Hypernormal Form Calculation for Triple-Zero Degeneracies

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Abstract

A computational approach to obtain normal forms for equilibrium points of three-dimensional autonomous systems, having a linear degeneracy corresponding to a triple-zero eigenvalue, is presented. Also, we provide the explicit expressions for the normal form coefficients, and analyze some additional simplifications that can be achieved.

The results are applied in the analysis of bifurcation behaviours in an autonomous electronic oscillator.

1 Introduction

The normal form theory is an useful tool to build, for the analysis of a given dynamical system, another one which is equivalent and easier to study. Typically, when one is dealing with a nonhyperbolic situation, the full consideration of nonlinear terms in the system is required. So, for each degeneracy in the linear part, it is very relevant to determine the nonlinear terms that can be removed by means of successive changes of variables, in order to obtain the simplest equivalent system which gives account of the original dynamics.

For the most frequent bifurcation cases, normal forms have been obtained (see [8], [10]). Here, following a line of previous works (see [7] and references therein) we will give a computational approach to build normal forms corresponding to a triple zero eigenvalue in the linear part. This situation was already considered in [10] but our approach seems to be more interesting from the point of view of applications: we give explicit expressions for the coefficients of the normal form and the changes of variables can be easily computed. Another important feature is that the algorithms used are very efficient when implemented in standard computer algebra systems.

Bull. Belg. Math. Soc. 6 (1999), 357–368

2 Normal Forms and Lie Transforms

Let us consider a dynamical system in $\mathbb{R}^3$ with an equilibrium point at the origin, whose linearization matrix has a triple zero eigenvalue with geometric multiplicity one. We will assume that the linearization matrix is put in Jordan form:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

The basis of our approach is to make the changes of variables in successive steps, without affecting in some sense previous steps. More explicitely, let us denote $\mathcal{H}_j$ the linear space of tridimensional vector homogeneous polynomials in three variables of degree j. Suppose that we perform the near-identity transformation

$$x = \tilde{x} + \phi_k(\tilde{x}),\ \phi_k \in \mathcal{H}_k,\ k \geq 2,$$

in the system

$$\dot{x} = Ax + \sum_{j\geq 2} F_j(x),\ F_j \in \mathcal{H}_j. \tag{2}$$

It is not difficult to show that the transformed system can be written as $\dot{\tilde{x}} = A\tilde{x} + \sum_{j\geq 2} \tilde{F}_j(\tilde{x})$ where $\tilde{F}_j \in \mathcal{H}_j$, and the following relations hold:

- $\tilde{F}_j = F_j$ for $j = 2, 3, \ldots, k-1$ and
- $\tilde{F}_k = F_k - L_k\phi_k$.

Here, $L_k : \mathcal{H}_k \longrightarrow \mathcal{H}_k$ is the linear operator defined by $L_k\phi_k(x) = D\phi_k(x)Ax - A\phi_k(x)$. This operator is called the *homological* operator.

If F_k belongs to $\mathcal{R}_k$, the range of L_k, the terms of degree k can be eliminated with an appropiate choice of ϕ_k. Otherwise, one can split $\mathcal{H}_k = \mathcal{R}_k \oplus \mathcal{C}_k$ (where $\mathcal{C}_k$ denotes a complementary subspace for $\mathcal{R}_k$ to be later selected explicitely) and write

$$F_k = F_k^r + F_k^c \text{ with } F_k^r \in \mathcal{R}_k,\ F_k^c \in \mathcal{C}_k.$$

It is easy to see that there exists ϕ_k such that $L_k\phi_k = F_k^r$. Then, we can obtain $\tilde{F}_k = F_k^c \in \mathcal{C}_k$. Thus, it is clear that the choice of $\mathcal{C}_k$ will determine the structure of normal forms to be achieved.

