional K-subspaces*

$$\{x = 0, y = x \begin{bmatrix} u+\rho t & \gamma t \\ t & u \end{bmatrix} \ \forall t, u \in K\}$$

is a spread in $PG(3,K) \iff z^2 + z\rho - \gamma \neq 0 \ \forall \ z \in K$ where x and y are 2-vectors.

Let $\pi_{\rho,\gamma}$ denote the corresponding translation plane.

(2) *$\pi_{\rho,\gamma}$ is a Desarguesian plane $\iff$ both ρ and γ are in the center of K.*

(3) *$\pi_{\rho,\gamma}$ is a semifield plane which admits the elation group*

$$E_u = \left\langle \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall u \in K \right\rangle.$$

The component orbits union the axis $x = 0$ of E define a set C_u of pseudo-regulus nets that share exactly the component $x = 0$ and each of these nets may be derived to produce an associated translation plane.

(4) *$\pi_{\rho,\gamma}$ also admits the elation group*

$$T_t = \left\langle \begin{bmatrix} 1 & 0 & \rho t & \gamma t \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall t \in K \right\rangle.$$

The component orbits union the axis $x = 0$ of E defines a set C_t of pseudo-regulus nets that share exactly the component $x = 0$, and each of these nets may be derived to produce an associated translation plane.

(5) *$\pi_{\rho,\gamma}$ also admits the homology group*

$$H = \left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix} \forall u \in K^* \right\rangle.$$

The component orbits union the axis $(y = 0)$ and the coaxis $(x = 0)$ define a set R of pseudo-regulus nets that share two components $x = 0, y = 0$ and each of these nets may be derived to produce an associated translation plane.

(6) *For each derived translation plane, the kernel of the plane is* $Z(K)$.

In particular, if K *is not a field then the plane derived from a Desarguesian plane does not have its spread in* $PG(3, K)$ *and hence cannot be considered a Hall plane even in this case.*

Also, note that if K *is infinite-dimensional over* $Z(K)$ *then the derived planes are infinite-dimensional over their kernels.*

Proof: By previous results, we obtain a spread $\iff$ equation $z^2t+z\rho t-\gamma t = \phi_z(t)$ is bijective for each z in K . Hence, we obtain a spread $\iff$ $z^2 + z\rho - \gamma$ is always nonzero for z in K.

This proves that a spread is obtained under the stated conditions.

Direct computation shows that the spread is multiplicative $\iff$ ρ and $\gamma \in Z(K)$ and furthermore multiplicative inverses exist provided $z^2 + z\rho - \gamma \neq 0$ for z in K.

From Johnson [17], it follows easily that

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in K$$

is a derivable net and also a pseudo-regulus net in $PG(3, K)$ (see also Johnson [22] and the previous results on pseudo-reguli). Clearly, any nontrivial image of a component by E_u union the axis of E_u is then a pseudo-regulus net and similarly, any nontrivial image of a component by H union the axis and coaxis of H is a pseudo-regulus net.

It remains to show that any nontrivial image of a component by T_t union the axis of T_t is a pseudo-regulus net in $PG(3, K)$. Change bases by the matrix

$$\begin{bmatrix} 1 & \rho & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \gamma^{-1} & 0 \end{bmatrix}.$$

This turns T_t into E_u.

Although at first glance the following seems trivial, it remains to show that when $x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in K$ is derived, there is a constructed translation plane. As we mentioned previously, there are derivable nets in affine planes such that derivation of these nets do not produce a corresponding affine plane. On the other hand, derivation by a regulus in a spread in $PG(3, H)$, for H a field, does always produce an affine plane but this result cannot necessarily be used as our spread is in $PG(3, K)$ for K a skewfield.

In order that the constructed structure is, in fact, an affine plane, it must be that the subplanes of this derivable net are Baer subplanes of the translation plane. Since the net is derivable, the subplanes must be point-Baer subplanes. It remains to show that they are also line-Baer subplanes.

The affine subplanes sharing the parallel classes incident with the zero vector are $\rho_{a,b} = \{(a\alpha, b\alpha, a\beta, b\beta) \ \forall \ \alpha, \beta \in K\}$ and a and b not both zero.

We note from the previous section that any left 2-dimensional K-subspace defines a line-Baer subplane. If either a or $b = 0$ then $\rho_{a,b}$ is a left 2-dimensional K-subspace. Hence, assume $ab \neq 0$.

Take any line $y = x \begin{bmatrix} u+\rho t & \gamma t \\ t & u \end{bmatrix} + (c,d)$. It must be shown that this line intersects $\rho_{a,b}$ projectively. Clearly, we may assume that $t \neq 0$ as if $t = 0$ then a line of the net containing the subplane is obtained and the line projectively intersects the subplane.

