Codimension two singularities of sliding vector fields

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Abstract

The main aim of this paper is to study the behavior of the so called *Sliding Vector Fields* around an equilibrium point. Such systems emerge from ordinary differential equations on $\mathbb{R}^3$ with discontinuous right-hand side. In this work an analysis of generic codimension two bifurcation diagram is performed by given a complete topological study of its phase portrait as well as the respective normal forms.

1 Introduction

The main aim of this paper is to study a class of codimension two singularities of the so called Sliding Vector Fields (SVF). Such systems emerge from ordinary differential equations on $\mathbb{R}^3$ with discontinuous right-hand side (see for instance [F] and [U]). In our approach we assume that these discontinuities occur on the $2-sphere$ $M = S^2$ and the rules for defining the solution orbits of such ODE are made via Filippov's convention (see [F]). In [T3] all the codimension one singularities were analyzed and we refer to it for the necessary background. In this work a singularity analysis of generic codimension-two bifurcation diagrams is performed by giving a complete topological study of its phase portrait as well as the respective normal forms.

In what follows we give some preliminaries and basic definitions.

Let $p \in M$ and $f : (\mathbb{R}^3, M) \to (\mathbb{R}, 0)$ be a C^∞ representation of M at p, with $df(p) \neq 0$. So M is the separating boundary of the regions $M_+ = \{f > 0\}$ and $M_- = \{f < 0\}$.

1991 *Mathematics Subject Classification* : 58F21, 34C05.

Key words and phrases : sliding vector field, singularity, bifurcation, normal form.

Denote by Ξ^r the set of all germs at p of C^r vector fields on $\mathbb{R}^3, p$ endowed with the C^r topology with r big enough for our purposes.

Let G^r be the set of all germs at p of vector fields Z on $\mathbb{R}^3$ defined by:

$$X(q),\ if\ q \in M_+$$

$$Y(q),\ if\ q \in M_-$$

where X, Y are in Ξ^r and on M the solution curves of Z obey Filippov's rules.

We use the notation $Z = (X, Y)$ for an element of G^r.

The *Sliding Region* (denoted by $R(X, Y)$) is the region in M where both vector fields X and Y point toward M; in this case the solutions of Z through points of M follow the orbits of the (sliding) vector field $F = F(X, Y)$. This vector field is tangent to M and is defined at $q \in M$ by the vector $F(q) = m - q$, such that m is the point where the segment joining $(q + X(q))$ and $(q + Y(q))$ is tangent to M. Observe that if $X(q)$ and $Y(q)$ are linearly dependent then q is a critical point of F. We call $F = F(X, Y)$ the $Sliding\ Vector\ Field$ (SVF) associated to (X, Y). Moreover, the boundary of $R(X, Y)$ ($\partial R(X, Y)$) can be non smooth, due to the existence of corners on it (see [T2]). It must be said that, a simple calculation shows that $F = F(X, Y)$ can be smoothly extended beyond the boundary of SR and each corner of $F(X, Y)$ is a critical point of this vector field. In this way each corner is a well distinguished singularity of the SVF. Our main interest is to classify a class of two-parameter families of local SVF on M which present such singularities. We mention that, there are generically so many topological types (at least 12) of them and we concentrate in this work, the attention to those ones we think are the most important.

For $Z = (X, Y)$ in G^r we have:

a) The set $\partial R(X, Y)$ is characterized by the points in M where X or Y is tangent to the surface. We denote by S_X (resp.S_Y) the set (in M) where X (resp. Y) is tangent to M. Generically S_X and S_Y are union of circles. So, $\partial R(X, Y)$ is the union of parts of S_X and S_Y;

b) Throughout the paper, we consider coordinates (x, y, z) around a point $p \in \partial R(X, Y)$ such that $f : \mathbb{R}^3, M \to \mathbb{R}, 0$ is given by $f(x, y, z) = z$.

1.1.Definition. Let $Z_0 = (X_0, Y_0)$ and $Z = (X, Y)$ be in G^r and $p \in Cl(R(X_0, Y_0))$. We say that Z_0 is C^0 $M - equivalent$ to Z at p if there is a neighborhood U of p in M, such that $F(X_0, Y_0)\mid_U$ is C^0 equivalent to $F(X, Y)\mid_U$.

Let $X \in \Xi^r$ and $p \in M$.

1.2.Definition. We say that p is an $M - singular\ point$ (resp. $M - regular\ point$) of X if $Xf(p) = 0$ (resp. $Xf(p) \neq O$).

1.3.Definition. We say that p is a $fold$ (resp. $cusp$) singularity of X if $Xf(p) = 0$ and $X^2f(p) \neq 0$ (resp. $Xf(p) = X^2f(p) = 0$ and $\{df(p), dXf(p), dX^2f(p)\}$ are linearly independent).

The cusp points are isolated points located at extremes of the curves of fold points.

For any $g : M \to \mathbb{R}$ denote by H_g the Hamiltonian field associated to the mapping.

