On the Classical d-Orthogonal Polynomials Defined by Certain Generating Functions , I

Youssèf Ben Cheikh Khalfa Douak*

Abstract

The purpose of this work is to present some results on the d-orthogonal polynomials defined by generating functions of certain forms to be specified below. The resulting polynomials are natural extensions of some classical orthogonal polynomials. The first part of this study is motived by the recent work of Von Bachhaus [21] who showed that, among the orthogonal polynomials, only the Hermite and the Gegenbauer polynomials are defined by the generating function $G\big[2xt - t^2\big]$. Here we generalize this result in the context of d-orthogonality, by considering the polynomials generated by $G\big[(d+1)xt - t^{d+1}\big]$, where d is a positive integer. We obtain that the resulting polynomials are d-symmetric (Definition 1.2) and "classical" in the Hahn's sense. We provide some examples to illustrate the results obtained and show that they involve certain known polynomials. Finally, we conclude by giving some properties of the zeros of these polynomials as well as a $(d+1)$-order differential equation satisfied by each polynomial. In forthcoming paper [2] we will consider the polynomials generated by $e^t\Psi(xt)$.

1 Introduction and preliminary results

Let $G[z]$ be analytic at $z = 0$ and has the expansion

$$G[z] = \sum_{n=0}^{\infty} a_n z^n\,, \quad a_n \neq 0\,,\ n \geq 0. \tag{1.1}$$

*The corresponding author

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Now, let d be an arbitrary positive integer with $d \geq 1$. Define the monic polynomials $P_n(.;d)$, $n = 0, 1, \ldots,$ by the generating function

$$G\Big[(d+1)xt - t^{d+1}\Big] = \sum_{n=0}^{\infty} c_n P_n(x;d) t^n\,, \qquad c_n \neq 0\,,\ n \geq 0. \tag{1.2}$$

Also we put

$$G_d(x,t) = G\Big[(d+1)xt - t^{d+1}\Big]. \tag{1.3}$$

Since $P_n(.;d)$ is monic, we have $c_n = (d+1)^n a_n$, $n \geq 0$.

Without any requirements of orthogonality, Rainville [18] has studied these polynomials when $d = 1$ and obtained that the two families of Legendre and Hermite polynomials are contained in this class of polynomials. Later, Srivastava and others (see, e. g. [19] and the references therein) also investigated the polynomials g_n^p, $n = 0, 1, \ldots,$ known as Gould-Hopper [12] polynomials, and defined by the generating function $\exp\Big(pxt - t^p\Big) = \sum_{n=0}^{\infty} g_n^p(x) t^n/n!$, where p is a positive integer. He showed that these polynomials are contained in the more general class of polynomials defined by (1.2).

Recently, when $d = 1$, Von Bachhaus [21] has considered the orthogonal polynomials defined by the generating function $G_1(x,t) = G\Big[2xt - t^2\Big]$. He stated that

Theorem 1.1 [21] *The only orthogonal polynomials generated by* $G_1(x,t) = G\Big[2xt - t^2\Big]$ *are the Hermite and the Gegenbauer polynomials.*

At first, he proved that the polynomials P_n, $n \geq 0$, satisfy a second-order differential equation. Next, by comparing this differential equation with the Bochner result [3], and after discussion of all the possible cases, he obtained the above result. Note in passing that this result was briefly mentioned in the work of Sister Celine, see Fasenmyer [10].

In the present paper, we investigate the d-orthogonal polynomials defined by the generating function (1.2) and intereste in some particular cases of the polynomials obtained and certain of their interesting properties. First of all, we give an alternative way of obtaining the Von Bachhaus result and next we consider the following problem:

(P): *Find all d-orthogonal polynomials defined by the generating function* $G_d(x,t) = G\Big[(d+1)xt - t^{d+1}\Big]$.

