Minimal Centroaffine Immersions of Codimension Two

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Abstract

Minimal equi-centroaffine immersions of codimension two are characterized as solutions of a certain variational problem. We determine the moduli space of such immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$ whose induced connection and affine fundamental form coincide with the ones of the Clifford torus.

1 Introduction

In equi-centroaffine differential geometry, the theory of hypersurfaces has a long history. However, relatively little has been achieved in the study of equi-centroaffine immersions with higher codimensions. It is the purpose of this paper to study such immersions with codimension two.

For a centroaffine immersion into the affine space, the position vector yields its *first canonical* normal vector field. A standard method of choosing a *second* one was proposed in 1950 by Lopšic (see Walter [7]). Recently, reorganizing geometry of equi-centroaffine immersions of codimension two, Nomizu and Sasaki [4] proposed another fruitful choice. Adopting the latter one, we take the prenormalized Blaschke normal field as the second canonical normal vector field. Fixing two normal vector fields, we can define the induced volume form and consider a variational problem of the volume. Then we say that an equi-centroaffine immersion of codimension two is *minimal* if the volume is extremal under any variation having no part in the direction of the position vector infinitesimally. We prove that an equi-centroaffine

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immersion is minimal if and only if the trace of the affine shape operator with respect to the prenormalized Blaschke normal field vanishes identically (Theorem 3.2).

It is also remarked that in equi-affine geometry there are methods to induce a volume form without fixing special normal vector fields (see Dillen, Mys, Verstraelen and Vrancken [3]) and that in centroaffine geometry, instead of equi-centroaffine geometry, there are new approaches for immersions of codimension two (for example, see Scharlach [6]).

In Section 4, we illustrate some examples (Examples 4.1, 4.2 and 4.4). In particular, the Clifford torus ϕ gives rise to a minimal equi-centroaffine immersion of $\mathbb{R}^2$ into $\mathbb{R}^4$. We remark that the pair of its induced connection and its affine fundamental form is a trivial statistical structure (for example, see Amari [1]). In fact, the affine fundamental form of ϕ is an indefinite flat metric on $\mathbb{R}^2$, and the induced connection is its Levi-Civita connection. We then determine the set $\mathfrak{M}_\phi$ of $SL(4;\mathbb{R})$-congruence classes of minimal equi-centroaffine immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$ whose induced connection and affine fundamental form coincide with the ones of the Clifford torus ϕ respectively (Theorem 4.3).

In Section 5, the duality for minimal equi-centroaffine immersions is studied. We prove that an equi-centroaffine immersion is minimal if and only if its dual is minimal (Proposition 5.1). We also have a representation formula of the dual immersion for a given minimal equi-centroaffine immersion and its affine fundamental form (Proposition 5.2).

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2 Preliminaries

We recall the basic equi-centroaffine geometry developed in Nomizu and Sasaki [4]. Let f be an immersion of an $n(\geq 2)$-dimensional manifold M into $\mathbb{R}^{n+2}$. Throughout this paper, we identify an $\mathbb{R}^{n+2}$-valued function with a section of $f^{-1}T\mathbb{R}^{n+2}$ by

$$C^\infty(M;\mathbb{R}^{n+2}) \ni \xi = {}^t(\xi^1,\ldots,\xi^{n+2}) \mapsto \sum_{i=1}^{n+2} \xi^i \left(\frac{\partial}{\partial x^i}\right)_f \in f^{-1}T\mathbb{R}^{n+2}.$$

We call $\xi \in \Gamma(f^{-1}T\mathbb{R}^{n+2})$ a *transversal vector field* of f if at each point $x \in M$, the tangent space $T_{f(x)}\mathbb{R}^{n+2}$ is decomposed as

$$T_{f(x)}\mathbb{R}^{n+2} = f_*T_xM \oplus \mathbb{R}\xi_x \oplus \mathbb{R}f(x).$$