A singularity $p \in M$ of $F = F(X,Y)$ is classified by the following list: 1) $p \in Int(R(X,Y))$ and it is a critical point of F; 2) p is a tangency point between $\partial R(X,Y)$ and F; 3) p is a corner of $\partial R(X,Y)$.

We deal in this work just with the last case.

The classification of the singularities of a SVF, $F(X,Y)$, is based mainly on the relative positions of the following three objects: X, Y and M.

For $Z = (X,Y)$ in G^r and p in M, we define:

(t_1) : We say that p is a $t_1 - singularity$ of $F(X,Y)$ if: i) p is a fold point of X and Y; ii) The contact between S_X and S_Y at p is cubic. We impose the following extra conditions: iii) $(XY(f)YX(f))(p) \neq (YX(f)XX(f))(p)$ and iv) $(YX(f)(p) \neq YY(f))(p)$. Let $\Sigma_{2,1}$ be the set of all vector field in G^r having p as a t_1-singularity.

(t_2) : We say that p is a $t_2 - singularity$ of $F(X,Y)$ if: i) p is a fold point of X; ii) p is a cusp point of Y; iii) the contact between S_X and S_Y at p is quadratic. We impose the following extra conditions: iv) $HH_{Xf}Yf(p) \neq 0$, $XYf(p) \neq 0$, $YXf(p) \neq 0$ and $(YXf(p) + XYf(p)) \neq 0$. As above consider the set $\Sigma_{2,2}$.

(t_3) We say that p is a $t_3 - singularity$ of $F(X,Y)$ if: i) p is a cusp point of X and Y; ii) S_X and S_Y are in general position at p. We impose the following extra conditions: iii) $XYf(p) \neq 0$, $YXf(p) \neq 0$ and $(YXf + XYf)(p) \neq 0$. As above we consider the set $\Sigma_{2,3}$.

(t_4) We say that p is a $t_4 - singularity$ of $F(X,Y)$ if: i) $Y(p) = 0$; ii) p is a fold point of X; iii) S_X and S_Y are in general position at p. We impose the following extra generic conditions: iv) $H_{XYf}Xf(p) \neq 0$ and $H_{YYf}Xf(p) \neq 0$; v) p is hyperbolic critical point of Y, with distinct eigenvalues and with the corresponding eigenspaces transverse to M at p. So we have the set $\Sigma_{2,4}$.

In this paper, we refer to a $t_i - singularity$ as a $codimension - two$ singularity, for some $i = 1, 2, 3, 4$.

2 Statement of the main result

Theorem A:
a) The set $\Sigma_2 = \bigcup_{i=1}^{4}(\Sigma_{2,i})$ is a C^{r-3} codimension two submanifold of G^r;
b) Let $Z_{\alpha,\beta}$ be a 2-parameter family of vector fields in G^r, $(r > 3)$ for which the following properties hold: i) the map $(p, \alpha, \beta) \rightarrow Z_{\alpha,\beta}(p)$ is transverse to the variety $\mathbb{R}^3 \times \Sigma_2$ at $(p, 0, 0)$; ii) The point p is a codimension 2 singularity of $Z_{0,0}$. Then all topological types of $Z_{\alpha,\beta}$ are classified and the respective normal forms are exhibited.

3 The t_1-singularity

3.1. Proposition. Let $Z_0 \in G^r$ and $p_0 \in M$. Assume that p_0 is a t_1-singularity of Z_0. Then there exist neighborhoods U of Z_0 in G^r, V of p_0 in M and a C^{r-3} mapping $h : U, Z_0 \rightarrow \mathbb{R}^2, 0$ such that: i) $dh(Z_0)$ is surjective; ii) $h(Z) = 0$ if and only if Z has a t_1-singularity p(Z) in V; iii) there is a codimension-one variety Σ_1 in $\mathbb{R}^2, 0$ which describes the bifurcation set of G^r nearby Z_0.

Proof:

Let $Z_0 = (X_0, Y_0)$. We choose coordinates (x,y,z) around $p_0 = 0$, such that

$f(x,y,z) = z$, $X_0 = (X_1, X_2, X_3)$ and $Y_0 = (Y_1, Y_2, Y_3)$,

with $X_0 f(x,y,0) = X_3(x,y,0) = y$ and $Y_0 f(x,y,0) = y - x^3$.

This implies that $(X_0^2 f)(x,y,0) = X_2(x,y,0)$, $(Y_0^2 f)(x,y,0) = Y_2(x,y,0) + 3x^2 Y_1(x,y,0)$, $X_2(0) = b \neq 0$ and $Y_2(0) = d \neq 0$.

Call $X_1(0) = a$ and $Y_1(0) = c$.

We may select neighborhood U of Z_0 in G^r and C^r functions $\phi, \rho : U \times J \to \mathbf{R}$ $y = \phi(Z,x)$ and $y = \rho(Z,x)$, which are solutions of $Xf = 0$ and $Yf = 0$ respectively, with $J = (-\epsilon, \epsilon)$ and $Z = (X,Y)$ in U.

Define the mapping $\tau(Z,x) = \phi(Z,x) - \rho(Z,x)$.

