On flag transitive $c.c^*$-geometries admitting a duality

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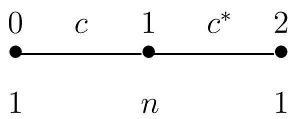
Abstract

We continue the classification of flag–transitive $c.c^*$–geometries (Γ, G) started in [Ba1, Ba2]. We consider those geometries which admit a duality. We show that, if Γ is not covered by a truncated Coxeter complex of type D_n and if the stabilizer of a point G_p has no regular normal subgroup, then (Γ, G) is one of 4 exceptional examples or the stabilizer G_p is a linear group $L_2(q)$, where $(q-1) \equiv 0(4)$ or $q = 2^r$, r even or G_p is a unitary group. Moreover, we reduce the problem to determine the geometries with $G_p \cong U_3(q)$ to the problem to determine those with $G_p \cong L_2(q)$, $q = p^r$, r even. We apply our results to flag–transitive $C_2.c$–geometries having exactly two points on a line.

1 Introduction.

We follow [Bue2] for the terminology and notation of diagram geometry. A $c.c^*$-*geometry* is a geometry with diagram as follows:

$(c.c^*)$



where n is a positive integer, called the *order* of the geometry. The integers above the nodes are the types. As usual we also call the elements of type 0 points, those of type 1 lines and those of type 2 circles. We recall that the stroke

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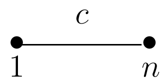
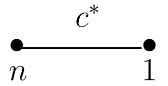


means the class of circular spaces with $n+2$ points and



has the dual meaning. We also recall that a circular space is a complete graph with at least three vertices, viewed as a geometry of rank 2 with vertices and edges as points and lines, respectively. Let $\mathcal{P}$ be the set of points and $\mathcal{C}$ the set of circles of Γ. A *duality* of Γ is an incidence preserving bijective map of Γ which exchanges the points and circles.

A class of $c.c^*$–geometries is provided by the semibiplanes, where a *semibiplane* is a connected incidence structure satisfying:

(i) any two points are incident with 0 or 2 common blocks;
(ii) any two blocks are incident with 0 or 2 common points.
(see for example [Wi]). A semibiplane where each pair of points is incident with exactly two blocks, is called *biplane*.

In [Ca] and [CaKa] biplanes Γ are considered which admits some polarity, i.e. a duality of order two. They also assume that all points of Γ are absolute points. There are biplanes having a polarity without any or only some absolute points, for example the unique biplanes with 4 or 11 points, respectively. In this paper we classify the flag-transitive $c.c^*$-geometries which admit a duality. As an application we obtain a classification of the flag-transitive $C_2.c$-geometries with thin lines. The main result of this paper is used in the general classification of flag-transitive $c.c^*$-geometries, see [BaBue].

Notation. Let G act flag–transitively on Γ and let $\{p, l, c\}$ be a maximal flag. Then for $x \in \{p, l, c\}$, we denote by G_x the stabilizer of x in G and by K_x the kernel of the action of G_x on Γ_x, the residue of x in Γ. In order to simplify the notation we will also write G_0, G_1 and G_2 instead of G_p, G_l and G_c. As usual we denote by B the stabilizer of the maximal flag $\{p, l, c\}$ and abbreviate $G_i \cap G_j$ by $G_{i,j}$.

Let $\overline{\Gamma} = \Gamma(G, (G_0, G_1, G_2))$ be the group geometry whose objects of type i are the cosets of G_i in G, $0 \leq i \leq 2$; incidence being non trivial intersection. As $\overline{\Gamma}$ is isomorphic to Γ we are sometimes identifying both geometries.

According to [Ba1] the stabilizer of a point, G_0, is a doubly transitive permutation group, so either an affine or an almost simple group, [Ca]. The known examples of $c.c^*$-geometries are given in [Ba2], but see also [BaPa] or [BaBue].