Let D denote the standard flat connection of $T\mathbb{R}^{n+2}$ and $f^{-1}T\mathbb{R}^{n+2}$, and Det the standard volume form of $\mathbb{R}^{n+2}$. Associated with (f,ξ) we have the identities

$$\begin{cases} D_X f_*Y = f_*\nabla_X Y + h(X,Y)\xi + T(X,Y)f, \\ D_X\xi = -f_*SX + \tau(X)\xi + P(X)f, \quad \text{for} \quad X,Y \in \Gamma(TM) \end{cases} \tag{2.1}$$

and define

$$\theta(X_1,\ldots,X_n) := \mathrm{Det}(f_*X_1,\ldots,f_*X_n,\xi_x,f(x)) \quad \text{for} \quad X_j \in T_xM, x \in M.$$

On M, we then have a torsion-free affine connection ∇, symmetric $(0,2)$-tensor fields h and T, a $(1,1)$-tensor field S, 1-forms τ and P, and a volume form θ. In this paper, we call $S \in \Gamma(\operatorname{End} TM)$ the *affine shape operator* of (f,ξ), and $h \in \Gamma(T^*M \otimes T^*M)$ the *affine fundamental form* of (f,ξ).

Lemma 2.1 (Nomizu and Sasaki [4]). *Let $(f;\xi_0) : M \to \mathbb{R}^{n+2}$ be an immersion with a transversal vector field. If the affine fundamental form is nondegenerate, then there exists a transversal vector field ξ of f such that*

$$\begin{cases} h \quad \text{is nondegenerate}, \\ \tau = 0, \\ \omega_h = \theta, \\ \operatorname{tr}_h\{(X,Y) \mapsto T(X,Y) + h(SX,Y)\} = 0, \end{cases} \tag{2.2}$$

where ω_h is the volume form with respect to the pseudo-Riemannian metric h. Moreover, such ξ is uniquely determined up to sign.

We then call such a transversal vector field ξ the *prenormalized Blaschke normal vector field* of f. For simplicity, we call an immersion fixed with the prenormalized Blaschke normal vector field a *centroaffine* immersion, and denote it by f instead of (f,ξ). Two centroaffine immersions $f_1, f_2 : M \to \mathbb{R}^{n+2}$ are said to be *congruent* if there exists a special linear transformation $A \in SL(n+2;\mathbb{R})$ such that $f_2 = Af_1$.

Remark 2.2. The second condition of (2.2) is equivalent to the condition that the induced volume form is parallel with respect to the induced connection: $\nabla\theta = 0$.

3 A Variational-Theoretic Approach

Let $f : M \to \mathbb{R}^{n+2}$ be a centroaffine immersion. We consider a smooth variation $F : M \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+2}$ of f satisfying

$$\begin{cases} \quad f_t \ := F(\cdot, t) : M \to \mathbb{R}^{n+2} \quad \text{is a centroaffine immersion}, \\ \quad f_0 \ = f, \\ \quad f_t \ = f \quad \text{outside a compact set}, \\ \text{and} \\ \quad \nu \ = 0, \end{cases} \tag{3.1}$$

where $\nu \in C^\infty(M)$ is defined by

$$F_*\left(\frac{\partial}{\partial t}\right)_{(x,0)} = f_* V_x + v(x)\xi_x + \nu(x) f(x), \qquad V \in \Gamma(TM), v \in C^\infty(M).$$

For a variation F, we denote by θ_t the volume form induced by f_t.

Definition 3.1. A centroaffine immersion $f : M \to \mathbb{R}^{n+2}$ is said to be *minimal* if for an arbitrary variation F satisfying (3.1),

$$\left.\frac{d}{dt}\right|_{t=0} \int_M \theta_t = 0.$$

Theorem 3.2. *A centroaffine immersion $f : M \to \mathbb{R}^{n+2}$ is minimal if and only if the trace of the affine shape operator vanishes identically:* $\operatorname{tr} S = 0$.

In fact, the first variational formula is given by

$$\left.\frac{d}{dt}\right|_{t=0} \int_M \theta_t = -\frac{n}{n+2} \int_M v \operatorname{tr} S\theta. \tag{3.2}$$

Proof. Let $\Xi(x,t) = (\xi_t)_x$ be the prenormalized Blaschke normal vector field of f_t. We define $Z \in \Gamma(TM)$, a and $\alpha \in C^\infty(M)$ by

$$\Xi_*\left(\frac{\partial}{\partial t}\right)_{(x,0)} = f_* Z_x + a(x)\xi_x + \alpha(x) f(x).$$