It satisfies $\tau(Z_{(0,0)}) = \frac{\partial \tau}{\partial x}(Z_0, 0) = \frac{\partial^2 \tau}{\partial x^2}(Z_0,0) = 0$ and

$$\frac{\partial^3 \tau}{\partial x^3}(Z_0, 0) \neq 0.$$

Assume for instance that the last inequality is positive.

Let $x = \eta(Z)$ be the solution of

$$\frac{\partial^2 \tau}{\partial x^2} = 0.$$

Define the mapping $h : U \to \mathbf{R^2}$ by $h = (h_1, h_2)$ where $h_1(Z) = \tau(\eta(Z), Z)$ and $h_2(Z) = \frac{\partial \tau}{\partial x}(\eta(Z), Z)$.

Observe that:

i)$h(Z) = 0$ if and only if $Z \in \Sigma_{2,1}$ (this means that the point $P(Z) = \eta(Z)$ is a t_1-singularity of Z in a small neighborhood of 0 in M);

ii) $dh(Z_0)$ is surjective.

The last assertion can be checked by taking the following family in U :

$Z_{\alpha,\beta} = (X_{\alpha,\beta}, Y_{\alpha,\beta})$ with $Z_{0,0} = Z_0$, $X_{\alpha,\beta} = X_0$ and $Y_{\alpha,\beta} = (Y_1, Y_2, y - (x^3 + \alpha x + \beta))$.

We have $\tau(x,\alpha,\beta) = x^3 + \alpha x + \beta$, $\eta(\alpha,\beta) = 0$, $\tau(0,\alpha,\beta) = \beta$, $\frac{\partial \tau}{\partial x}(0,\alpha,\beta) = \alpha$.

Now parts i) and ii) of the proposition become immediate.

We now proceed the proof of part iii). We have that:

a) if $h_2(Z) = 0$ then $\frac{\partial \tau}{\partial x}(x, Z) = 0$ if and only if $x = \eta(Z)$;

b) if $h_2(Z) > 0$ then $\frac{\partial \tau}{\partial x}(x, Z) > 0$ for every x in J;

c) if $h_2(Z) < 0$ then there are associated with Z, two points in J, $\eta_1 = \eta_1(Z)$ and $\eta_2 = \eta_2(Z)$, satisfying $\eta_1 < \eta(Z) < \eta_2$ $\frac{\partial \tau}{\partial x}(\eta_1, Z) = \frac{\partial \tau}{\partial x}(\eta_2, Z) = 0$, $\frac{\partial^2 \tau}{\partial x^2}(\eta_1, Z) < 0$ and $\frac{\partial^2 \tau}{\partial^2 x}(\eta_2, Z) > 0$;

d) the correspondence $Z \to \eta_i(Z)$ is C^{r-3}, i=1,2;

e) each $\eta_i(Z)$ converges to $\eta(Z)$ in the class C^1.

f) $(\tau(\eta_1(Z)))^2 + (\tau(\eta_2(Z)))^2 \neq 0$.

Consider the open set in G^r given by $U_1 = \{Z; h_2(Z) < 0\}$ and the real function g defined in U_1 by:

$g(Z) = g_1(Z) g_2(Z)$ where $g_i = \tau(\eta_i(Z), Z)$ with $i = 1, 2$.

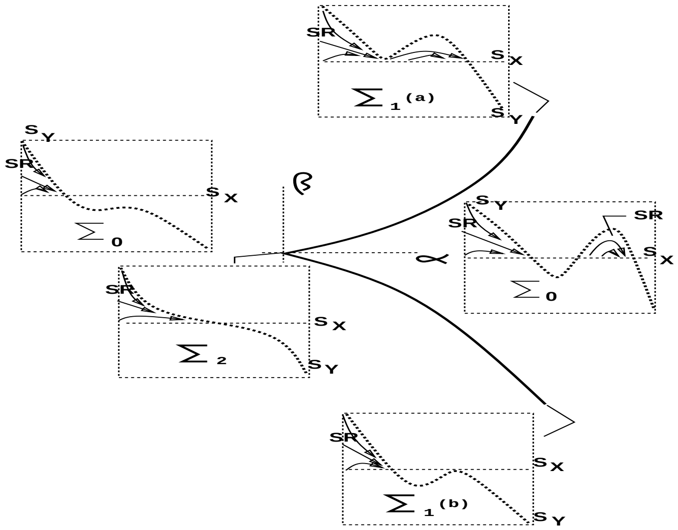

Figure 1: Unfolding of a t_1-singularity

Observe now, that Z has a codimension-one singularity in a neighborhood of p_0 in M (which is given by either η_1 or η_2) if and only if $g(Z) = 0$. Moreover, the bifurcation set in U is described by $Cl\{g^{-1}(0)\}$.

This finishes the proof of the proposition. ∎

3.2. Corollary. $\Sigma_{2,1}$ is a C^{r-3} codimension two submanifold of G^r.

3.3. Remark. Following the general form of the family $Z_{\alpha,\beta}$ given above, we deduce that $\eta_1(\alpha,\beta) = (\frac{-\alpha}{3})^{\frac{1}{2}}$, $\eta_2(\alpha,\beta) = -(\frac{-\alpha}{3})^{\frac{1}{2}}$, and the bifurcation set of the family is characterized by $\alpha^3 = \beta^2$ (see Fig.1).