Another key observation is that the above procedure can be accomplished in a recursive way by means of an algorithmic scheme described in detail elsewhere (see [9], [1]). Here, we only present a slight adaptation which is more convenient for our purpose. For that, let us introduce a Lie bracket operation defined as $[U, V] = DU \cdot V - DV \cdot U$ for arbitrary tridimensional functions U, V. Let consider the succession of functions defined by

$$\begin{aligned} W_{1,1} &= 1!F_2, \\ W_{k,1} &= k!F_{k+1} + \sum_{j=0}^{k-2} \begin{pmatrix} k-1 \\ j \end{pmatrix} (k-j-1)!\,[F_{k-j}, U_j]\,, \quad k \geq 2, \\ W_{k,l} &= W_{k,l-1} + \sum_{j=0}^{k-l} \begin{pmatrix} k-l \\ j \end{pmatrix} [V_{k-j-1,l-1}, U_j]\,, \quad 2 \leq l \leq k, \end{aligned}$$

where $U_{k-1} \in \mathcal{H}_{k+1}$ is selected such that

$$L_{k+1}\left(U_{k-1}\right) = W^r_{k,k} \quad \text{(the projection of } W_{k,k} \text{ onto } \mathcal{R}_k\text{)}, \tag{3}$$

and $V_{k,l} = W_{k,l} - W^r_{k,k}$, for $l = 1, \ldots, k$. Notice that $W_{k,l} \in \mathcal{H}_{k+1}$ for all l. The normal form for the system (2) can be obtained from the elements $W_{k,k}$:

$$\tilde{F}_{k+1} = \frac{1}{k!}\Pi_{k+1}\left(W_{k,k}\right) = \frac{1}{k!}W^c_{k,k},$$

where Π_{k+1} denotes the projection operator onto $\mathcal{C}_{k+1}$. This treatment is very advantageous because the succession $\{W_{k,l}\}$ can be organized in a Lie triangle, easy to implement in a symbolic language

$$\begin{array}{|c|c|c|c|c|c|l}
\hline
W_{1,1} & & & & & & \to U_0 \\
\hline
W_{2,1} & W_{2,2} & & & & & \to U_1 \\
\hline
W_{3,1} & W_{3,2} & W_{3,3} & & & & \to U_2 \\
\hline
\vdots & \vdots & \vdots & \ddots & & & \\
\hline
W_{k,1} & W_{k,2} & W_{k,3} & \cdots & W_{k,k} & & \to U_{k-1} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots &
\end{array}$$

The elements in the row k have degree $k+1$ and $U_{k-1}, \tilde{F}_{k+1}$ are obtained from the diagonal element $W_{k,k}$.

The above procedure is valid for any linearization matrix A. In next section, we will take advantage of specific structure of linear part A given in (1). It should also be noticed that the normal form will not necessarily be unique. On the one hand, we have some freedom in the choice of $\mathcal{C}_k$. Furthermore, if dim Ker$L_{k+1} > 0$, we can also introduce some parameters, describing the general solution of equations (3). As we will show below, selecting adequately these parameters, we are able to annihilate upper degree terms in the normal form.

3 Computing Normal Forms

For the effective calculation of normal forms, let us consider the system (2) written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \sum_{k\geq 2} \begin{pmatrix} f_k(x,y,z) \\ g_k(x,y,z) \\ h_k(x,y,z) \end{pmatrix}. \tag{4}$$

Our first task is to define explicitely the spaces $\mathcal{C}_k$ and to perform the computations indicated in the previous section, which includes the resolution of the linear equations (3), for each $k \geq 2$. For that, it is very efficient to use a linear space setting, defining canonical bases for $\mathcal{H}_k$ and obtaining the corresponding matrix representation of operator L_k. With these ideas, and selecting reverse lexicographic order for elements in the bases, a computer algebra code has been written in REDUCE 3.2. We will now discuss some specific aspects for our case (for more details, see [6]).

Let us introduce, to choose $\mathcal{C}_k$ for this case, the integers $m = \left\lfloor \frac{k}{2} \right\rfloor$ and $m' = \left\lfloor \frac{k-1}{2} \right\rfloor$, where $\lfloor \cdot \rfloor$ denotes the *floor* function. The orthogonal complementary subspace to $\mathcal{R}_k$ (with respect to a suitable inner product, see [2]) is the subspace of dimension $\left\lfloor \frac{3(k+1)}{2} \right\rfloor$ given by

$$\begin{aligned} \mathrm{Ker} L_k^* \;&=\; \mathrm{span}\left\{ p(j)\begin{pmatrix} x \\ y \\ z \end{pmatrix}, 0 \le j \le m'; \right. \\ &\left. p(j)\begin{pmatrix} 0 \\ x \\ y \end{pmatrix}, 0 \le j \le m';\; p(j)\begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}, 0 \le j \le m \right\}, \end{aligned} \tag{5}$$