The main result obtained here is the following:

Theorem 1.2 *The only d-orthogonal polynomials generated by* (1.2) *are the classical d-symmetric polynomials.*

When $d = 1$, we rediscover the Von Bachhaus result.

The paper is divided into four sections. Following the introduction and preliminaries necessary for the sequel, we give, in Section 2, another simple proof of the Von Bachhaus Theorem based only on the use of the recurrence relation and the Hahn's property [13]. In Section 3, we state and solve the above problem **(P)** for any integer d with $d \geq 1$. Our method of solving this problem is also based only on the recurrence relations and the use of Hahn's property. We obtain that the resulting polynomials are the classical d-symmetric polynomials and show that there are 2^d solutions of this problem. Two typical families are singled out. The first is the Hermite type polynomials and the second is the ultraspherical type polynomials. Finally, we conclude the paper by giving, in Section 4, some properties of the polynomials obtained.

Before discussing the above problem, let us recall some preliminary results which we need below. Throughout this paper, we assume that $\{P_n\}_{n\geq 0}$ is a sequence of monic polynomials $\Big(P_n(x) = x^n + \dots\Big)$ and $\{u_n\}_{n\geq 0}$ its dual sequence defined by $\Big\langle u_n, P_m\Big\rangle = \delta_{n,m}\,; n,m \geq 0$, where $\Big\langle\,,\,\Big\rangle$ are the duality brackets between the vector space of polynomials with complex coefficients and its dual.

Definition 1.1 *The polynomial sequence $\{P_n\}_{n\geq 0}$ is called a d-orthogonal polynomial sequence (d-OPS) with respect to the d-dimensional functional $\boldsymbol{\mathcal{U}} = {}^t(u_0, \ \dots, u_{d-1})$ if it fulfils* [15, 20]

$$\begin{cases} \Big\langle u_k\ ,\ P_m P_n\Big\rangle = 0\,, & m \geq dn + k + 1,\ n \geq 0, \\ \Big\langle u_k\ ,\ P_n P_{dn+k}\Big\rangle \neq 0\,, & n \geq 0, \end{cases} \tag{1.4}$$

for each integer k with $k = 0, 1, \dots, d-1$.

Note that, when $d = 1$, we meet again the ordinary regular orthogonality. In this case, $\{P_n\}_{n\geq 0}$ is an orthogonal polynomial sequence (OPS).

The remarkable characterization of the d-OPS is that they satisfy a $(d+1)$-order recurrence relation [20] which we write in the form

$$P_{m+d+1}(x) = (x - \beta_{m+d})P_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} P_{m+d-1-\nu}(x)\,,\ m \geq 0, \tag{1.5}$$

with the initial conditions

$$\begin{cases} P_0(x) = 1 \quad , \quad P_1(x) = x - \beta_0 \quad \text{and} \quad \text{if} \quad d \geq 2: \\ P_n(x) = (x - \beta_{n-1})P_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} P_{n-2-\nu}(x)\,,\ 2 \leq n \leq d, \end{cases} \tag{1.6}$$

and the regularity conditions

$$\gamma_{n+1}^0 \neq 0\,, \quad n \geq 0. \tag{1.7}$$

When $d = 1$, the recurrence (1.5) with (1.6) is reducible to the well-known second-order recurrence relation

$$\begin{cases} P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ n \geq 0\,; \\ P_0(x) = 1,\ P_1(x) = x - \beta_0. \end{cases} \tag{1.8}$$