3.4. Remark. In the coordinates given above, the general form of the Sliding Vector field $F_0 = F(X_0, Y_0)$ (associated with Z_0) is the following:

$$F_0(x, y, 0) = ((a-c)y - ax^3, (b-d)y - bx^3) + o[(x, y)]^3$$

3.5. Corollary. Let $Z_{\alpha,\beta}$ be a 2-parameter family of vector fields in G^r for which the following properties hold: i) $Z_{0,0}$ has a t_1-singularity p_0 in V; ii) the mapping: $(\alpha,\beta) \to Z_{\alpha,\beta}$ is transverse to $\Sigma_{2,1}$ at $(0,0)$. Then the C^0 normal form of the corresponding Sliding Vector Field is:

$F_{\alpha,\beta}(x, y) = ((a-c)y - ax^3 - a(\alpha x + \beta), (b-d)y - bx^3 - b(\alpha x + \beta))$ where $X_{\alpha,\beta}(x, y, 0) = (a, b, y - x^3)$, $Y_{\alpha,\beta}(x, y, 0) = (c, d, y - x^3 - \alpha x - \beta)$ with $b \neq 0$, $c \neq 0$, $a \neq c$ and $b \neq d$.

3.6. Remark. In the above coordinates, we have the following expressions:
$X_0X_0f(0) = b$, $X_0Y_0f(0) = a$, $Y_0X_0f(0) = c$ and $Y_0Y_0f(0) = d$.

4 The t_2-singularity

4.1. Proposition. Let $Z_0 \in G^r$ and $p_0 \in M$. Then Proposition 3.1 holds if p_0 is a t_2-singularity of Z_0.

Proof:

Let $Z_0 = (X_0, Y_0)$.

Choose coordinates (x,y,z) such that $X_0 = (X_1, X_2, X_3)$, $Y_0 = (X_1, X_2, X_3)$, $X_0 f(x, y, 0) = X_3(x, y, 0) = y$ and

$Y_0 f(x, y, 0) = Y_3(x, y, 0) = y - x^2$. So from the hypotheses we get

$$X_0 X_0 f(0) = X_2(0) = b_0 \neq 0,$$

$$Y_0 Y_0 f(x, y, 0) = Y_2(x, y, 0),$$

$Y_1(0) = 0$ and $Y_2(0) = 0$.

Let

$$Y_1(x, y, 0) = a_1 x + a_2 y + o|(x, y)|^2$$

and

$$Y_2(x, y, 0) = b_1 x + b_2 y + o|(x, y)|^2.$$

It follows that: I) $b_1 \neq 0$, provided that $Det\{df(0), dY_0 f)(0), dY_0 Y_0 f(0)\}$ is non-zero; II) $a_1 \neq 0$, provided that $HH_{X_0 f} Y_0 f(0) \neq 0$. Denote $X_1(0) = a_0$.

As before, for $Z = (X, Y)$ assume that the singular sets of X and Y are given by $S_X = \{y = \phi(x)\}$ and $S_Y = \{y = \rho(x)\}$ respectively and define the mapping

$$\tau(x, Z) = \phi(x) - \rho(x)$$

with x being in small interval around 0 and Z being in a neighborhood U of Z_0 in G_r.

We have

$$\tau(0, Z_0) = \frac{\partial \tau}{\partial x}(0, Z_0) = 0$$

and

$$\frac{\partial \tau}{\partial x^2}(0, Z_0) \neq 0$$

(say > 0).

First of all, let $x = \eta(Z)$ be the solution of $\frac{\partial \tau}{\partial x}(x, Z) = 0$. So:
i) if $\tau(\eta(Z), Z) = 0$ then for every $x \neq 0$, we have $\tau(x, Z) > 0$.
ii) if $\tau(\eta(Z), Z) < 0$then there exist, associated to Z, two points $x_1 = \eta_1(Z)$ and $x_2 = \eta_2(Z)$ with $x_1 < \eta(Z) < x_2$, $\eta(x_1, Z) = \eta(x_2, Z) = 0$, $\frac{\partial \eta}{\partial x}(x_1, Z) < 0$ and $\frac{\partial \eta}{\partial x}(x_2, Z) > 0$. Moreover the correspondence $\{Z \to x_i\}$ is smooth for $i = 1, 2$.
iii) if $\tau(\eta(Z), Z) > 0$ then for every x we have that $\tau(x, Z) > 0$.

Now, it is clear that there exists a smooth function

$$P : U \to V$$

(V being a neighborhood of p_0 in M) where for each $Z = (X, Y)$ in U, P(Z) is the cusp point of Y in V.

Finally, on defines the desired C^{r-3} mapping

$$h : U, Z_0 \to \mathbf{R^2}, \mathbf{0},$$

$h = (h_1, h_2)$ by $h_1(Z) = \tau(\eta(Z), Z)$ and $h_2(Z) = H_{Xf}Yf(P(Z))$.