where L_k^* is the homological operator corresponding to the matrix A^* and $p(j) = x^{k-2j-1}(y^2 - 2xz)^j$. A simpler complementary subspace to $\mathcal{R}_k$ can be obtained by means of a slight modification in the previous basis, writing

$$\begin{aligned} \mathrm{Ker} L_k^* \;&=\; \mathrm{span}\left\{ p(j)\begin{pmatrix} x \\ y \\ z \end{pmatrix}, 0 \le j \le m';\; p(j)\begin{pmatrix} 0 \\ x \\ y \end{pmatrix}, 0 \le j \le m'; \right. \\ &\left. p(0)\begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix};\; p(j)\begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} + 2p(j-1)\begin{pmatrix} x \\ y \\ z \end{pmatrix}, 1 \le j \le m \right\}. \end{aligned} \tag{6}$$

From here, it is not difficult to pass to the following complementary subspace

$$\begin{aligned} \mathcal{C}_k \;&=\; \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ x^{k-j-1} z^{j+1} \end{pmatrix}, 0 \le j \le m' ; \right. \\ &\left. \begin{pmatrix} 0 \\ 0 \\ x^{k-2j-1} y^{2j+1} \end{pmatrix}, 0 \le j \le m';\; \begin{pmatrix} 0 \\ 0 \\ x^{k-2j} y^{2j} \end{pmatrix}, 0 \le j \le m \right\}, \end{aligned} \tag{7}$$

by using the orthogonality between the elements of the bases (6) and (7) (see [3]).

Thus, the normal form we will compute has the following structure:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= \sum_{k \ge 2} \left\{ \sum_{j=0}^{m'} \left(a_j^{(k)} x^{k-j-1} z^{j+1} + b_j^{(k)} x^{k-2j-1} y^{2j+1} \right) + \sum_{j=0}^{m} c_j^{(k)} x^{k-2j} y^{2j} \right\}. \end{aligned} \tag{8}$$

In the computation of the above coefficients, we can take into account the possibility of solving equations (3) including an arbitrary element of $\mathrm{Ker} L_k$, which turns out to be

$$\mathrm{Ker} L_k = \mathrm{span}\left\{ q(j)\begin{pmatrix} x \\ y \\ z \end{pmatrix},\; q(j)\begin{pmatrix} y \\ z \\ 0 \end{pmatrix} : 0 \le j \le m' ;\; q(j)\begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix} : 0 \le j \le m \right\},$$

where $q(j) = z^{k-2j-1}(y^2 - 2xz)^j$. Doing so, and denoting $\lambda_{k,j}^{(i)}$ the free coordinates in $\mathrm{Ker}L_k$, the REDUCE code gives us the following general expressions for the normal form coefficients up to third degree:

$$\begin{aligned}
a_0^{(2)} &= f_{xx} + g_{xy} + h_{xz},\\
b_0^{(2)} &= g_{xx} + h_{xy},\\
c_0^{(2)} &= h_{xx}/2,\\
c_1^{(2)} &= (2f_{xx} + 2g_{xy} + h_{yy})/2,\\
a_0^{(3)} &= (-16\lambda_{2,0}^{(1)}h_{xx} + 8\lambda_{2,1}^{(3)}h_{xx} - 4f_{yz}h_{xx} - f_{yy}g_{xx} - 2f_{xz}g_{xx} + 4f_{xy}g_{xy} + 4f_{xy}h_{xz}\\
&\quad -2f_{xx}g_{yy} - 4f_{xx}h_{yz} - 4g_{zz}h_{xx} - 2g_{yz}g_{xx} + 2g_{yy}g_{xy} + 2g_{yy}h_{xz} - 3g_{xx}h_{zz}\\
&\quad +4f_{xxx} + 4g_{xxy} + 4h_{xxz})/8,\\
a_1^{(3)} &= (8\lambda_{2,0}^{(1)}f_{xx} + 8\lambda_{2,0}^{(1)}g_{xy} + 16\lambda_{2,0}^{(1)}h_{yy} - 24\lambda_{2,0}^{(1)}h_{xz} - 24\lambda_{2,1}^{(3)}f_{xx} - 24\lambda_{2,1}^{(3)}g_{xy} - 8\lambda_{2,1}^{(3)}h_{yy}\\
&\quad -8\lambda_{2,1}^{(3)}h_{xz} + 8f_{yz}f_{xx} + 8f_{yz}g_{xy} + 4f_{yz}h_{yy} + 2f_{yy}f_{xy} + 3f_{yy}g_{yy} - 2f_{yy}g_{xz}\\
&\quad +2f_{yy}h_{yz} - 4f_{xz}f_{xy} - 4f_{xz}g_{yy} + 4f_{xy}g_{yz} + 2f_{xy}h_{zz} + 4f_{xx}g_{zz} + 4g_{zz}g_{xy}\\
&\quad +4g_{zz}h_{yy} - 4g_{zz}h_{xz} + 4g_{yz}g_{yy} + 8g_{yz}h_{yz} + 3g_{yy}h_{zz} - 2g_{xz}h_{zz} + 6h_{zz}h_{yz}\\
&\quad -4f_{xyy} + 8f_{xxz} - 4g_{yyy} + 8g_{xyz} - 4h_{yyz} + 8h_{xzz})/16,\\
b_0^{(3)} &= (4f_{yy}h_{xx} + 12f_{xy}g_{xx} + 4f_{xy}h_{xy} - 12f_{xx}g_{xy} - 4f_{xx}h_{yy} - 8f_{xx}h_{xz} - 12g_{yz}h_{xx}\\
&\quad +3g_{yy}g_{xx} + 6g_{yy}h_{xy} - 6g_{xz}g_{xx} - 8g_{xy}h_{xz} - 6g_{xx}h_{yz} - 8h_{zz}h_{xx} - 8h_{yz}h_{xy}\\
&\quad +4g_{xxx} + 4h_{xxy})/8,\\
b_1^{(3)} &= (-8\lambda_{2,0}^{(1)}g_{xx} - 8\lambda_{2,0}^{(1)}h_{xy} + 8\lambda_{2,0}^{(2)}h_{xx} + 40\lambda_{2,1}^{(3)}g_{xx} + 40\lambda_{2,1}^{(3)}h_{xy} - 4f_{zz}h_{xx} - 16f_{yz}g_{xx}\\
&\quad -8f_{yz}h_{xy} + 6f_{yy}f_{xx} + 6f_{yy}g_{xy} - 4f_{yy}h_{xz} - 12f_{xz}f_{xx} - 4f_{xz}g_{xy} - 2f_{xy}g_{yy}\\
&\quad -4f_{xy}g_{xz} + 4f_{xy}h_{yz} - 4f_{xx}g_{yz} - 2f_{xx}h_{zz} - 4g_{zz}g_{xx} - 12g_{zz}h_{xy} - 20g_{yz}g_{xy}\\
&\quad -12g_{yz}h_{yy} + 16g_{yz}h_{xz} + g_{yy}^2 + 2g_{yy}g_{xz} - 2g_{yy}h_{yz} - 8g_{xz}h_{yz} - 6g_{xy}h_{zz} - 12h_{zz}h_{yy}\\
&\quad +8h_{zz}h_{xz} + 4g_{xyy} - 8g_{xxz} + 4h_{yyy} - 8h_{xyz})/24,\\
c_0^{(3)} &= (6f_{xy}h_{xx} - 6f_{xx}h_{xy} + 3g_{yy}h_{xx} - 6g_{xx}h_{xz} - 6h_{yz}h_{xx} + 2h_{xxx})/12,\\
c_1^{(3)} &= (-16\lambda_{2,0}^{(1)}h_{xx} + 28\lambda_{2,1}^{(3)}h_{xx} - 12f_{yz}h_{xx} + f_{yy}g_{xx} + 2f_{yy}h_{xy} - 6f_{xz}g_{xx} + 4f_{xy}g_{xy}\\
&\quad -4f_{xx}g_{xz} - 4f_{xx}h_{yz} - 8g_{zz}h_{xx} - 6g_{yz}g_{xx} - 8g_{yz}h_{xy} + 4g_{yy}g_{xy} + 2g_{yy}h_{yy}\\
&\quad -4g_{xz}g_{xy} - 4g_{xy}h_{yz} - 3g_{xx}h_{zz} - 6h_{zz}h_{xy} - 2h_{yz}h_{yy} + 4f_{xxx} + 4g_{xxy} + 2h_{xyy})/4.
\end{aligned}$$

We remark that the interest of the above approach is to be able of generating the expressions for specific systems, without having to substitute values in the previous formulas.