In [Ba2] the following has been shown. If G_0 is an almost simple group, then G_0 is a group of Lie–type of rank 1 in its natural action or $G_0 \cong L_3(2), L_2(11), A_7$ of degree $7, 11, 15$, respectively, or Γ is covered by a $\{1, \ldots, n-3\}$–truncation of the dual Coxeter complex of type D_n, $Tr(\Delta_n)$. If $G_0 \cong L_3(2)$, $L_2(11)$, A_7, then $G \cong U_3(3)$, M_{12}, M_{22} or $3M_{22}$ and Γ has 36, $144, 176$ or 352 points, respectively, see [Ba2, Theorems A and B]. Theorem B of [Ba2] moreover states, if G_0 is a group of Lie–type of rank 1 in its natural action and if Γ is of order at most 20, then

$soc(G_0) \cong L_2(q)$ and one of the following holds:

(i) $q = 4,\ G \cong L_2(11)$ and Γ has 11 points;

(ii) $q = 5,\ E(G) \cong A_6$ or $3A_6$ and Γ has 6 or 18 points, respectively;

(iii) $q = 9,\ E(G) \cong L_3(4)$ or $2L_3(4)$ and Γ has 56 or 112 points, respectively;

(iv) $q = 11,\ E(G) \cong M_{12}$ and Γ has 144 points.

See also the paper of Grams and Meixner [GM], where they studied some of the geometries assuming $n \leq 10$.

Notice, that all known flag–transitive $c.c^*$–geometries having an almost simple stabilizer of a point admit a duality except the example (iv).

The flat flag–transitive $c.c^*$–geometries are determined in [BaPa], where *flat* means that each point is incident with each circle. They are either a gluing of two copies of an affine n–dimensional space over $GF(2)$ or the flat JvT-geometry (the geometry which is listed in (ii) having 6 points).

In [BaBue] the following situation is considered. Suppose G_0 is not an affine group. If (Γ, G) is a minimal not known example with respect to its order and for that order with respect to the number of points, then G is a group of Lie-type [BaBue, Theorems 1 and 2].

We prove

Theorem 1.1. *Let (Γ, G) be a simply connected flag–transitive $c.c^*$–geometry. Suppose that Γ admits a duality α which fixes a flag of type $\{0, 2\}$. Then one of the following holds.*

(1) *Γ is the truncated dual Coxeter-complex of type D_n, $Tr(\Delta_n)$;*

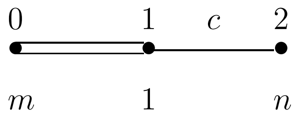
(2) *G_0 is an affine group;*

(3) *$soc(G_0) \cong L_2(q),\ (q-1) \equiv 0(4)$ or $q = 2^r,\ r$ even or $soc(G_0) \cong U_3(q)$. In both cases $soc(G_0)$ acts naturally on the circles in $res(x_0)$;*

(4) *$G_0 \cong L_3(2)$ (in its action of degree 7) and $G \cong U_3(3)$*
$G_0 \cong L_2(11)$ (in its action of degree 11) and $G \cong M_{12}$ or
$G_0 \cong A_7$ (in its action of degree 15) and $G \cong M_{22}$ or $2M_{22}$.

This theorem can be applied to flag–transitive $C_2.c$–geometries. A *$C_2.c$-geometry* is a geometry having the following diagram.

$(C_2.c)$



where m and n are positive integers. Here we call the elements of type 0, points, those of type 1 lines and those of type 2 quads.

Since any simply connected flag–transitive $C_2.c$–geometry gives rise to a simply connected flag–transitive $c.c^*$–geometry which admits a duality α which fixes a flag of type $\{0, 2\}$, (see Section 5), Theorem 1.1 yields the following result.

Corollary 1.2. *Let (Λ, G) be a flag–transitive $C_2.c$–geometry, whose lines are incident with exactly two points. Then one of the following holds.*

(1) *Λ is covered by the $\{1, ..., n-3\}$–truncation of the dual of the Coxeter complex of type C_n;*

(2) *G_0 is an affine group;*

(3) *$soc(G_0) \cong L_2(q)$, $(q-1) \equiv 0(4)$ or $q = 2^r$, r even or $soc(G_0) \cong U_3(q)$. In both cases $soc(G_0)$ acts naturally on the circles in $res(x_0)$;*

(4) *$G_0 \cong L_3(2)$ and $G \cong U_3(3) \times 2$*
$G_0 \cong L_2(11)$ (in its action of degree 11) and $G \cong Aut(M_{12})$ or
$G_0 \cong A_7$ and $G \cong Aut(M_{22})$ or $2Aut(M_{22})$.