We now proceed the characterization of the bifurcation set around Z_0 in G^r.

We have

a) $h(Z) = 0$ if and only if $\tau(\eta(Z), Z) = 0$ and $H_{Xf}Yf(P(Z) = 0$. The last inequality implies that $P(Z) = \eta(Z)$; the first one says that the curves S_X and S_Y have a quadratic contact at $(\eta(Z), Z))$. This means that $h(Z) = 0$ if and only if Z has a t_2-singularity in V.

b) If $\tau(\eta(Z), Z) < 0$ then there are two possibilities: b_1- $x_1 \neq P(Z)$ and $x_2 \neq P(Z)$; b_2- either $x_1 = P(Z)$ or $x_2 = P(Z)$. In the first case, Z has just codimension zero singularity in V; in the second case, Z has a codimension one singularity in V (see [T3]).

c) if $h_1(Z) = 0$ and $h_2(Z) \neq 0$ then $\eta(Z)$ is a codimension-one singularity of Z.

Define now the family

$$Z_{\alpha,\beta} = (X_{\alpha,\beta}, Y_{\alpha,\beta})$$

in G^r by

$$X_{\alpha,\beta} = X_0 + (0, 0, y + \alpha x + \beta)$$

and

$$Y_{\alpha,\beta} = Y_0.$$

In these coordinates we have

$$h_1(\alpha, \beta) = \beta - (\frac{\alpha^2}{2}) + h.o.t$$

and

$$h_2(\alpha, \beta) = \alpha + h.o.t.$$

Moreover, the cusp associated with the family is $P(\alpha, \beta) = (0, 0)$.

From the above expression of $h = (h_1, h_2)$, it is easy to show that $dh(Z_0)$ is surjective.

This finishes the proof of Proposition 4.1. ∎

4.2. Corollary. $\Sigma_{2,2}$ is a C^{r-3} codimension two submanifold of G^r.

4.3. Remark. The general form of the SVF, $F_0 = F(X_0, Y_0)$, is given by :

$$F_0(x, y, 0) = (a_0(y - x^2) - y(a_1x + a_2y), b_0(y - x^2) - y(b_1x + b_2y)) + h.o.t.$$

4.4. Corollary. Let $Z_{\alpha,\beta}$ be a 2-parameter family of vector fields in G^r for which the following properties hold : i) $Z_{0,0}$ has a t_2-singularity p_0 in V; ii) the mapping : $(\alpha, \beta) \to Z_{\alpha,\beta}$ is transverse to $\Sigma_{2,2}$ at (0,0). Then the C^0 normal form of the corresponding SVF is:

$$F_{\alpha,\beta}(x, y) = (a_0(y - x^2) - (a_1x + a_2y)(y + \alpha x + \beta), b_0(y - x^2) - (b_1x + b_2y)(y + \alpha x + \beta))$$

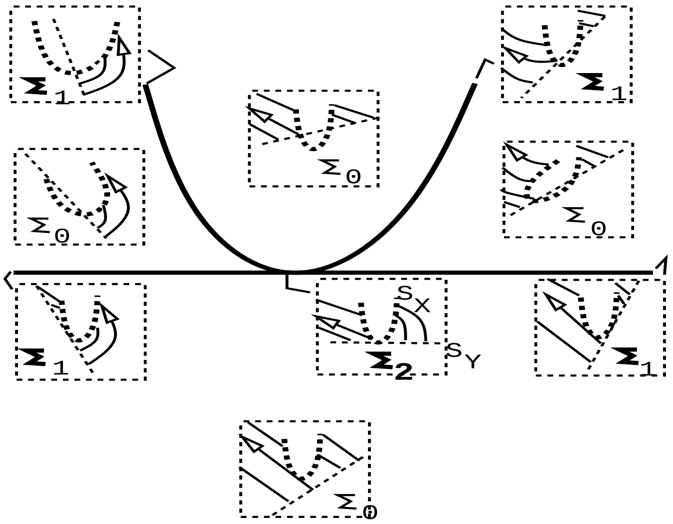

Figure 2: Unfolding of a t_2-singularity

, with $b_0 \neq 0$, $a_1 \neq 0$ and $b_1 \neq 0$. Moreover, the singular set of this family is expressed by $\beta(\alpha^2 - 4\beta) = 0$.

4.5. Remark. In addition to the above corollary we get that:
i) if $\alpha^2 = 4\beta$ then $F_{\alpha,\beta}$ has two singularities in V: a critical point which is a codimension one singularity and a cusp point which is a codimension zero singularity. This implies that $Z_{\alpha,\beta}$ is in Σ_1.
ii) if $\alpha^2 > 4\beta$ then the corresponding vector field has two critical point, both codimension zero singularities. Then $Z_{\alpha,\beta}$ is not in Σ_1.
iii) if $\alpha^2 < 4\beta$ then the vector field has no critical point in V; this implies that $Z_{\alpha,\beta}$ does not belong to Σ_1 .

5 The t_3-singularity

5.1. Proposition. Let $Z_0 \in G^r$. Then Proposition 3.1 holds if p_0 is a t_3-singularity of Z_0.

Proof:

Let $Z_0 = (X_0, Y_0)$. As above, consider the neighborhoods U and V in G^r and M respectively.