Now, let $\{Q_n\}_{n\geq 0}$ be the sequence of the monic derivatives defined by $Q_n(x) := (n+1)^{-1}P'_{n+1}(x)\,, n \geq 0$. According to the Hahn's property [13], if the sequence $\{Q_n\}_{n\geq 0}$ is also d-orthogonal, the sequence $\{P_n\}_{n\geq 0}$ is called "classical" d-OPS. In this case $\{Q_n\}_{n\geq 0}$ also satisfies a $(d+1)$-order recurrence relation

$$Q_{m+d+1}(x) = (x - \widetilde{\beta}_{m+d})Q_{m+d}(x) - \sum_{\nu=0}^{d-1} \widetilde{\gamma}_{m+d-\nu}^{d-1-\nu} Q_{m+d-1-\nu}(x)\,,\ m \geq 0, \tag{1.9}$$

with the initial and the regularity conditions

$$\begin{cases} Q_0(x) = 1 \quad , \quad Q_1(x) = x - \widetilde{\beta}_0 \quad \text{and if} \quad d \geq 2: \\ Q_n(x) = (x - \widetilde{\beta}_{n-1})Q_{n-1}(x) - \sum_{\nu=0}^{n-2} \widetilde{\gamma}_{n-1-\nu}^{d-1-\nu} Q_{n-2-\nu}(x) \, , \; 2 \leq n \leq d, \\ \widetilde{\gamma}_{m+1}^0 \neq 0 \, , \quad m \geq 0 \end{cases} \tag{1.10}$$

and it is d-orthogonal with respect to $\mathcal{V} = {}^t(v_0, \; \ldots, \; v_{d-1})$, where $\{v_n\}_{n\geq 0}$ is the dual sequence of $\{Q_n\}_{n\geq 0}$.

Finally, we will also recall the notion of d-symmetric polynomials. Let $\omega = \exp\Big(2i\pi/(d+1)\Big)$ and k be an integer with $0 \leq k \leq d$. By $\xi_k \; ; \; k = 0, 1, \ldots, d$, we denote the $d+1$ roots of unity, namely $\xi_k = \omega^k = \exp\Big(2ik\pi/(d+1)\Big)$.

Definition 1.2 [6] *The sequence $\{P_n\}_{n\geq 0}$ is called d-symmetric if it fulfils*

$$P_n(\omega x) = \omega^n P_n(x) \, , \; n \geq 0, \tag{1.11}$$

or, equivalently,

$$P_n(\xi_k x) = \xi_k^n P_n(x) \; ; \; k = 0, 1, \ldots, d \; ; \; n \geq 0. \tag{1.12}$$

When $d = 1$, then $\omega = -1$, this means that the sequence $\{P_n\}_{n\geq 0}$ is symmetric, that is to say $P_n(-x) = (-1)^n P_n(x)$, $n \geq 0$.

Theorem 1.3 [6] *Let $\{P_n\}_{n\geq 0}$ be a sequence of d-orthogonal polynomials with respect to the d-dimensional functional $\mathcal{U} = {}^t(u_0, \; \ldots, \; u_{d-1})$. Then the following statements are equivalent:*

(a) $\{P_n\}_{n\geq 0}$ is d-symmetric.

(b) $\{P_n\}_{n\geq 0}$ satisfies the (d+1)-order recurrence relation

$$\begin{cases} P_{n+d+1}(x) = x \, P_{n+d}(x) - \gamma_{n+1}^0 \, P_n(x) \; , \quad (\gamma_{n+1}^0 \neq 0), \quad n \geq 0, \\ P_n(x) = x^n \quad , \quad 0 \leq n \leq d. \end{cases} \tag{1.13}$$

In the other words, $\beta_n = 0$, $n \geq 0$ and $\gamma_{n+1}^\nu = 0$, $n \geq 0$ for $\nu = 1, \ldots, d-1$.

(c) For each integer j with $0 \leq j \leq d-1$, the moments of the linear functional u_j satisfy

$$(u_j)_{(d+1)n+i} = 0 \, ; \; i = 0, 1, \ldots, d \; ; \; i \neq j \; , \; n \geq 0. \tag{1.14}$$

$\Big($In this case the d-dimensional functional $\mathcal{U}$ is called d-symmetric$\Big)$.

This Theorem generalizes the well-known result about the symmetric orthogonal polynomials (see, e.g. [4]).