The method of proof of Theorem 1.1 will be as follows. According to [Ba2, Theorems A and B] we are only considering $soc(G_0)$ a group of Lie–type of rank 1. In [Ba2] the generators and relations of the groups acting flag–transitively on the truncation $Tr(\Delta_n)$ where given. For any $c.c^*$–geometry Γ with $soc(G_0) \cong L_2(q), q = p^r$, r odd and $p - 1 \not\equiv 0 \mod 4$ or $soc(G_0) \cong Sz(q)$ or $soc(G_0) \cong R(q)$ we show that these relations have to hold, which will prove Theorem 1.1. This will be done in Sections 2,3 and 6. In Section 4 we consider $soc(G_0) \cong U_3(q)$. We construct some subgeometry, which is again a flag–transitive $c.c^*$–geometry whose stabilizer of a point is isomorphic to $L_2(q)$. In Section 5 we discuss the relation between $c.c^*$ and $C_2.c$-geometries. Corollary 1.2 will be proved in Section 6.

2 Flag–transitive $c.c^*$–geometries.

First we list some known facts.

Lemma 2.1. [Ba1] *A group G acts flag–transitively on a $c.c^*$–geometry Γ, if and only if there are pairwise distinct subgroups $G_0, G_1, G_2 \leq G$, satisfying the following conditions:*

(1) *G_i is a doubly transitive permutation group on $\{G_{0,2}g, g \in G_i\}$, $i \in \{0, 2\}$.*

(2) *$B \trianglelefteq G_1$, $G_1/B \cong E_4$, $G_{1i}/B \cong \mathbb{Z}_2$ and $G_i = \langle a_i, G_{0,2}\rangle$, $a_i \in G_{1,i} \setminus B$, $i \in \{0, 2\}$, and $B = G_{0,1,2}$.*

(3) *$G_{0,2} \cap G_{0,2}^{a_i} = B$.*

(4) *$G = \langle G_0, G_2\rangle$.*

Proposition 2.2. [Ba1, Corollary 3.5] *The geometry Γ is covered by $Tr(\Delta_n)$ if and only if there exist a flag–transitive subgroup G of $Aut(\Gamma)$ and an isomorphism φ from G_0 onto G_2 such that φ centralizes $G_{0,2}$ and such that $(a_0 a_0^{\varphi})^2 = 1$ for some $a_0 \in N_{G_0}(B) \setminus B$.*

Lemma 2.3. *Suppose that G_0 is an almost simple group of Lie type of rank 1 and suppose $G_0 \not\cong R(3) \cong L_2(8) : 3$. Then $H = \langle soc(G_0), soc(G_2)\rangle$ acts flag–transitively on Γ.*

Proof. Let K be the two point stabilizer of G_0 in its doubly transitive permutation representation. As G_0 is an almost simple group of Lie type of rank 1 and $G_0 \not\cong R(3)$, we have $G_0 = Ksoc(G_0)$. Since the Borel subgroup B is a two point stabilizer in G_0 and in G_2 as well, we obtain $G_0 = soc(G_0)B$ and $G_2 = soc(G_2)B$. Hence $G = HB$, which yields that H acts flag–transitively. ∎

We are also able to prove the converse as is shown next.

Lemma 2.4. *Let Γ be simply connected and suppose that G_0 is isomorphic to $L_2(q)$, q odd, acting naturally on the circles in $res(x_0)$. Then Γ admits as group of automorphisms a group $\widetilde{G}$, such that $|\widetilde{G} : G| = 2$ and $\widetilde{G}_0 \cong PGL_2(q)$.*

Proof. We have

$$G_0 \cong L_2(q) \cong G_2,\ G_1 \cong \mathbb{Z}_{(q-1)/2}E_4, G_{i,1} \cong D_{2(q-1)/2},\ i = 0, 2,\ G_{0,2} \cong E_q\mathbb{Z}_{(q-1)/2}$$

and the Borel subgroup B is isomorphic to $\mathbb{Z}_{(q-1)/2}$. Due to [Ba2, Lemma (4.4)] there are elements $a_0, c, d, a_2 \in G$, such that

$$G_0 = \langle a_0, c, d\rangle,\ G_1 = \langle a_0, d, a_2\rangle,\ G_2 = \langle a_2, c, d\rangle, G_{0,2} = \langle c, d\rangle,$$

$$G_{i,1} = \langle a_i, d\rangle,\ i = 0, 2,\ B = \langle d\rangle,$$

where $o(d) = (q-1)/2, o(a_i) = 2, d^{a_i} = d^{-1},\ i = 0, 2$ and $o(c) = p$. Further there is an isomorphism $\phi : G_0 \rightarrow G_2$ with $\phi_{|G_{0,2}} = id$, cf. [Ba], and we may suppose $a_2 = a_0^{\phi}$. Since Γ is simply connected, G is the universal completion of this amalgam, [Pa1].