We choose coordinates (x, y, z) around $p_0 = 0$ such that $X_0 f(x, y, 0) = x$ and $Y_0 f(x, y, 0) = y$ where $X_0 = (X_1, X_2, X_3)$ and $Y_0 = (Y_1, Y_2, Y_3)$. This implies that:

$X_0 Y_0 f(0) = X_2(0)$, $Y_0 X_0 f(0) = Y_1(0)$ $X_0 X_0 f(x, y, 0) = X_1(x, y, 0)$ and $Y_0 Y_0 f(x, y, 0) = Y_2(x, y, 0)$.

Call $X_1(x, y, 0) = a_1 x + b_1 y + h.o.t$ and $Y_2(x, y, 0) = a_2 x + b_2 y + h.o.t.$

So, $b_1 \neq 0$ and $a_2 \neq 0$ provided that $\{df, dX_0 f, dX_0 X_0 f\}$ and $\{df, dY_0 f, dY_0 Y_0 f\}$, are linearly independent (at 0) respectively.

Call $Y_1(0) = a$ and $X_2(0) = b$.

As $Y_1 = Y_0 X_0 f$ and $X_2 = X_0 Y_0 f$ (at p_0), we get from the hypotheses that $a \neq 0$, $b \neq 0$ and $a + b \neq 0$.

As in Proposition 3.1, for each $Z = (X, Y) \in U$ assume that the singular sets of of X and Y are the graphs of $y = \phi(x)$ and $y = \rho(x)$ respectively. In the same way denote by c(X) and c(Y) the cusp points of X and Y contained in V, respectively. Observe that, c(X) (resp. C(Y)) is expressed by $XXf = 0$ (resp. $YYf = 0$).

The required mapping $h = (h_1, h_2) : U \to V$ is defined by $h_1 = XXf(P(Z))$ and $h_2 = YYf(P(Z))$ where P(Z) is expressed by the solution of $(\phi(x) - \rho(x)) = 0$.

Call by Σ_1' the variety in U characterized by the identities $P(Z) = c(X)$ or $P(Z) = c(Y)$ which are expressed by the equation $h_1 h_2 = 0$. It is clear that this variety lies in the bifurcation set of G^r.

Consider the family in G^r, $Z_{\alpha,\beta}$ given by:

$$X_{\alpha,\beta} = X_0 + (\alpha, 0, 0)$$

and

$$Y_{\alpha,\beta} = Y_0 + (0, \beta, 0).$$

We have that:

i) $h_1(\alpha, \beta) = \alpha + a_1 x + b_1 y + h.o.t;$
ii) $h_2(\alpha, \beta) = \beta + a_2 x + b_2 y + h.o.t;$
iii) $P(\alpha, \beta) = (0, 0);$
iv) The cusp points are given by

$$c(\alpha) = (0, -(\frac{\alpha}{b_1})) + h.o.t$$

and

$$c(\beta) = (-(\frac{\beta}{a_2}), 0)) + h.o.t.$$

Let us fix attention on the bifurcation set Σ_1. We have to distinguish the following subcase:

Distinguished saddle: p_0 is a saddle point of the associated SVF F_0 in such a way that, both invariant manifolds of $dF_0(0)$ meet the correspondent sliding region (we mention that, these invariant sets are tangent to S_{X_0} and S_{Y_0}; see Remark 5.3 below).

In this particular case, there exists a codimension-one manifold Γ of G^r contained in the open set $U_+ = \{Z : h_1(Z) > 0 \, and \, h_2(Z) > 0\}$ such that :

for Z=(X,Y) in Γ we have that c(X) and c(Y) are in the boundary of R(X,Y), the invariant manifolds of P(Z) are off R(X,Y) (in a very small neighborhood of the point) and the trajectories of the associated SVF, F(X,Y), passing through c(X) and c(Y) coincide. Moreover $\Sigma_{2,3} \in Cl\{\Gamma\}$; this situation is similar to that one in [T1] where a trajectory of a vector field is tangent to the boundary of a manifold at two distinct points. This situation has a further discussion in Remark 5.5 below. It follows that , $\Sigma_1 = \Sigma_1' \bigcup \Gamma$.

In all other cases, we have $\Sigma_1 = \Sigma_1'$.