We will prove that for $X_j \in \Gamma(TM)$ with $\theta(X_1, \dots, X_n) = 1$,

$$\left.\frac{d}{dt}\right|_{t=0} \theta_t(X_1, \dots, X_n) = \operatorname{div}_h V - v \operatorname{tr} S + a + \nu(n+1), \tag{3.3}$$

$$na = 2 \operatorname{div}_h V + \triangle_h v + \nu n - 2 \left.\frac{d}{dt}\right|_{t=0} \theta_t(X_1, \dots, X_n), \tag{3.4}$$

where ∇^h is the Levi-Civita connection of h, $\operatorname{div}_h V := \operatorname{tr}\{X \mapsto \nabla^h_X V\}$, and $\triangle_h$ is the Laplacian with respect to h.

The equations (3.3) and (3.4) imply

$$\left.\frac{d}{dt}\right|_{t=0} \theta_t(X_1, \dots, X_n) = \operatorname{div}_h V + \frac{1}{n+2}\triangle_h v - \frac{n}{n+2} v \operatorname{tr} S + n\nu,$$

from which we get the first variational formula (3.2), using $\nu = 0$ and Green's theorem.

We obtain (3.3) as follows:

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} \theta_t(X_1, \dots, X_n) \\
= &\left.\frac{d}{dt}\right|_{t=0} \operatorname{Det}(f_{t*}X_1, \dots, f_{t*}X_n, \xi_t, f_t) \\
= &\sum_i \operatorname{Det}(f_*X_1, \dots, f_*[\nabla_{X_i} V - vSX_i + \nu X_i], \dots, f_*X_n, \xi, f) \\
&+ \operatorname{Det}(f_*X_1, \dots, f_*X_n, a\xi, f) + \operatorname{Det}(f_*X_1, \dots, f_*X_n, \xi, \nu f) \\
= &\sum_i \theta(X_1, \dots, \nabla_{X_i} V, \dots, X_n) - v \sum_i \theta(X_1, \dots, SX_i, \dots, X_n) \\
&+ \nu n + a + \nu \\
= &\operatorname{div}_h V - v \operatorname{tr} S + a + \nu(n+1).
\end{aligned}$$

Here we have used the following identity (see [5, p.65]): For $X_i \in T_pM$ with $\theta(X_1, \dots, X_n) = 1$ and $V \in \Gamma(TM)$,

$$\operatorname{div}_h V = \operatorname{div}_\nabla V := \sum_{i=1}^n \theta(X_1, \dots, \nabla_{X_i} V, \dots, X_n). \tag{3.5}$$

To get (3.4), we compute the ξ-components of $D_X D_Y F_*(\partial/\partial t)$ and $D_{(\partial/\partial t)} D_X F_* Y$ at $t = 0$, where X and Y are vector fields on M. It follows that

$$\begin{aligned}
& \left\{ D_X D_Y F_* \frac{\partial}{\partial t}\bigg|_{t=0} \right\}^{\xi} \qquad (3.6)\\
= \;& \{D_X D_Y [f_* V + v\xi + \nu f]\}^{\xi} \\
= \;& \{D_X [f_*(\nabla_Y V - vSY + \nu Y) + (h(Y,V) + Yv)\xi \\
& +(T(Y,V) + vP(Y) + Y\nu) f]\}^{\xi} \\
= \;& h(X, \nabla_Y V) - vh(X, SY) + \nu h(X,Y) + Xh(Y,V) + XYv,
\end{aligned}$$

and

$$\begin{aligned}
& \left\{ D_{\frac{\partial}{\partial t}} D_X F_* Y \Big|_{t=0} \right\}^{\xi} \qquad (3.7)\\
= \;& \left\{ D_{\frac{\partial}{\partial t}(x,0)} \left[F_* \nabla^t_X Y + h_t(X,Y)\Xi + T_t(X,Y) F \right] \right\}^{\xi} \\
= \;& \left\{ D_{\nabla_X Y} F_* \frac{\partial}{\partial t}_{(x,0)} + \frac{d}{dt}\bigg|_{t=0} h_t(X,Y)\xi \right. \\
& + h(X,Y)(f_* Z + a\xi + \alpha f) \\
& \left. + \frac{d}{dt}\bigg|_{t=0} T_t(X,Y) f + T(X,Y)(f_* V + v\xi + \nu f) \right\}^{\xi} \\
= \;& h(\nabla_X Y, V) + \nabla_X Y v \\
& + \frac{d}{dt}\bigg|_{t=0} h_t(X,Y) + ah(X,Y) + vT(X,Y),
\end{aligned}$$

where $\{U\}^{\xi}$ denotes the ξ-component of $U \in \Gamma(f^{-1} T\,\mathbb{R}^{n+2})$.