Let e be a diagonal automorphism of G_0 such that $e^2 = d$. Then $c^e \in \langle c^{\langle d\rangle}\rangle$, $[d, e] = 1,\ e^{a_0} = e^{-1}$ and $a_0^e = a_0 d$.

We claim that e extends to an automorphism of G by setting $a_2^e = a_2 d$. For any relation $R(a_0, c, d, a_2)$ in G, we have to show that the relation $R(a_0^e, c^e, d^e, a_2^e)$ holds in G, as well.

By the existence of ϕ and by our choice of a_2 the map e defines not only an automorphism of G_0, but also an automorphism of G_2, $G_{0,1}$ and $G_{2,1}$. In G there is one further relation, namely $(a_0a_2)^2 = d^i$ for some $i \in \{1, ..., (q-1)/2\}$, cf. [Ba2, p.17] or Lemma 2.1. As $(a_0a_2)^{2e} = d^{ie} = d^i$ and $(a_0^e a_2^e)^2 = (a_0 d a_2 d)^2 = (a_0a_2)^2 = ((a_0a_2)^2)^e$, the map e extends to an homomorphism of G.

It remains to show that e is an automorphism of G. Obviously e is a surjection. Since e is surjective and G finite (see [Wi]), indeed e defines a bijection.

Hence e is an automorphism of G which normalizes G_0, G_1 and G_2 and therefore e induces an automorphism on Γ which proves the lemma. ∎

Observe that in Lemma 2.4 the condition that Γ is simply connected is neccessary. There is for example a quotient of $Tr(\Delta_{16})$ with 2^8 points, which admits as a point stabilizer $L_2(17)$, but not $PGL_2(17)$.

Using exactly the same argumentation as in Lemma 2.4 we obtain the following.

Lemma 2.5. *Let Γ be simply connected and suppose that G_0 is isomorphic to $U_3(q)$ with $(q+1, 3) = 3$. Then Γ admits as group of automorphisms a group $\widetilde{G}$, such that $|\widetilde{G} : G| = 3$ and $\widetilde{G}_0 \cong PGU_3(q)$.*

3 $c.c^*$–geometries with point stabilizer a linear, a Suzuki group or a group of Ree type.

Let $G = Aut(\Gamma)$.

Lemma 3.1. *Let (Γ, G) be a simply connected flag–transitive $c.c^*$–geometry. If Γ admits a polarity α which interchanges G_0 and G_2 and which centralizes $G_{0,2}$, then $(a_0a_0^\alpha)^4 = 1$ for any $a_0 \in N_{G_0}(B) \setminus B$.*

Proof. Set $a_2 = a_0^\alpha$. Then $(a_0a_2)^2 = (a_0a_2)^{2\alpha} = (a_2a_0)^2 = (a_0a_2)^{-2}$. Hence $(a_0a_2)^4 = 1$. ∎

Proposition 3.2. *Let (Γ, G) be a simply connected flag–transitive $c.c^*$–geometry and let $soc(G_0) \cong L_2(q), q = p^r$, r odd, $Sz(q)$ or $R(q)$. If Γ admits a duality α which fixes a flag of type $\{0, 2\}$, then there is a polarity π which acts trivially on $soc(G_0) \cap soc(G_2)$. Moreover, $(a_0a_0^\pi)^4 = 1$ for any $a_0 \in N_{G_0}(B) \setminus B$.*

Proof. Let α be chosen such that its order is a power of 2 and let $\{x_0, x_2\}$ be a flag, such that α interchanges x_0 and x_2. Hence α interchanges G_0 and G_2 and normalizes $G_{0,2}$.

For $soc(G_0) \not\cong L_2(q)$ set $H_i = soc(G_i)$ and for $soc(G_0) \cong L_2(q)$ set $H_i \cong PGL_2(q)$, respectively. By [Ba2] we may assume $G_0 \not\cong R(3) \cong L_2(8)3$ of degree 28. Note that according to Lemma 2.3 $\langle soc(G_0), soc(G_2)\rangle$ acts flag–transitively on Γ. So, by Lemma 2.4 $H = \langle H_0, H_2\rangle$ is a flag–transitive subgroup of G. Moreover, as H_i char G_i, the automorphism α interchanges H_0 and H_2.