2 Another proof of Von Bachhaus Theorem

Throught this section, we use the notation $P_n(x) = P_n(x;1)$ adopted by Von Bachhaus. First, we give the two following fundamental lemmas:

Lemma 2.1 [18] *If the sequence* $\{P_n\}_{n\geq 0}$ *is defined by* $G\Big[2xt-t^2\Big] = \sum_{n\geq 0} c_n P_n(x)t^n$, *then it satisfies*

$$nc_nP_n(x) + c_{n-1}P'_{n-1}(x) - c_n x P'_n(x) = 0, \quad n \geq 1. \tag{2.1}$$

Lemma 2.2 *If* $\{P_n\}_{n\geq 0}$ *is defined by* $G\Big[2xt - t^2\Big] = \sum_{n\geq 0} c_n P_n(x)t^n$, *then it is a symmetric sequence.*

Proof of Lemma 2.2 From $G_1(x,t) = G\Big[2xt - t^2\Big]$, it is easily verified that $G_1(-x,-t) = G_1(x,t)$. Then, using these changes in the right-hand sides, we get

$$\sum_{n\geq 0} c_nP_n(-x)(-1)^n t^n = \sum_{n\geq 0} c_nP_n(x)t^n.$$

By comparing the coefficients of t^n, we obtain

$$P_n(-x) = (-1)^n P_n(x), \quad n \geq 0.$$

Then $\{P_n\}_{n\geq 0}$ is symmetric.
This result may be also obtained by the use of the identity $G_1(-x,t) = G_1(x,-t)$.
■

Now, we suppose that $\{P_n\}_{n\geq 0}$ is an OPS. From the symmetry property, we deduce that the sequence $\{P_n\}_{n\geq 0}$ satisfies the second-order recurrence relation

$$\begin{cases} P_{n+2}(x) = xP_{n+1}(x) - \gamma_{n+1}P_n(x),\ n\geq 0\,; & \Big(\gamma_{n+1}\neq 0\ ,\ n\geq 0\Big)\,, \\ P_0(x) = 1,\ P_1(x) = x. \end{cases} \tag{2.2}$$

Therefore, to prove the Bachhaus result it suffices to show that $\{P_n\}_{n\geq 0}$ is classical, that is to say, the sequence $\{Q_n\}_{n\geq 0}$ is also orthogonal.

Proof of Theorem 1.1. Indeed, from (2.1), changing n by $n+1$, we get

$$P_{n+1}(x) = xQ_n(x) - \frac{nc_n}{(n+1)c_{n+1}}Q_{n-1}(x), \quad n \geq 0, \tag{2.3}$$

where $Q_n = (n+1)^{-1}P'_{n+1},\ n = 0,1,\dots,\ Q_{-1} = 0$.
Otherwise, by differentiating (2.2), we have

$$P_{n+1}(x) = (n+2)Q_{n+1}(x) + n\gamma_{n+1}Q_{n-1}(x) - (n+1)xQ_n(x),\ n \geq 0. \tag{2.4}$$

Substituting for P_{n+1} from (2.3) into (2.4) yields

$$\begin{cases} Q_{n+2}(x) = xQ_{n+1}(x) - \widetilde{\gamma}_{n+1}Q_n(x),\ n \geq 0; \\ Q_0(x) = 1,\ Q_1(x) = x, \end{cases} \tag{2.5}$$

where

$$\widetilde{\gamma}_{n+1} = \frac{n+1}{n+3}\Big[\gamma_{n+2} + \frac{c_{n+1}}{(n+2)c_{n+2}}\Big],\ n \geq 0. \tag{2.6}$$

Thus, the sequence $\{Q_n\}_{n\geq 0}$ is also orthogonal and then, by virtue of Hahn's property, $\{P_n\}_{n\geq 0}$ is classical. Because of the symmetry property, it is clear that only the Hermite and Gegenbauer (ultraspherical) polynomials are among the resulting polynomials. ∎

Remark 2.1. The coefficients $\widetilde{\gamma}_{n+1}$ defined by (2.6) are not zero for all $n \geq 0$. Indeed, this contradicts the regularity conditions (1.7). See Remark 3.1. below.