Hence, it remains to show that there exist $\alpha, \beta \in K$ such that

$$(a\alpha, b\alpha, a\alpha(u+\rho t) + b\alpha t + c, a\alpha\gamma t + b\alpha u + d) = (a\alpha, b\alpha, a\beta, b\beta).$$

Since $ab \neq 0$, we obtain the following equations:

$$\begin{aligned} a^{-1}(a\alpha\rho + b\alpha)t + a^{-1}c + \alpha u &= \beta, \\ b^{-1}a\alpha\gamma t + \alpha u + b^{-1}d &= \beta. \end{aligned}$$

If $\alpha = 0$ then there is an intersection with $x = 0$ and the given line at $(0,0,c,d)$. Hence, if $(c,d) = (a\beta, b\beta)$, there is an intersection with $\rho_{a,b}$ otherwise not. Thus, assume that $(c,d) \neq (a\beta, b\beta)$.

Hence, there exists such $\alpha, \beta \Longleftrightarrow$ there is a solution to

$$(\alpha\rho + a^{-1}b\alpha - b^{-1}a\alpha\gamma)t = -(a^{-1}c - b^{-1}d).$$

Note that

$$(a^{-1}c - b^{-1}d) = 0 \Longleftrightarrow (c,d) = (a\beta, b\beta) \text{ for some } \beta \in K.$$

Since we have assumed $t \neq 0$, there is a solution $\Longleftrightarrow$ ($\alpha\rho + a^{-1}b\alpha - b^{-1}a\alpha\gamma) \neq 0$. Since we can only have that $\alpha \neq 0$, so multiply by α^{-1}on the left to obtain $(\rho + \alpha^{-1}a^{-1}b\alpha - \alpha^{-1}b^{-1}a\alpha\gamma)$. Now multiply by $\alpha^{-1}a^{-1}b\alpha = x$ on the left to obtain the expression $x\rho + x^2 - \gamma$ and since this is $\neq 0$, we obtain a solution. Hence, each subplane $\rho_{a,b}$ is line-Baer and hence a Baer subplane.

It remains to show that the kernel of any derived plane Π is $Z(K)$. We may isolate on the net $x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ $\forall u \in K$ as all other nets may be transformed into this net either by a collineation of the plane or by a K-basis change.

We note that the Baer subplanes $\rho_{a,b}$ are components of the derived translation plane. As these are right K-spaces but are left invariant only under $Z(K)$ since a and b may vary over K, this shows that $Z(K)$ is a subfield of the kernel. If there exists an element g such that g is not in K and in the kernel of the plane, then the skewfield generated by K and g, $\langle K, g\rangle$, must fix all components of the translation plane which are not in the derivable net. Since all such components are 2-dimensional K-subspaces, this forces the components to be 1-dimensional $\langle K, g\rangle$ -subspaces. This implies that these components may be embedded into a Desarguesian affine plane Σ coordinatized by $\langle K, g\rangle$.

At this point, it might be well to state a more general result on derivation.

Proposition 20. *Let D be a pseudo-regulus net with components lines of $PG(3,K)$, for K a skewfield. Let V_4 denote the associated left K-vector space.*

If T is a line of $PG(3,K)$ which as a vector subspace of V_4 is contained in the union of the components of D then T intersects each component of D and defines a Baer subplane of the net whose infinite points are exactly those of D.

Proof: Represent the components of D in standard form $x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ $\forall$ $u \in K$. Any line T is a 2-dimensional left vector space. Certainly, if K is finite then T intersects each component of D. Hence, assume that K is infinite. T intersects each component in a 0-dimensional or 1-dimensional K-vector subspace. Hence, there are infinitely many components which T intersects in 1-dimensional K -vector subspaces. Choose any three of these as $x = 0, y = 0, y = x$ and choose a new basis, if necessary, to preserve the standard form. Hence, relative to this basis, $T = \langle (s, t, 0, 0), (0, 0, s, t) \rangle$ where not both s and t of K are zero. Since T is a left 2-dimensional K -subspace, the 1-dimensional subspace $T_u = \langle (s, t, us, ut) \rangle$ for $u \in K$ must be contained in some component of D. Clearly, this component is $y = x \begin{bmatrix} s^{-1}us & 0 \\ 0 & t^{-1}ut \end{bmatrix}$. Thus, $s^{-1}us = t^{-1}ut = u^{\sigma}$ $\forall$ $u \in K$. So, T must intersect each component $y = x \begin{bmatrix} u^{\sigma} & 0 \\ 0 & u^{\sigma} \end{bmatrix}$ $\forall$ $u \in K$ and since σ is an automorphism of K, it follows that T must intersect each component of D in a 1 -dimensional vector subspace so there is a spread induced on T which forces T to be a Desarguesian affine Baer subplane of D whose infinite points are exactly those of D.