The conclusion of the proposition is immediate. ∎

5.2. Corollary. $\Sigma_{2,3}$ is a C^{r-3} codimension two submanifold of G^r.

5.3. Remark. The general form of the SVF associated to Z_0 is

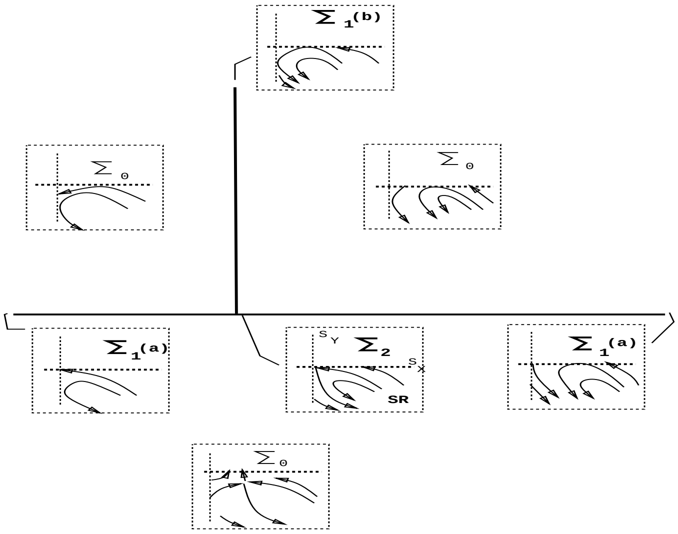

Figure 3: Unfolding of a t_3-singularity (distinguished saddle case)

$$F_0(x, y, 0) = (y(a_1x + b_1y) - ax, by + x(a_2x + b_2y)) + h.o.t.$$

We have the following result.

5.4. Corollary. The point p_0 is a hyperbolic critical point of the vector field F_0, with real and distinct eigenvalues and having the associated invariant manifolds tangent to the curves $\{X_0f = 0\}$ and $\{Y_0f = 0\}$; moreover these contacts are quadratic.

5.5 Remark. From 5.3, we have the following C^0 normal form of the SVF, associated to $Z_{\alpha,\beta}$:

$$F_{\alpha,\beta}(x, y, 0) = (-ax + \alpha y + y(a_1x + b_1y), \beta x + by + x(a_2x + b_2y)),$$

with $a \neq 0$, $b \neq 0$, $a \neq b$, $a_2 \neq 0$ and $b_1 \neq 0$. Depending on the nature of the critical point (saddle or node) of F_0 and on the relative position of the associated invariant manifolds we get different unfoldings of $F_{\alpha,\beta}$. If $\alpha = 0$ and $\beta \neq 0$ then $F_{\alpha,\beta}$ has one invariant manifold tangent to $\{X_{\alpha,\beta}f = 0\}$ and the other is transverse to $\{Y_{\alpha,\beta}\}$. In the last case, the vector field has another singularity (which is of codimension 0) defined by the cusp point on $\{X_{\alpha,\beta}f = 0\}$. The case $\alpha \neq 0$ and $\beta = 0$ is similar. When the codimension two singularity is a distinguished saddle a straightforward computation shows that the variety Γ is expressed by $\beta = \frac{\alpha^2}{(a+b)} + hot$ with $\alpha < 0$ and $\beta < 0$.

6 The t_4-singularity

6.1. Proposition. Let $Z_0 \in G^r$. Then Proposition 3.1 holds if p_0 is a t_4-singularity of Z_0.

Proof:

As before, let $Z_0 = (X_0, Y_0)$ be in G^r and (x, y, z) be coordinates around p_0 in $\mathbb{R}^3$ such that:

$X_0f(x,y,0) = X_3 = x$ and $Y_0f(x,y,0) = Y_3(x,y,0) = y$.

It follows, from the definition, that

$X_0X_0f(0) = X_1(0) = a \neq 0$, $X_0Y_0f(0) = X_2(0) = b \neq 0$ and $H_{XYX}Xf(0) = \partial\frac{Y_1}{\partial x}(0) = a_1 \neq 0$ and $H_{YYf}Xf(0) = a_2 \neq 0$.

Denote $Y_1(x,y,0) = a_1x + b_1y + h.o.t$ and $Y_2(x,y,0) = a_2x + b_2y + h.o.t.$

As above, let U and V be neighborhoods of Z_0 and p_0 respectively.

Define the C^r mapping $q : U, Z_0 \to \mathbf{R^3}, \mathbf{p_0}$ where $q(Z) = (q_1(Z), q_2(Z), q_3(Z))$ is the critical point of Y nearby p_0 in R^3, with Z=(X,Y).

Denote by P(Z) the point in V, which is the intersection between S_X and S_Y, for each Z=(X,Y) in U.

From [ST], it follows that there is, associated to Z=(X,Y), a point $c(Z) = (c_1(Z), c_2(Z))$ in V such that:

i) if $q_3(Z) = 0$ then $c(Z) = q(Z)$;

ii) if $q_3 \neq 0$ then c(Z) is a cusp point of Y;

iii) all points in S_Y, different from c(Z), are fold points of Y;

iv) the correspondence $Z \to c(Z)$ is C^{r-2}.

The required mapping $h = (h_1, h_2)$ is defined by

$h_1(Z) = YYf|S_Y(P(Z))$, and $h_2(Z) = YXf|S_Y(P(Z))$ with Z=(X,Y).

Observe that:

a) $h_1(Z) = 0$ if and only if $P(Z) = c(Z)$: in our coordinates this means that the second component of Y at P(Z) is zero;

b) the identity $h_2(Z) = 0$ says that the first component of Y at P(Z) is zero.

c) so $h(Z) = 0$ if and only if P(Z) is a critical point of Y;

d)the variety Σ_1 in this case is described by the union of the following sets: $h_1(Z) = 0$ and $g(Z) = 0$ where

$$g(Z) = (YXf|S_Y)(c(Z))$$

with c(Z) in the boundary of the sliding region . This means that either $P(Z)$ is a cusp point of Y or $c(Z)$ is a critical point (in $Cl\{SR\}$) of Y but different from $P(Z)$.