4 From Normal to Hypernormal Forms

From the above expressions, we see that, under certain hypothesis, one can choose adequately the parameters λ's, in order to annihilate some normal form coefficients of order greater than three. Then, we can obtain reduced normal forms, called hypernormal forms, see [11].

Before studying this possibility, we will show that something similar can be achieved for the second degree terms. The key idea is to perform linear changes depending on parameters, namely:

$$\begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix} = e^{B\gamma} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \ \gamma \in \mathbb{R}, \tag{9}$$

where B is a matrix belonging to $\mathcal{Z}_A$, the centralizer of A, i.e., $AB = BA$. Using that $e^{B\gamma}A = Ae^{B\gamma}$, it is easily obtained that the transformed system has the same linear part.

In our case, we have

$$\mathcal{Z}_A = \text{span}\left\{ I, A, C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

and then, we can write $B = \alpha I + \beta A + \delta C$. Our goal is to perform the linear change (9), later put the resulting system in normal form and finally, select the parameters α, β, δ and γ in order to annihilate some terms in the normal form. It can be shown (see [6]) that the parameters α, β and γ are not essential, i. e., they do not provide any simplification. So, we will take $\alpha = \beta = 0$, $\gamma = 1$, and then $B = \delta C$. The change is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{-B} \begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix}. \tag{10}$$

Applying to the transformed system the results of Section 2, we obtain the following expressions for the second order normal form coefficients:

$$\begin{aligned} a_0^{(2)} &= f_{xx} + h_{xz} + g_{xy}, \qquad & b_0^{(2)} &= h_{xy} + g_{xx}, \\ c_0^{(2)} &= h_{xx}/2, & c_1^{(2)} &= \delta h_{xx} + f_{xx} + h_{yy}/2 + g_{xy}. \end{aligned}$$

Then, if $h_{xx} \neq 0$, we can achieve $c_1^{(2)} = 0$ by selecting δ adequately. Moreover, we can obtain further simplifications in higher order terms by means of the constants λ's:

Theorem 4.1 *Let us consider the system (4), and assume that $h_{xx} \neq 0$. Then, a hypernormal form up to third order is*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= a_1 xz + a_2 xy + a_3 x^2 + b_1 xz^2 + b_2 x^2 y + b_3 x^3. \end{aligned}$$

Proof:

Denote by

$$F_k(x,y,z) = \begin{pmatrix} f_k(x,y,z) \\ g_k(x,y,z) \\ h_k(x,y,z) \end{pmatrix},$$

the k–degree terms of the system (4), and suppose that the second-order terms are already put in hypernormal form, i. e.,

$$F_2 = \begin{pmatrix} 0 \\ 0 \\ a_1xz + a_2xy + a_3x^2 \end{pmatrix}.$$

The triangular scheme in this cases becomes

F_2	
$2F_3 + [F_2, U_0]$	$2F_3 + 2[F_2, U_0]$

where U_0 satisfies $L_kU_0 = \Pi_2(F_2) = 0$ (and therefore $\tilde{F}_2 = F_2 \in \mathcal{C}_2$). Selecting U_1 adequately, we can obtain the expression for $\tilde{F}_3$ from the last diagonal element:

$$\tilde{F}_3 = \frac{1}{2!}\Pi_3(2F_3 + 2[F_2, U_0] - L_3U_1) = F_3^c + \Pi_3([F_2, U_0]).$$