Now we return to the proof of the statement above.

Let D denote the derivable net of the original translation plane and let M denote the complementary net. Then $\pi = \pi_{\rho,\gamma} = D \cup M$ and $\Sigma = N \cup M$ for some net N. It follows that D and N are replacements for each other and note that all components are at least K -subspaces. Assume that D and N are not equal. Then each component T of N which is is not a component of D is a 2-dimensional K-subspace which intersects each component of D either trivially or in a 1 -dimensional K-subspace and is contained in the net D. By the above proposition, it follows that T is an affine Baer subplane of D sharing all infinite points with D. Hence, a subnet N^* of N is defined by the set of Baer subplanes of D so that π and Σ are derivates of each other. Since the Baer subplanes of D are left 2-dimensional subspaces, it follows that K is a field. Since N and N^* cover the same points, $N^* = N$.

Thus, it follows that D and N are either equal or one is the opposite regulus of the other and, in the latter case, K must be a field. In this situation, the original plane is Pappian and the derived plane is Pappian.

However, the derived plane will now admit a full point-Baer homology group or a full point-Baer elation which acts regularly on the components of the pseudo-regulus net other than the axis and coaxis or axis respectively. It follows that the kernel of one of the Baer subplanes is K and we have seen that this would force the axis Baer subplane to be an L-space where L is the kernel of the plane. In other words, the kernel can't be as large as a skewfield coordinatizing the entire space since the superplane contains a kernel group acting transitively on the nonzero points of any component. Hence, the kernel of the derived plane is $Z(K) = K$ in this situation.

The other situation is when π is actually Desarguesian and coordinatized by $L = \langle K, g \rangle$ and derivable and there is an element g not in K which fixes each Baer subplane of the derivable net D. Hence, g induces an element of the kernel on each Baer subplane. Since the kernel of each Baer subplane is K^{opp} it follows that g must be in $Z(K)$ acting as a kernel homology of the derived plane which is a contradiction to our assumptions. This completes the proof of the theorem.

Example 1. *Let K be any quaternion skewfield defined over a field F such that there exist nonsquares in F. Then there exist nonsquares γ in K.*

Hence, $x^2-\gamma$ is nonzero $\forall x\in K$.

Let $z=a+be_1+ce_2+de_3$ be a typical element in K with a,b,c,d in F where e_i for $i=1,2,3,4$ satisfy the conditions of Pickert p. 160 table 39 [29].

Then $z^2=(a^2+b^3c_1+c^2c_2-d^2c_1c_2)+b^2e_1+c^2e_2+d^2e_3$ where c_i for $i=1,2$ are the constants of table 39. So, if for example, γ is nonsquare in F then γe_i are all nonsquares in K.

There are quaternion skewfields of any characteristic (see e.g. Cohn [5] p.292, Pickert [29], section (6.3) and Yaqub [32] lemmas 2,3,4) and there exist skewfields of characteristic 2 such that there exist nonsquares in F.

Definition 12. *Any translation plane $\pi_{\rho,\gamma}$ shall be called a skew-Desarguesian plane and any plane derived as above from $\pi_{\rho,\gamma}$ shall be called a skew-Hall plane.*

Remark 8. *We noted above when there is a full K_o-point-Baer homology or full K_o-point-Baer elation group of order >2 acting on a translation plane with spread in $PG(3,K)$ then this forces K to be a field.*

In the skew-Hall planes, there are both full K-point-Baer homology groups and elation groups for subplanes with kernel $K=K_o$ and K is not necessarily commutative. On the other hand, the kernel of the associated translation plane is $Z(K)$. It would appear that the existence of full K_o-point-Baer elation or homology groups forces the kernel of a corresponding translation plane to be a field. However, this has been proved only for spreads in $PG(3,K)$.

6 Double-covers.

In Biliotti-Johnson [3], the concept of double-covers is developed for spreads in $PG(3,K)$, where K is a field. Here, we consider this in a more general setting.

Definition 13. *Let π denote a conical translation plane with spread in $PG(3,K)$. Let C denote a normal set of pseudo-reguli which share a common line L and whose union is the spread for π. If there exists a second normal set of pseudo-reguli $C^*\neq C$ which share a common line L^* whose union is the spread for π, we shall call $\{C,C^*\}$ a double-cover for π. If $L^*=L$ and $\{C,C^*\}$ is a normal set of pseudo-reguli then we call $\{C,C^*\}$ a normal, double-cover with common line.*

So, we see that the skew-Desarguesian planes admit normal double-covers with a common line. The converse is also true as we now point out.