It remains to prove the surjectivity of $dh(Z_0)$. Take the family $Z_{\alpha,\beta}$ in G^r such that: i) $X_{\alpha,\beta} = X_0$;

ii) $Y_{\alpha,\beta} = Y_0 + (\alpha, \beta, 0)$.

We can check that:

$h_1(\alpha,\beta) = \beta + h.o.t$, $h_2(\alpha,\beta) = \alpha + h.o.t$ and

$g(\alpha,\beta) = (\frac{\alpha}{a_1}) - (\frac{\beta}{a_2}) + h.o.t.$

From this, we deduce that $dh(Z_0)$ is surjective and a straightforward calculation shows that the bifurcation set of the family is described by $\beta(a_2\alpha - a_1\beta) + h.o.t = 0$ for $sgn(a_1)\alpha < O$.

The proof of the proposition is now immediate. ∎

6.2. Corollary. $\Sigma_{2,4}$ is a C^{r-3} codimension two submanifold of G^r.

6.3. Remark. The general form of the SVF, $F_0 = F(X_0, Y_0)$ (following the proof of 6.1) is:

$$F_0(x,y,0) = (-ay + a_1x^2 + b_1xy, -by + a_2x^2 + b_2xy) + h.o.t.$$

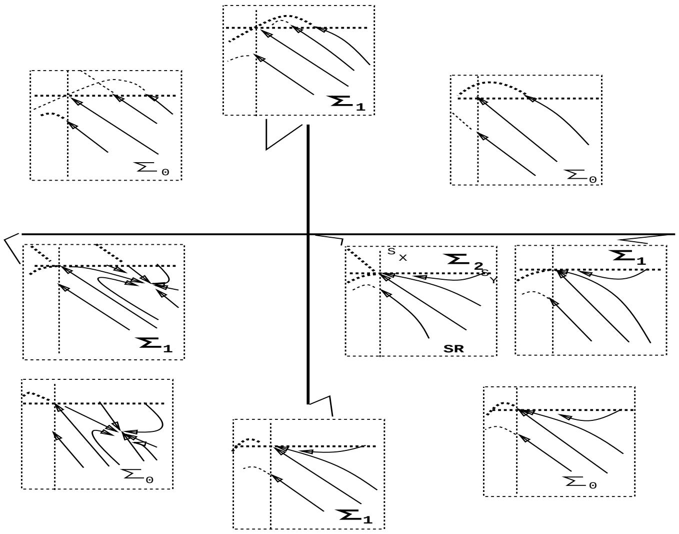

Figure 4: Unfolding of a t_4-singularity

The eigenvalues of $dF_0(0)$ are $\eta_1 = 0$ and $\eta_2 = -b$ and the respective eigenspaces are expressed by $y = 0$ and $y = -(\frac{b}{a})x$. So the singularity in question is a saddle node in which the associated center manifold is tangent (quadratic contact) to S_Y. Depending on the position of this manifold we have different phase portraits of the vector field.

6.4. Corollary. Let $Z_{\alpha,\beta}$ be a 2-parameter family of vector fields in G^r for which the following properties hold: i) $Z_{0,0}$ has a t_4-singularity ; ii) the mapping

$$: (\alpha, \beta) \to Z_{\alpha,\beta}$$

is transverse to $\Sigma_{2,4}$ at (0,0). Then the C^0 normal form of the corresponding SVF is:

$$F_{\alpha,\beta}(x, y) = (my + \alpha x - kx^2, \epsilon y - \beta x - mx^2),$$

with $\epsilon = \pm 1$, $k \neq 0$, $m \neq 0$, $n \neq 0$ and $k \neq mn$.

6.5. Remark. In addition to the above corollary we mention that, the bifurcation set of this family is given by $\beta\alpha = 0$. It is clear, there are different cases to be analyzed, depending on the values of m, n and k; these values reflect the position of the center manifold with respect to S_Y (exterior or interior tangency) and the position between the other invariant (hyperbolic) manifold with respect the sliding region. In Figure 4, we illustrate the case where this manifold meets the interior of SR and the center manifold has a interior tangency with S_Y. In order, if $\beta = 0$ and $\alpha \neq 0$, then the origin is a hyperbolic critical point of $F_{\alpha,\beta}$ having one eigenspace tangent to $S_{Y_{\alpha,\beta}}$. If $\alpha = 0$ then the origin is a saddle-node of $F_{\alpha,\beta}(x, y)$ with eigenspaces transverse to $S_{X_{\alpha,\beta}}$ and $S_{Y_{\alpha,\beta}}$.

7 Proof of Theorem A

Part a) of the theorem follows from 3.2, 4.2, 5.2 and 6.2 and Part b) follows from 3.1, 3.5, 4.1, 4.4, 5.1, 5.4, 6.1 and 6.4. ∎

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