To study how F_3^c may be simplified, let us define the linear operator $M : \mathrm{Ker}L_2 \longrightarrow \mathcal{C}_3$ by $M(U) = \Pi_3([\phi_2^c, U])$. The matrix representation of M, considering the basis of L_2:

$$\left\{ v_1 = z\begin{pmatrix} x \\ y \\ z \end{pmatrix}, v_2 = z\begin{pmatrix} y \\ z \\ 0 \end{pmatrix}, v_3 = z\begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix}, v_4 = (y^2 - 2xz)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

is given by

	v_1	v_2	v_3	v_4
x^2z	$-4a_3$	0	0	$2a_3$
xz^2	$\frac{-3a_1}{2}$	0	0	$\frac{-a_1}{2}$
x^2y	0	0	0	0
x^3	0	0	0	0
y^3	$\frac{-a_2}{3}$	$\frac{2a_3}{3}$	0	$\frac{5a_2}{3}$
xy^2	$-8a_3$	0	0	$14a_3$

6×4

Under the hypothesis $a_3 \neq 0$ (or equivalently $h_{xx} \neq 0$), this matrix has rank 3, and a complementary subspace to the range of M is given by

$$\mathrm{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ xz^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2y \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^3 \end{pmatrix} \right\}.$$

Therefore, these are just the terms that appear in the third-order hypernormal form of the statement of the theorem. ∎

Similar ideas can be used to obtain different hypernormal forms under adequate hypothesis:

Theorem 4.2 *If $h_{xx} \neq 0$, a fifth-order hypernormal form for the system (4) is*

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= a_1xz + a_2xy + a_3x^2 + b_1xz^2 + b_2x^2y + b_3x^3 + c_1x^3z + c_2x^4 + c_3x^2y^2,
\end{aligned}$$

where $c_1 = 0$ or $c_3 = 0$.

Finally, for systems with $\mathbb{Z}_2$–symmetry we have obtained the following result:

Theorem 4.3 *If the system (4) has $\mathbb{Z}_2$–symmetry, and $h_{xxx} \neq 0$ (or equivalently $c_0^{(3)} \neq 0$), a hypernormal form up to fifth order is given by*

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= b_1x^2z + b_2xz^2 + b_3x^2y + b_4y^3 + b_5x^3 + c_1x^2z^3 + c_2x^4y + c_3x^5 + c_4x^3y^2.
\end{aligned}$$

5 Application to an Autonomous Electronic Oscillator

In this last section, we consider the system

$$\begin{aligned}
r\dot{x} &= -(\beta+\nu)\,x + \beta y - A_3x^3 + B_3(y-x)^3 - A_5x^5 + B_5(y-x)^5, \\
\dot{y} &= \beta x - (\beta+\gamma)\,y - z - C_3y^3 - B_3(y-x)^3 - C_5y^5 - B_5(y-x)^5, \\
\dot{z} &= y,
\end{aligned} \tag{11}$$

governing the behaviour of an autonomous electronic circuit widely analysed (see [4], [5] and references therein, where pitchfork, Hopf, Takens–Bogdanov and Hopf–pitchfork bifurcations of the equilibrium at the origin have been studied). The linear degeneracy of the origin corresponding to a triple zero eigenvalue occurs at the critical values $-\nu_c = \beta_c = -\gamma_c = \sqrt{r}$ and $-\nu_c = \beta_c = -\gamma_c = -\sqrt{r}$ (see figure 1). Here we focus our attention on the first case. We begin making a linear change of variables

$$\begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix} = P \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\beta+\nu}{r} & \frac{\beta}{r} & 0 \\ \left(\frac{\beta+\nu}{r}\right)^2 + \frac{\beta^2}{r} & -\frac{\beta}{r}\left(\frac{\beta+\nu}{r} + \beta + \gamma\right) & -\frac{\beta}{r} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

bringing the linear part of system (11) to the form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{pmatrix},$$

where

$$\varepsilon_1 = -\frac{\beta+\nu}{r}, \quad \varepsilon_2 = -\frac{r + \beta\nu + \beta\gamma + \nu\gamma}{r}, \quad \varepsilon_3 = -\frac{r\beta + r\gamma + \beta + \nu}{r}.$$