Since $[X, \partial/\partial t] = 0$ for any $X \in \Gamma(TM)$ and D is flat, we get $D_X D_Y F_*(\partial/\partial t) = D_{(\partial/\partial t)} D_X F_* Y$, from which (3.6) and (3.7) imply that

$$\begin{aligned}
ah(X,Y) \;=\; & \{h(X, \nabla_Y V) + h(Y, \nabla_X V) + (\nabla_X h)(Y,V)\} \\
& - v\{h(X,SY) + T(X,Y)\} + \{\mathrm{Hess}_h\, v(X,Y) - K_X Y v\} \\
& + \nu h(X,Y) - \frac{d}{dt}\bigg|_{t=0} h_t(X,Y),
\end{aligned}$$

where $\mathrm{Hess}_h\, v$ denotes the Hessian of v with respect to h: $\mathrm{Hess}_h\, v(X,Y) := \nabla^h_X v_* Y - v_* \nabla^h_X Y$, and $K_X Y := K(X,Y) := \nabla_X Y - \nabla^h_X Y$. We take the trace of the equation above with respect to h, noting the following identity (see [5, p. 145]):

$$\frac{d}{dt}\bigg|_{t=0} \theta_t(X_1, \ldots, X_n) = \frac{1}{2}\,\mathrm{tr}_h \frac{d}{dt}\bigg|_{t=0} h_t.$$

We then obtain (3.4) by using (2.2), the Codazzi equation (4.4) and the following identity (see [5, p. 50, 53]):

$$\mathrm{tr}\, K_X = -\frac{1}{2}\,\mathrm{tr}_h(\nabla_X h) = h(\mathrm{tr}_h K, X) = 0. \qquad (3.8)$$

This completes the proof of Theorem 3.2. ∎

4 Examples

In this section, we illustrate some basic examples of minimal centroaffine immersions of codimension two.

Example 4.1. For $M := \mathbb{R}^2 \ni x = (x^1, x^2)$, we set $f(x) := (x^1, x^2, (1/2)\{(x^1)^2 - (x^2)^2\}, 1)$. Then $f(M)$ lies in a hyperplane and the prenormalized Blaschke normal vector field is given by $\xi(x) := (0, 0, 1, 0)$, which implies that the affine shape operator vanishes at each point, in particular, $f : M \to \mathbb{R}^4$ is a minimal centroaffine immersion.

Example 4.2. The Clifford torus $\phi(x^1, x^2) = 1/\sqrt{2}(\cos\sqrt{2}x^1, \sin\sqrt{2}x^1, \cos\sqrt{2}x^2, \sin\sqrt{2}x^2)$ gives rise to a minimal centroaffine immersion into $\mathbb{R}^4$, as well as a compact flat minimal surface in the unit 3-sphere, regarded as an object in Euclidean differential geometry (see [2, p. 87]). We remark that the affine shape operator of ϕ does not vanish.

We can calculate the prenormalized Blaschke normal vector field and ∇, h, T, S, P of the Clifford torus ϕ : For $\partial_i := \partial/\partial x^i$,

$$\begin{aligned}
&\xi = \frac{1}{\sqrt{2}}(-\cos\sqrt{2}x^1, -\sin\sqrt{2}x^1, \cos\sqrt{2}x^2, \sin\sqrt{2}x^2),\\
&\nabla^{\phi}_{\partial_i}\partial_j = 0, \quad h^{\phi} = (dx^1)^2 - (dx^2)^2,\\
&T^{\phi} = -\left\{(dx^1)^2 + (dx^2)^2\right\}, \quad S^{\phi} = dx^1 \otimes \partial_1 - dx^2 \otimes \partial_2,\\
&P^{\phi} = 0.
\end{aligned} \tag{4.1}$$