Since $H_0 \cong PGL_2(q), Sz(q)$ or $R(q)$ we have $H_{0,2} = O_p(H_{0,2}) : B$ and $B \cong \mathbb{Z}_{q-1}$.

Set $Q = O_p(H_{0,2})$. For $\Phi(Q)$ the Frattini subgroup of Q we have $\overline{Q} = Q/\Phi(Q) \cong E_q$ and B acts regularly on $\overline{Q}^* = \overline{Q} \setminus \{1\}$, cf. [HuIII, XI, 3.1 and 13.2].

As α normalizes Q and $H_{0,2}$, the group B^α is a complement in $H_{0,2}$ to Q. Hence by the Theorem of Schur–Zassenhaus there is a $q \in Q$ with $[B, q\alpha] \leq B$.

We have $\overline{Q} : B \cong \Gamma L_1(q)$. Hence $q\alpha \in N_{GL_r(p)}(B) \cong \mathbb{Z}_{q-1} : \mathbb{Z}_r$ [HuI, II, 7.3].

Assume $q\alpha \notin B$ as an automorphism of $\overline{Q}$. As r is odd, $O_2(N_{GL_r(p)}(B)) = O_2(B)$. Since α induces on $\overline{Q}$ an automorphism whose order is a power of 2, we have $\alpha \in O_2(B)$ in contradiction to our assumption.

Thus $q\alpha \in B$ as an automorphism of $\overline{Q}$ and there exists an $b \in B$ such that $[\overline{Q}, qb\alpha] = 1$ and due to the Three–Subgroup–Lemma $[B, qb\alpha] \leq C_B(\overline{Q}) = 1$. Set $\pi = qb\alpha$. We claim that $[H_{0,2}, \pi] = 1$. The restriction of π on $H_{0,2}$ induces an automorphism β on $H_{0,2}$, which normalizes B. Hence due to [Ba2] β can be extended to some automorphism γ of H_2. Then γ centralizes the semidirect product X of $\overline{Q}$ with B. Hence $\gamma \in C_{Aut(H_0)}(B) = B$ and $\gamma \in C_{N_{Aut(H_0)}(Q)}(\overline{Q}) = Q$, so $\gamma \in B \cap Q = 1$ and $\gamma = 1$. This gives $\beta = 1$ and $[H_{0,2}, \pi] = 1$ as claimed.

As $qb \in H_{0,2}$ the automorphism π interchanges H_0 and H_2. Since $C_{Aut(H_i)}(H_{0,2}) = 1$, it follows that π^2 acts trivially on H_i, for $i = 0, 2$. Thus $\pi^2 = 1$. This proves the first part of the proposition.

As π is of order 2, it interchanges a_0 and a_2. Now Lemma 3.1 implies the last part of the assertion. ∎

Corollary 3.3. *Let Γ be simply connected and suppose that $soc(G_0) \cong L_2(q)$, $(q-1) \equiv 2(4)$ or $q = 2^r$, r odd or $soc(G_0) \cong Sz(q)$. If Γ admits a duality α which fixes a flag of type $\{0,2\}$, then Γ is $Tr(\Delta_n)$.*

Proof. We may assume $G_i = soc(G_i)$ for $i = 0, 2$. By Proposition 3.2 $(a_0 a_0^\alpha)^4 = 1$ for any $a_0 \in N_{G_i}(B) \setminus B$. Since $(a_0 a_0^\alpha)^2 \in B$ and $O_2(B) = 1$, in fact $(a_0 a_0^\alpha)^2 = 1$. Thus Proposition 2.2 yields the assertion. ∎

Next let us consider $soc(G_0) \cong R(q)$. Then the Borel subgroup B is of even order. Hence, we have to use more elaborated arguments as for $L_2(q)$, $(q-1) \equiv 2(4)$ or $q = 2^r$, r odd or $Sz(q)$ to show that Γ is covered by $Tr(\Delta_n)$. We first use the fact that $(a_0 a_0^\alpha)^4 = 1$ for the polarity α (Proposition 3.2) which reduces the problem to the determination of the $c.c^*$-geometries with point stabilizer isomorphic to $R(3)$. The latter was already done in [Ba2].

Let $G = \langle soc(G_0), soc(G_2)\rangle \leq Aut(\Gamma)$. By Lemma 2.3 G acts flag–transitively on Γ and we have

$$G_0 \cong G_2 \cong R(q),\ G_1 \cong \mathbb{Z}_{q-1}E_4,\ G_{0,2} = O_p(G_{0,2})B,\ B \cong \mathbb{Z}_{q-1}$$

and $G_{i,1} \cong D_{2(q-1)}$, $i = 0, 2$.