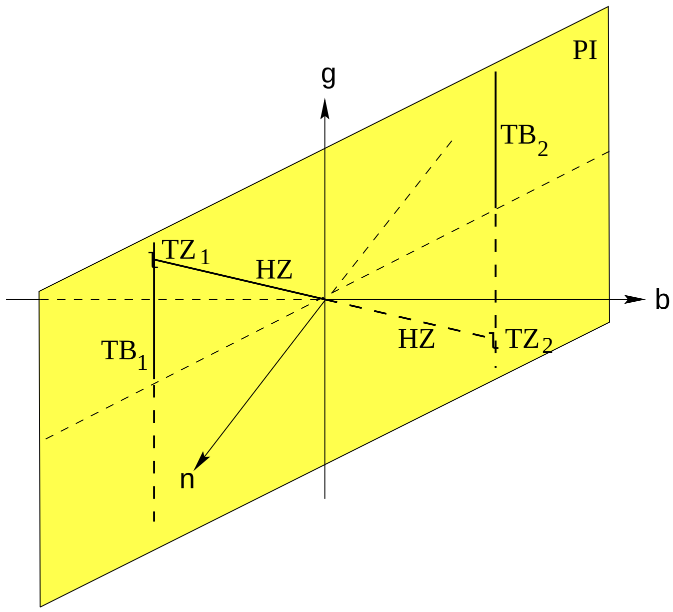

Figure 1: Partial bifurcation set of system 11 in the ν–β–γ parameter space where several bifurcations of codimension 1 (PI, pitchfork), 2 (TB_1, TB_2, Takens–Bogdanov; HZ, Hopf–zero) and 3 (TZ_1, TZ_2, triple–zero) appear.

These expressions allow to verify the transversality condition:

$$\frac{\partial(\varepsilon_1, \varepsilon_2, \varepsilon_3)}{\partial(\nu, \beta, \gamma)} = \frac{2}{r^{\frac{3}{2}}} \neq 0 \quad \text{at} \quad \nu = \nu_c, \beta = \beta_c, \gamma = \gamma_c,$$

i. e., the change of parameters $\nu, \beta, \gamma \leftrightarrow \varepsilon_1, \varepsilon_2, \varepsilon_3$ is invertible.

Next, we compute the hypernormal form up to third order for the system (11) for the critical values of the parameters. To this end, two linear changes of variables are in order: the first one taking the linear part into the Jordan form, and the second one of the form indicated in (10). Globally, we make the linear change

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P^{-1} e^{B} \begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \delta \\ 0 & \sqrt{r} & 0 \\ \sqrt{r} & 0 & \sqrt{r}(\delta - 1) \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix},$$

obtaining a system of the form (4). The expressions for the third-order normal form coefficients are

$$\begin{aligned} a_0^{(3)} &= -3\frac{rB_3 + A_3 + B_3}{r}, & a_1^{(3)} &= 3\frac{r^2 B_3 + r^2 C_3 - 2r\delta B_3 + rB_3 - 2\delta A_3 - 2\delta B_3}{2r}, \\ b_0^{(3)} &= \frac{6B_3}{\sqrt{r}}, & b_1^{(3)} &= \frac{2B_3(r - 2\delta)}{\sqrt{r}}, \\ c_0^{(3)} &= -\frac{A_3 + B_3}{r}, & c_1^{(3)} &= 3\frac{-3rB_3 + 2\delta A_3 + 2\delta B_3 - 2A_3 - 2B_3}{r}. \end{aligned}$$

We will take $\delta = \frac{r}{2}$ in order to annihilate the coefficient $b_1^{(3)}$ (note that other elections of δ permit to annihilate $a_1^{(3)}$ or $c_1^{(3)}$ under adequate hypothesis for A_3, B_3). The