Let $\mathfrak{M}_\phi$ be the set of congruence classes of minimal centroaffine immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$ whose induced connection ∇ and affine fundamental form h coincide with the ones of the Clifford torus ϕ respectively:

$$\nabla = \nabla^{\phi}, \quad h = h^{\phi}.$$

We can determine the moduli space $\mathfrak{M}_\phi$ as follows:

Theorem 4.3. *The set $\mathfrak{M}_\phi$ is parameterized by the set of pairs (λ, μ) of smooth functions on $\mathbb{R}^2$ satisfying*

$$\frac{\partial\lambda}{\partial x^1} = \frac{\partial\mu}{\partial x^2}, \qquad \frac{\partial\lambda}{\partial x^2} = \frac{\partial\mu}{\partial x^1}. \tag{4.2}$$

Proof. *Step 1.* First, we recall the Gauss, Codazzi and Ricci equations in a general setting. Let $f : M \to \mathbb{R}^{n+2}$ be a centroaffine immersion. Then the following hold: For any $X, Y, Z \in \Gamma(TM)$,

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY - T(Y,Z)X + T(X,Z)Y, \tag{4.3}$$
$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z), \tag{4.4}$$
$$(\nabla_X T)(Y,Z) + h(Y,Z)P(X) = (\nabla_Y T)(X,Z) + h(X,Z)P(Y), \tag{4.5}$$
$$(\nabla_X S)Y + P(X)Y = (\nabla_Y S)X + P(Y)X, \tag{4.6}$$
$$h(X,SY) = h(Y,SX), \tag{4.7}$$
$$T(X,SY) - T(Y,SX) = dP(X,Y), \tag{4.8}$$

where R denotes the curvature tensor field of the induced connection ∇.

Conversely, if a torsion-free affine connection ∇, symmetric $(0,2)$-tensor fields h and T, a $(1,1)$-tensor field S and a 1-form P on a simply-connected manifold M satisfy the equations above, then there exists an immersion with a transversal vector field $(f,\xi) : M \to \mathbb{R}^{n+2}$ such that ∇, h, T, S, P coincide with the geometric quantities induced as in (2.1) with $\tau = 0$. Moreover, (f,ξ) is uniquely determined up to $GL(n+2;\mathbb{R})$-motion.

Step 2. For any centroaffine immersion $f \in \mathfrak{M}_\phi$, we set

$$T = T_{ij}dx^i dx^j, \quad S = S_i^j dx^i \otimes \partial_j, \quad P = P_i dx^i.$$

It follows from (4.1) and the Gauss equation (4.3) that

$$S_1^1 = -T_{22}, \quad S_2^1 = -T_{21}, \quad S_1^2 = T_{12}, \quad S_2^2 = T_{11}.$$

Using the assumption $0 = \operatorname{tr} S = S_1^1 + S_2^2$, we then determine S and T in the form

$$S = \begin{bmatrix} -\lambda & -\mu \\ \mu & \lambda \end{bmatrix}, \qquad T = \lambda\left\{(dx^1)^2 + (dx^2)^2\right\} + 2\mu dx^1 dx^2. \tag{4.9}$$

By the Codazzi equations (4.5) and (4.6), we have

$$\begin{cases} \partial_1 T_{22} - \partial_2 T_{12} &= P_1 = & -\partial_1 S_2^2 + \partial_2 S_1^2, \\ \partial_1 T_{21} - \partial_2 T_{11} &= P_2 = & \partial_1 S_2^1 - \partial_2 S_1^1, \end{cases}$$

which implies (4.2) and

$$P = 0. \tag{4.10}$$

In consequence, we obtain a correspondence of $f \in \mathfrak{M}_\phi$ to (λ, μ).

Step 3. We now show that the correspondence above is bijective. For a given pair (λ,μ) of smooth functions satisfying (4.2), we define a torsion-free affine connection ∇, symmetric $(0,2)$-tensor fields h and T, a $(1,1)$-tensor field S and a 1-form P on $\mathbb{R}^2$ by (4.1), (4.9) and (4.10). It can be checked that they satisfy the Gauss, Codazzi and Ricci equations (4.3) – (4.8), and consequently, there exists a corresponding immersion with a transversal vector field $(f_0,\xi_0) : \mathbb{R}^2 \to \mathbb{R}^4$.