Corollary 3.4. *Let Γ be simply connected and suppose $soc(G_0) \cong R(q)$. If Γ admits a duality α which fixes a flag of type $\{0,2\}$, then Γ is $Tr(\Delta_n)$.*

Proof. If $G_0 \cong R(3)$ of degree 28, then according to [Ba2, page 19] Γ is isomorphic to $Tr(\Delta_{26})$. Therefore by Lemma 2.3 we may assume $G_0 = soc(G_0)$. According to Proposition 3.2 we may assume that α is a polarity, which exchange G_0 and G_2 and centralizes $G_{0,2}$. Let $g \in B$ be an involution. Then $(a_0 a_0^\alpha)^2 \in \langle g\rangle$ by Proposition 3.2.

Let H_0 be a subgroup of G_0 isomorphic to $R(3)$ and let a_0 be chosen such that $a_0 \in H_0$. Set $H = \langle H_0, H_0^\alpha\rangle$. Then, as $g \in H_0$, the group geometry

$$\Gamma(H, (H_0,\ \langle a_0, H_0 \cap B, a_0^\alpha\rangle, H_0^\alpha))$$

is a $c.c^*$-geometry with point stabilizer isomorphic to $R(3)$. By ([Ba2, page 19]) $(a_0 a_0^\alpha)^2 = 1$, so Proposition 2.2 yields that Γ is covered by $Tr(\Delta_n)$. ∎

Remark 3.5. *If (Γ, G) is a $c.c^*$-geometry which does not admit a duality and whose stablizer of a point is isomorphic to $R(q)$, then we would be able to determine Γ under the assumption that we know all flag-transitive $c.c^*$-geometries with point stabilizer a linear group $L_2(q)$, using the same method as described in the next section. There we consider an element g of order $q+1$, here we would have to consider an involution.*

4 $c.c^*$–geometries with point stabilizer a unitary group.

If G_0 is a unitary group, then $q = p^r$ with r even. Therefore we can not use the idea of the proof of Proposition 3.2 to determine the flag-transitive $c.c^*$-geometries whose point stabilizers are unitary groups. But we can reduce this problem to the problem to determine the $c.c^*$-geometries whose point stabilizers are isomorphic to $L_2(q)$.

Let $soc(G_0) \cong U_3(q)$. By Lemma 2.3 we may assume $G_0 \cong G_2 \cong PGU_3(q)$. Then,

$$G_0 \cong G_2 \cong PGU_3(q),\ G_1 \cong \mathbb{Z}_{(q^2-1)}E_4,,\ G_{0,2} = O_p(G_{0,2})B,\ B \cong \mathbb{Z}_{(q^2-1)}$$

and $G_{i,1} \cong D_{2(q^2-1)},\ i = 0, 2.$

Let $g \in B$ be an element of order $q+1$ and let B be the stabilizer of the flag $\{x_0, x_1, x_2\}$ in Γ. Then g fixes $q+1$ circles on the point x_0 and $q+1$ points on the circle x_2 [HuI, II. (10.12)]. Moreover, $E(K_i) \cong L_2(q)$, where $K_i = C_{G_i}(g)/\langle g\rangle$ $(i = 0, 2)$.

Let Δ be the subgeometry of Γ whose set of elements of type i are the elements of Γ of type i which are fixed by g, $i = 0, 2$, and whose lines are the lines of Γ, whose residue is fixed elementwise by g. Let incidence be the one inherited from Γ. For $i = 0, 2$ set $C_i = E(K_i)$.

Lemma 4.1. *Let $\widetilde{\Delta}$ be a connected component of Δ containing the flag $\{x_0, x_1, x_2\}$. Then $\widetilde{\Delta}$ is a c.c*–geometry with flag–transitive group of automorphism $C = \langle C_0, C_2\rangle$ and point stabilizer $C_0 \cong L_2(q)$.*

Proof. Let y_0 be a point in $\widetilde{\Delta}$. Let y_2, z_2 be two circles being incident with the point y_0. Then there is exactly one line y_1 in Γ in $res(y_0)$ which is incident with y_2 and z_2. Hence y_1 is fixed by g and, as a line is incident with two points and two circles, g fixes the residue of y_1 elementwise. So $y_1 \in \widetilde{\Delta}$.