3 Generalization of the above result in the d-orthogonality case

In this section, we make similar investigations by considering the analogous problem:
(P): *Find all d-orthogonal polynomials defined by the generating function* (1.2).
We first need the two fundamental lemmas:

Lemma 3.1 *If $\{P_n(.;d)\}_{n\geq 0}$ is defined by $G\Big[(d+1)xt - t^{d+1}\Big] = \sum_{n=0}^{\infty} c_n P_n(x;d)t^n$ then it is a d-symmetric sequence.*

Proof. Indeed, from $G_d(x,t) = G\Big[(d+1)xt - t^{d+1}\Big]$, it is easily verified that

$$G_d(\omega x, t) = G_d(x, \omega t).$$

Make the same changes of the two variables x and t in the right-hand side of (1.2), we get

$$\sum_{n\geq 0} c_n P_n(\omega x; d)t^n = \sum_{n\geq 0} c_n P_n(x;d)\omega^n t^n.$$

By identification we obtain that

$$P_n(\omega x; d) = \omega^n P_n(x;d), \quad n \geq 0.$$

Then $\{P_n(.;d)\}_{n\geq 0}$ is d-symmetric. ∎

Lemma 3.2 *If the sequence $\{P_n(.;d)\}_{n\geq 0}$ is defined by* (1.2) *, then it satisfies*

$$\begin{aligned} &P_{n+d+1}(x;d) = xQ_{n+d}(x;d) - \frac{(n+1)c_{n+1}}{(n+d+1)c_{n+d+1}} Q_n(x;d), \quad n \geq 0, \\ &P_{n+1}(x;d) = xQ_n(x;d),\ n = 0,1,\ldots,d-1. \end{aligned} \tag{3.1}$$

Proof. By differentiating the generating function (1.2) with respect to t and x, we easily obtain the following partial differential equation:

$$t\frac{\partial G_d}{\partial t} - (x - t^d)\frac{\partial G_d}{\partial x} = 0. \tag{3.2}$$

Now, using the right-member of (1.2), we have

$$\frac{\partial G_d}{\partial t} = \sum_{n\geq 0}(n+1)c_{n+1}P_{n+1}(x;d)t^n, \tag{3.3}$$

and

$$\frac{\partial G_d}{\partial x} = \sum_{n\geq 0} c_{n+1}P'_{n+1}(x;d)t^{n+1} = \sum_{n\geq 0}(n+1)c_{n+1}Q_n(x;d)t^{n+1}. \tag{3.4}$$

Substituting (3.3) and (3.4) into (1.2), we obtain

$$\begin{aligned}\sum_{n\geq 0} c_{n+1}P_{n+1}(x;d)t^{n+1} &= (x-t^d)\sum_{n\geq 0}(n+1)c_{n+1}Q_n(x;d)t^{n+1}\\ &= \sum_{n\geq 0}(n+1)c_{n+1}xQ_n(x;d)t^{n+1} - \sum_{n\geq d}(n-d+1)c_{n-d+1}Q_{n-d}(x;d)t^{n+1}.\end{aligned} \tag{3.5}$$

By comparing the coefficients of t^n and shifting indices the relation (3.1) follows immediatly. ■

Now, we return to our problem and prove the main result.