The pair (f_0,ξ_0) satisfies the conditions in (2.2) at least except the third equation: $\omega_h = \theta$. We then choose a constant r so that $(rf_0, r\xi_0)$ satisfies that condition at the origin o of $\mathbb{R}^2$. It should be remarked that $(rf_0, r\xi_0)$ and (f_0,ξ_0) are $GL(4;\mathbb{R})$-congruent, and hence induce the same geometric quantities ∇, h, T, S, τ, P. Remark 2.2 concludes that $(rf_0, r\xi_0)$ satisfies the third equation of (2.2) at each point as well as the other conditions. Setting $(f,\xi) := (rf_0, r\xi_0)$, we obtain a minimal centroaffine immersion f with prenormalized Blaschke normal vector field ξ corresponding to (λ,μ). This completes the proof of Theorem 4.3. ∎

We remark that the Clifford torus ϕ corresponds to the pair $(-1,0)$ of constant functions. The set of two smooth functions on $\mathbb{R}^2$ satisfying (4.2) is so large that we have many minimal centroaffine immersions whose induced connection and affine fundamental form are given by (4.1). It contrasts with the higher dimensional cases: Nomizu and Sasaki [4] proved that two centroaffine immersions are congruent if the induced connections and the affine fundamental forms respectively coincide, and if the dimension n of the source manifold is greater than two.

Example 4.4. A holomorphic curve $f(z) = (z, z^2/2)$ gives rise to a minimal centroaffine immersion of $D = \{z \in \mathbb{C}; \Re z > 0, \Im z > 0\}$ into $\mathbb{C}^2 = \mathbb{R}^4$. The prenormalized Blaschke normal vector is given by $\xi(z) = \sqrt{2}\sqrt{-1}|z|^{-3}(z, 2z^2)$.

5 Duality

We first recall the duality for centroaffine immersions of codimension two, which is introduced by Nomizu and Sasaki [4]. Let $\mathbb{R}_{n+2}$ denote the dual space of $\mathbb{R}^{n+2}$. For a given centroaffine immersion $f : M \to \mathbb{R}^{n+2}$, we define maps $v, w : M \to \mathbb{R}_{n+2}$ by

$$\begin{cases} v(x)(\xi_x) &= 1, \\ v(x)(f(x)) &= 0, \\ v(x)(f_*X) &= 0, \end{cases} \qquad \begin{cases} w(x)(\xi_x) &= 0, \\ w(x)(f(x)) &= 1, \\ w(x)(f_*X) &= 0, \end{cases} \tag{5.1}$$

for each $x \in M$ and $X \in T_xM$. These maps are well-defined, since $\{\xi_x, f(x), f_*X_1, \ldots, f_*X_n\}$ is a basis of $\mathbb{R}^{n+2}$.

Proposition 5.1. (i) *For a given centroaffine immersion $f : M \to \mathbb{R}^{n+2}$, the map $v : M \to \mathbb{R}_{n+2}$, defined as above, is a centroaffine immersion with prenormalized Blaschke normal vector field w, which is called the dual immersion of f.*
(ii) *If the affine shape operator of f vanishes identically, the image of v lies on a hyperplane of $\mathbb{R}_{n+2}$.*
(iii) *The centroaffine immersion f is minimal if and only if so is v.*

Proof. The proposition is essentially proved in [4]. We denote the geometric quantities associated with f by $\nabla, h, T, S, \tau = 0, P$, and those associated with (v, w) by $\nabla^*, h^*, T^*, S^*, \tau^*, P^*$. By definition, we get the following (see Lemmas 3.1 - 3.3, [4]):

$$\begin{cases} v_*X(\xi_x) &= 0, \\ v_*X(f(x)) &= 0, \\ v_*X(f_*Y) &= -h(X,Y). \end{cases} \qquad \begin{cases} w_*X(\xi_x) &= -P(X), \\ w_*X(f(x)) &= 0, \\ w_*X(f_*Y) &= -T(X,Y). \end{cases} \tag{5.2}$$

$$h^* = h, \tag{5.3}$$
$$T^*(X,Y) = -h(SX,Y),$$
$$Z\,(h(X,Y)) = h(\nabla_Z X, Y) + h(X, \nabla^*_Z Y),$$
$$P^* = -P,$$
$$\tau^* = 0,$$
$$h^*(S^*X, Y) = -T(X,Y). \tag{5.4}$$