Thus by the definition of the lines in $\widetilde{\Delta}$, the residue of a point is a complete graph with at least 3 vertices. Then the same holds for the residue of a circle. Thus, as the residue of a line is a generalized *two*–gon, $\widetilde{\Delta}$ is a *c.c**–geometry.

In $\widetilde{\Delta}$ the residue of a plane (resp. point) contains $q+1$ points (resp. planes) on which C_2 (resp. C_0) acts faithfully. Therefore, C_i acts doubly transitively on $Res(x_i)$ $(i = 0, 2)$ and C acts flag-transitively on $\widetilde{\Delta}$. ■

Suppose that we know all flag-transitive simply connected geometries of type *c.c** with point stabilizer a linear group $L_2(q)$. We conjecture that they are truncations $Tr(\Delta_{q-1})$ for $q \geq 27$. Let (Γ, G) be a flag-transitive *c.c**-geometry with $soc(G_0) \cong U_3(q)$ and let $a_0 \in N_{G_0}(B) \setminus B$. Then, if the conjecture holds, the previous Lemma implies that there is an $a_2 \in N_{G_2}(B) \setminus B$ such that $(a_0a_2)^2 \in \langle g\rangle$ and that $\widetilde{\Delta}$ would be a quotient of $Tr(\Delta_{q-1})$. Therefore, $C_G(g)$ would have as a factor either $2^qL_2(q)$ or $2^{(q-1)/2}L_2(q)$. In [BaBue] we consider the minimal not known examples whose stabilizer of a point is not an affine group; minimal with respect to the order n and for that n with a minimal number of points. We proved

Theorem 4.2. *[BaBue, Corollary (1.3)] Let (Γ, G) be a flag-transitive c.c*-geometry. Suppose that (Γ, G) is a minimal not known example whose stabilizer of a point is not an affine group. Then G is a group of Lie-type.*

Moreover, it is well known that a *c.c**-geometry of order n has at most 2^{n-1} points and is isomorphic to $Tr(\Delta_{n-1})$ if and only if Γ has exactly 2^{n-1} points.

Hence, if the conjecture holds, then it remains to determine those groups of Lie-type G, which contains a subgroup U isomorphic to $U_3(q)$ such that $|G : U| < 2^{q^3}$ and which possess an element g of order $q+1$ whose centralizer in G has a factor as described above.

5 Relation between $c.c^*$ and $C_2.c$–geometries.

To any $c.c^*$–geometry Γ a $C_2.c$–geometry is related in a natural way. Relate to a $c.c^*$–geometry Γ the following rank three geometry $\Lambda = \Lambda(\Gamma)$.
Let Δ be the point–circle incidence graph of Γ and call a quadrangle of Δ geometric if its four vertices are incident with a common line in Γ. Define the rank three geometry $\Lambda = \Lambda(\Gamma)$ as follows. Take as points, lines and quads of Λ the vertices, the edges and the geometric quadrangles of Δ, respectively.

Lemma 5.1. *The geometry Λ is of type $C_2.c$. Further if Γ is flag–transitive and possesses a duality which fixes a flag of type $\{0, 1, 2\}$, then Λ is flag–transitive, as well.*

Proof. It is straightforward to check that Λ is of type $C_2.c$.

If G acts flag–transitively on Γ, then G extended by the duality acts transitively on the maximal flags of Λ. ∎

Observe that those $C_2.c$–geometries whose point–line truncation is a bipartite graph are exactly those obtained from a $c.c^*$–geometry. If Λ is a $C_2.c$–geometry whose point–line truncation is a bipartite graph, then we get a $c.c^*$–geometry Γ by taking as set of points one part, as set of circles the other part of the bipartition and as set of lines the quads of Λ. A point is incident to a circle if and only if they are on a common line in Λ and incidence between points (circles) and lines is defined by inclusion.

Lemma 5.2. *Let Λ be a $C_2.c$–geometry with two points on a line. Then the point–line truncation of the universal cover $\widetilde{\Lambda}$ of Λ is a bipartite graph.*

Proof. This follows immediately from [Neu] or [Ri, Theorem 1]. ∎

Corollary 5.3. *Each simply connected $C_2.c$–geometry with two points on each line is related to exactly one simply connected $c.c^*$–geometry up to exchange of 'points' and 'circles'. Moreover, if Λ is related to a $c.c^*$–geometry than each cover of Λ is related to a $c.c^*$–geometry.* ∎

Recall the two properties of geometries. The first condition is equivalent to the Intersection Property in [Bue1].