Proof of Theorem 1.2. As the sequence $\{P_n(.;d)\}_{n\geq 0}$ is also d-OPS, according to Lemma 3.1.and Theorem 1.3., it satisfies the recurrence relation

$$\begin{cases} P_{n+d+1}(x;d) = x\,P_{n+d}(x;d) - \gamma^0_{n+1}\,P_n(x;d)\ ,\ n\geq 0,\\ P_n(x;d) = x^n \quad ,\quad 0\leq n\leq d,\end{cases} \tag{3.6}$$

with the regularity conditions (1.7).
Now, by differentiating (3.6) and shifting indices, we obtain

$$\begin{aligned}P_{n+d+1}(x;d) = (n+d+2)Q_{n+d+1}(x;d) + (n+1)\gamma^0_{n+2}Q_n(x;d)\\ - (n+d+1)xQ_{n+d}(x;d),\ n\geq 0.\end{aligned} \tag{3.7}$$

From (3.1) and (3.7), we obtain that the sequence $\{Q_n(.;d)\}_{n\geq 0}$ also satisfies the following $(d+1)$-order recurrence relation:

$$\begin{cases} Q_{n+d+1}(x;d) = xQ_{n+d}(x;d) - \widetilde{\gamma}^0_{n+1}Q_n(x;d)\ ,\quad n\geq 0,\\ Q_n(x;d) = x^n \quad ,\quad 0\leq n\leq d,\end{cases} \tag{3.8}$$

where

$$\widetilde{\gamma}^0_{n+1} = \frac{n+1}{n+d+2}\left[\gamma^0_{n+2} + \frac{c_{n+1}}{(n+d+1)c_{n+d+1}}\right],\ n\geq 0. \tag{3.9}$$

Clearly, the sequence $\{Q_n(.;d)\}_{n\geq 0}$ also form a d-OPS and then $\{P_n(.;d)\}_{n\geq 0}$ is "classical" in the Hahn's sense.
As shown below, the coefficients $\widetilde{\gamma}^0_{n+1}$, $n\geq 0$, are not zero, because this contradicts the regularity conditions (1.7). ■

In the sequel, we will determine all the solutions of our problem. We start by giving and solving the system satisfied by the coefficients γ^0_{n+1} and $\widetilde{\gamma}^0_{n+1}$, $n\geq 0$.

From (3.7) and (3.8) we obtain

$$P_{n+d+1}(x;d) = Q_{n+d+1}(x;d) + \Big((n+1)\gamma_{n+2}^0 - (n+d+1)\widetilde{\gamma}_{n+1}^0\Big)Q_n(x;d),\ n \geq 0. \tag{3.10}$$

Next, from the last and the recurrence relations satisfied by $\{P_n(.;d)\}_{n\geq 0}$ and $\{Q_n(.;d)\}_{n\geq 0}$ and after some calculations, we obtain that the coefficients γ_{n+1}^0 and $\widetilde{\gamma}_{n+1}^0$ fulfill the following system:

$$(n+d+2)\widetilde{\gamma}_{n+1}^0 = (n+d)\widetilde{\gamma}_n^0 + (n+1)\gamma_{n+2}^0 - (n-1)\gamma_{n+1}^0\ ,\ n \geq 1, \tag{3.11}$$

$$(d+2)\widetilde{\gamma}_1^0 = \gamma_2^0 + \gamma_1^0, \tag{3.12}$$

$$(n+2d)\widetilde{\gamma}_{n+d}^0\widetilde{\gamma}_n^0 = 2(n+d)\gamma_{n+d+1}^0\widetilde{\gamma}_n^0 - n\gamma_{n+d+1}^0\gamma_{n+1}^0\ ,\ n \geq 1. \tag{3.13}$$

Remark 3.1. By virtue of (3.13) and the regularity conditions (1.7), it is clear that $\widetilde{\gamma}_{n+1}^0 \neq 0\ ,\ n \geq 0$. Indeed, if $\widetilde{\gamma}_{n_0}^0 = 0$ for $n_0 \geq 1$, then $\gamma_{n_0+d+1}^0\gamma_{n_0+1}^0 = 0$, which is contradictory.