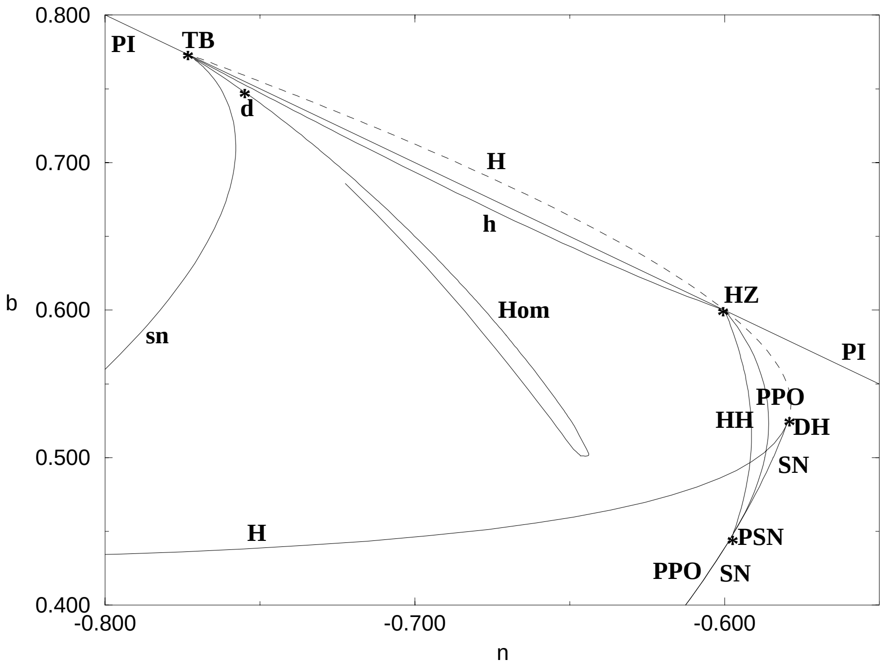

Figure 2: Partial bifurcation set in the ν–β plane for $\gamma = -0.6$, $r = 0.6$, $A_3 = 0.3286$, $B_3 = 0.9336$, $C_3 = A_5 = B_5 = C_5 = 0$. Four codimension 2 bifurcation points (TB, Takens–Bogdanov; HZ, Hopf–zero; DH, degenerate Hopf bifurcation of the origin; PSN, pitchfork–saddle–node of periodic orbits) and several codimension 1 bifurcation curves (PI, pitchfork of equilibria; H, Hopf of the origin; h, Hopf of the nontrivial equilibria; Hom, homoclinic orbit; SN and sn, saddle–node bifurcations of periodic orbits; HH, torus bifurcation; PPO, pitchfork of periodic orbits) are drawn. The point d on Hom marks the beginning of the Šil'nikov region.

fifth-order hypernormal form of Theorem 4.3 has been computed, but the lengthy expressions for the coefficients prevent us to show them here.

The bifurcation analysis of this hypernormal form constitutes the starting point in the numerical study carried out. The linear degeneracies of the origin arising in this system are of codimension one (pitchfork and Hopf bifurcations), two (Takens–Bogdanov and Hopf–pitchfork) and three (triple–zero).

We have investigated numerically the bifurcation set in the ν–β plane for a value of $\gamma = -0.6$, relatively close to the triple–zero point (see figure 2). We have obtained a nondegenerate Takens–Bogdanov bifurcation TB (where a pitchfork bifurcation curve of equilibria PI intersects with a subcritical Hopf bifurcation of the origin, labelled H). From TB, three codimension 1 bifurcation curves appear: a curve h of supercritical Hopf bifurcation of nontrivial equilibria, a curve Hom of homoclinic connections and a curve sn of saddle–node bifurcation of periodic orbits. The homoclinic curve Hom enters quickly into the Šil'nikov region —this occurs at point d— giving rise to the appearance of complex periodic and aperiodic behaviour. Finally, it will disappear in a *T-point* spiralling around it (we haven't drawn this spiral for the sake of clarity because the T-point is very close to TB). There is another codimension 2 organizing centre: HZ, corresponding to a Hopf–pitchfork bifurcation. From such a point, several codimension 1 curves emerge: the curve h (which joins TB and HZ), a curve PPO of pitchfork bifurcation of periodic orbits and a curve HH of torus bifurcation. There is also a degenerate Hopf bifurcation point DH (below it, H is supercritical), and a saddle–node bifurcation of periodic orbits SN appears. This bifurcation curve coalesces with PPO at PSN (a double–one codimension 2 bifurcation point). This point will be the end of the torus bifurcation curve HH cited above.

The richness of this bifurcation set shows the complex situations that may arise in the vicinity of the triple–zero degeneracy point. Numerical simulation works have taken advantage of the analytical results provided by the study of its normal form.

Acknowledgements. This work was partially supported by the *Ministerio Español de Educación y Ciencia* in the frame of DGICYT project PS90–0139 and by the *Consejería de Educación y Ciencia de la Junta de Andalucía*.

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