(IP) For any two elements x and y, the set of points incident with x and y coincide, if not empty, with the set of points incident with some element z, which is incident with both x and y;

and

(LL) Each pair of points is incident with at most one line.

Lemma 5.4. *The following statements are equivalent.*

(i) *(LL) holds in* Γ.

(ii) *(IP) holds in* Γ.

(iii) *(IP) holds in* $\Lambda(\Gamma)$.

(iv) Δ *is a rectagraph, that is each path of length three lies in a unique quadrangle.*

Proof. According to [Pa2, Lemma (7.25)] (i) and (ii) are equivalent.

Now we want to prove the euivalence of (iii) and (i). Assume first (i). Since in Λ the points and the lines are the vertices and the edges of Δ and since Δ is a bipartite graph, (IP) holds for x or y a point or a line. Two quads intersect in $0, 1, 2$ or 4 points, since in Δ each path of length 3 lies in exactly one geometric quadrangle. If two different quads intersect in exactly two points, then, by (i), these two points are incident with a line. Hence (iii) holds. Assume that (i) does not hold. Then there are two points, which are incident with at least two lines. Hence in $\Lambda(\Gamma)$ there are two points at distance two which are incident with two quads. Thus for these two quads the Intersection Property does not hold, i.e. (iii) implies (i).

Finally we claim that Δ is a rectagraph under the assumption of (i). Since in Δ each path of length 3 lies in a geometric quadrangle, we have to show that Δ only contains geometric quadrangles. Assume that $\{c_1, c_2, c_3, c_4\}$ are the vertices of a non–geometric quadrangle, c_i being a neighbour of c_{i+1}. Then c_1, c_2, c_3 as well as c_1, c_4, c_3 lie in a geometric quadrangle. Hence in Γ the elements c_1 and c_3 are incident with two lines. Since c_1 and c_3 are either points or circles, in both cases the property (LL) does not hold in contradiction to our assumption. If Δ is a rectagraph, then obviously (i) holds. This proves the lemma. ■

Remark. Let Λ be a $C_2.c$–geometry, whose lines are incident with exactly two points. Then obviously (IP) holds if and only if the point–line truncation of Λ is a rectagraph.

6 Proofs of Theorem 1.1 and Corollary 1.2.

In order to show Theorem 1.1 we still need the following statement.

Lemma 6.1. *Let* Γ *be a flag–transitive* $c.c^*$*–geometry. Suppose that* Γ *possesses a duality, which fixes a flag of type* $\{0, 2\}$. *Then the universal cover* $\widetilde{\Gamma}$ *of* Γ *possesses as well a duality which fixes a flag of type* $\{0, 2\}$.

Proof. Associate to Γ the $C_2.c$–geometry $\Lambda = \Lambda(\Gamma)$. Then by Lemma 5.1 Λ is flag–transitive as well. According to Lemma 5.3 the universal cover $\widetilde{\Lambda}$ of Λ is related to the universal cover $\widetilde{\Gamma}$ of Γ. As $Aut(\widetilde{\Lambda}) = Corr(\widetilde{\Gamma})$ the assertion follows. ■

Remark 6.2. *As pointed out by Pasini [pers. comm.] Lemma 6.1 is part of a general fact: Any correlation of a geometry can always be lifted to a correlation of the universal cover.*

Proof of Theorem 1.1.
According to [Ba2, Theorems A and B] else (1),(2) or (4) holds or $soc(G_0)$ is a simple group of Lie–type of rank 1. Hence assume that $soc(G_0)$ is a group of Lie–type of rank 1. Now Lemma 6.1 yields that we may assume Γ to be simply connected. Then the Corollaries 3.3 and 3.4 prove the assertion. ∎

Proof of Corollary 1.2.
Let $\widetilde{\Lambda}$ be the universal cover of Λ. Then by Lemma 5.3 $\widetilde{\Lambda} = \widetilde{\Lambda}(\Gamma)$ for some simply connected $c.c^*$–geometry. Moreover Γ is flag–transitive admitting a duality, which fixes a flag $\{p, l, c\}$ of type $\{0, 1, 2\}$ and the stabilizer of a point is isomorphic to G_0.

Notice, if Γ is $Tr(\Delta_n)$, then $\widetilde{\Lambda}$ is the $\{1, ..., n-3\}$–truncation of an apartment of type C_n. Therefore, application of Theorem 1.1 proves the corollary. ∎

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