We omit the proof of (i) and (ii). To get (iii), we only have to remark that

$$\operatorname{tr} S = -\operatorname{tr}_h T = -\operatorname{tr}_{h^*} T = \operatorname{tr} S^*, \tag{5.5}$$

which follows from the fourth condition of (2.2), (5.3) and (5.4). ∎

Proposition 5.1 (iii) means that there exists a natural correspondence between two minimal centroaffine immersions. In the following proposition, we demonstrate the correspondence for a given immersion f together with affine fundamental form h. To this end, we define $e_1 \wedge \cdots \wedge e_{n+1} \in \mathbb{R}_{n+2}$ for $e_j \in \mathbb{R}^{n+2}$ by

$$e_1 \wedge \cdots \wedge e_{n+1}(\eta) := \operatorname{Det}(\eta, e_1, \ldots, e_{n+1}) \quad \text{for} \quad \eta \in \mathbb{R}^{n+2}.$$

Proposition 5.2. *Let $f : M \to \mathbb{R}^{n+2}$ be a minimal centroaffine immersion with affine fundamental form h. Then the dual immersion v and the prenormalized Blaschke normal vector field w are given by*

$$v(x) = \frac{n}{\operatorname{Det}(\triangle_h f, f, e_1, \ldots, e_n)} f \wedge e_1 \wedge \cdots \wedge e_n(x), \tag{5.6}$$
$$w(x) = \frac{-1}{\operatorname{Det}(\triangle_h f, f, e_1, \ldots, e_n)} \triangle_h f \wedge e_1 \wedge \cdots \wedge e_n(x), \tag{5.7}$$

where $X_1, \ldots, X_n$ are local frame fields on M and $e_j := f_* X_j$.

Proof. *Step 1*. Let ξ be the prenormalized Blaschke normal vector field of f. First, we prove that

$$v(x) = \frac{1}{\operatorname{Det}(\xi, f, e_1, \ldots, e_n)} f \wedge e_1 \wedge \cdots \wedge e_n(x), \tag{5.8}$$
$$w(x) = \frac{-1}{\operatorname{Det}(\xi, f, e_1, \ldots, e_n)} \xi \wedge e_1 \wedge \cdots \wedge e_n(x). \tag{5.9}$$

Indeed, we can easily check by (5.1) and (5.2) that the right hand side $\widetilde{v}(x)$ of (5.8) satisfies

$$f(\widetilde{v}) = 0, \quad f_* X_j(\widetilde{v}) = 0, \quad \xi(\widetilde{v}) = 1,$$

from which we conclude $\widetilde{v} = v$. In the same fashion, we can prove (5.9).
Step 2. We show that

$$\xi = \frac{1}{n}[\operatorname{tr} S f + \triangle_h f]. \tag{5.10}$$

Indeed, by (5.3) and (5.4) we calculate

$$\begin{aligned} \operatorname{Hess}_h f(X, Y) &= D_X f_* Y - f_* \nabla^h_X Y \\ &= f_* \nabla_X Y + h(X, Y)\xi + T(X, Y) f - f_* \nabla^h_X Y \\ &= f_* K_X Y + h(X, Y)\xi + T(X, Y) f, \end{aligned}$$

which implies by (2.2)

$$\triangle_h f = \operatorname{tr}_h \operatorname{Hess}_h f = f_* \operatorname{tr}_h K + n\xi - \operatorname{tr} S f.$$

Therefore, we obtain (5.10) by (3.8) and (5.5).
Step 3. By (5.10),we have $\xi = (1/n)\triangle_h f$, from which (5.8) and (5.9) imply (5.6) and (5.7). ∎

Remark 5.3. Let $f : \mathbb{R}^2 \to \mathbb{R}^4$ be a minimal centroaffine immersion whose induced connection and affine fundamental form coincide with the ones of the Clifford torus ϕ respectively: $f \in \mathfrak{M}_\phi$. Then we can check that f is *self-dual*, that is, by definition, the geometric quantities ∇, h, T, S, τ and P associated with f and those associated with its dual immersion coincide respectively.

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