Theorem 21. *Let π be any conical translation plane with spread in $PG(3,K)$, for K a skewfield. If π admits a normal, double-cover with common line then π is a skew-Desarguesian plane.*

Proof: Represent π in the form $x=0, y=x\begin{bmatrix} u+g(t) & f(t) \\ t & u \end{bmatrix}$ $\forall u,t\in K$ with respect to the cover C and choose the common line to be $x=0$. Let D be any pseudo-regulus net in C^*. By our assumptions, if we choose the same basis for $x = 0$ with any basis for the entire space then a choice of lines of $D - \{x = 0\}$ as $y = 0, y = x$ forces D to be in standard form. Moreover, D can share at most two components with any net of C. Let λ denote the subset of K so that D share two components with the regulus net

$$\{x = 0, y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u \end{bmatrix} \text{ for } t \text{ fixed and } \forall u \in K\}.$$

By previous results, we may assume that 0 is in λ. That is, we may assume that D shares $x = 0$ and $y = 0$ with the standard net in C. Hence, there exists a subset λ of K such that the components of D have the form

$$x = 0, y = 0, y = x \begin{bmatrix} u_t + g(t) & f(t) \\ t & u_t \end{bmatrix} \forall t \in \lambda$$

and for each t, u_t is in K. Choose a basis change which fixes $x = 0$ pointwise and leaves $y = 0$ invariant and maps

$$y = x \begin{bmatrix} u_{t_1} + g(t_1) & f(t_1) \\ t_1 & u_{t_1} \end{bmatrix} = xM$$

onto $y = x$ for $t_1 \in K - \{0\}$. Let $\sigma = \begin{bmatrix} M & 0_2 \\ 0_2 & I_1 \end{bmatrix}$ and change basis applying the mapping σ. Then, it follows that

$$\begin{bmatrix} u_{t_1} + g(t_1) & f(t_1) \\ t_1 & u_{t_1} \end{bmatrix}^{-1} \begin{bmatrix} u_t + g(t) & f(t) \\ t & u_t \end{bmatrix} = \begin{bmatrix} h(t) & 0 \\ 0 & h(t) \end{bmatrix} \forall t \in \lambda.$$

Since we obtain the standard pseudo-regulus net, it follows that $\{h(t) \mid t \in \lambda\} = K$. In other words, the mapping $t \mapsto h(t)$ is surjective from λ to K. Since, differences of matrices of D are nonsingular, it follows that the indicated mapping is also injective. Hence, $\lambda = K$ and

$$D = \{x = 0, y = xM_D \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in K$$

and M_D a nonsingular matrix depending only on $D\}$. Another way to see this is to apply a previous result which states that if two normalizing pseudo-reguli share two components then they share a homology group of the form

$$\left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix} \ \forall u \in K - \{0\} \right\rangle.$$

Furthermore, when we choose D to be represented in the standard form then all pseudo-regulus in C^* have the form $x = 0, y = x \begin{bmatrix} u + g^*(t) & f^*(t) \\ t & u \end{bmatrix}$ for functions g^* and f^*.

Notice that we have shown that any pseudo-regulus net of C^* must share two components with each pseudo-regulus net of C. There are also two elation groups E_C and E_{C^*} where

$$E_c = \left\langle \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall\, u \in K \right\rangle = \left\langle \begin{bmatrix} I_2 & uI_2 \\ 0_2 & I_2 \end{bmatrix} \forall\, u \in K \right\rangle .$$

It follows that

$$E_{C^*} = \left\langle \begin{bmatrix} I_2 & M_D u I_2 \\ 0_2 & I_2 \end{bmatrix} \forall\, u \in K \right\rangle .$$

Since the pseudo-reguli from C share two components with pseudo-reguli from C^*, it follows that $E_C \cap E_{C^*} = \langle 1 \rangle$. Let $M_D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then there is a component of the form $y = x \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ which is not in the standard net. Hence, it follows that neither c nor b can be zero. Taking the image of $y = 0$ under E_{C^*} and then E_C, in turn, we obtain $y = x \begin{bmatrix} at & bt \\ ct & dt \end{bmatrix}$ and then $y = x \begin{bmatrix} at+u & bt \\ ct & dt+u \end{bmatrix}$. Let $dt + u = v$ and $s = ct$ to obtain components of the form $y = x \begin{bmatrix} \rho s + v & \gamma s \\ s & v \end{bmatrix}$ where $\rho = (a-d)c^{-1}$ and $\gamma = bc^{-1}$. This completes the proof of the theorem.

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