To solve this system, we pose:

$$\widetilde{\gamma}_n^0 = \gamma_{n+1}^0 \frac{n}{n+d}\vartheta_n\ , \quad \vartheta_n \neq 0\ , \quad n \geq 1. \tag{3.14}$$

Thus, substituting for $\widetilde{\gamma}_n^0$ from (3.14) into (3.11)-(3.13), the above system becomes

$$(n+1)\Big[(n+d+2)(\vartheta_{n+1}-1)+1\Big]\gamma_{n+2}^0 = (n+d+1)\Big[n(\vartheta_n-1)+1\Big]\gamma_{n+1}^0\ ,\ n \geq 1, \tag{3.15}$$

$$\Big[(d+2)(\vartheta_1-1)+1\Big]\gamma_2^0 = (d+1)\gamma_1^0, \tag{3.16}$$

$$\vartheta_{n+d} + \frac{1}{\vartheta_n} = 2\ , \quad n \geq 1. \tag{3.17}$$

The Riccati equation (3.17) plays an important role in the solution of the above system. Indeed, for k fixed $\Big(k = 0, 1, \ldots, d-1\Big)$, by replacing n with $dn+k$, the last equation becomes

$$\vartheta_{d(n+1)+k} + \frac{1}{\vartheta_{dn+k}} = 2\ ,\ n \geq 0 \quad \Big(\text{with } n \neq 0 \text{ if } k = 0\Big). \tag{3.18}$$

Thus, for each k, it is easy verified that this equation has the following solutions:

$$\begin{cases} 1\ ,\ n \geq 0; \\ \vartheta_{dn+k} = \frac{n+\lambda_k+1}{n+\lambda_k}\ ,\ n \geq 0, \end{cases} \qquad \Big(n \neq 0 \text{ if } k = 0\Big) \tag{3.19}$$

where λ_k are d arbitrary parameters with $\lambda_0 \neq -1, -2, \ldots$ and $\lambda_k \neq 0, -1, -2, \ldots$ for $k = 1, 2, \ldots, d-1$.

Thus, Eq. (3.18) has exactly 2^d solutions denoted $\mathcal{S}_{d,i}\ ,\ i = 1, \ldots, 2^d$. Each solution $\mathcal{S}_{d,i}$ is in fact a d-uplet of the solutions (3.19) of the form:

$$\Big(\vartheta_{dn},\ \vartheta_{dn+1},\ \ldots,\ \vartheta_{dn+d-1}\Big).$$

In the sequel, we adopt the following classification of the solutions $\mathcal{S}_{d,i}$:

$\mathcal{S}_{d,i}$:	ϑ_{dn}	ϑ_{dn+1}	ϑ_{dn+2}	$\dots$	ϑ_{dn+d-2}	ϑ_{dn+d-1}
$\mathcal{S}_{d,1}$:	1	1	1	$\dots$	1	1
$\mathcal{S}_{d,2}$:	1	1	1	$\dots$	1	$\frac{n+\lambda_{d-1}+1}{n+\lambda_{d-1}}$
$\mathcal{S}_{d,3}$:	1	1	1	$\dots$	$\frac{n+\lambda_{d-2}+1}{n+\lambda_{d-2}}$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\mathcal{S}_{d,2^d-1}$:	$\frac{n+\lambda_0+1}{n+\lambda_0}$	$\frac{n+\lambda_1+1}{n+\lambda_1}$	$\frac{n+\lambda_2+1}{n+\lambda_2}$	$\dots$	$\frac{n+\lambda_{d-2}+1}{n+\lambda_{d-2}}$	1
$\mathcal{S}_{d,2^d}$:	$\frac{n+\lambda_0+1}{n+\lambda_0}$	$\frac{n+\lambda_1+1}{n+\lambda_1}$	$\frac{n+\lambda_2+1}{n+\lambda_2}$	$\dots$	$\frac{n+\lambda_{d-2}+1}{n+\lambda_{d-2}}$	$\frac{n+\lambda_{d-1}+1}{n+\lambda_{d-1}}$

Table 1.

Consequently, we have 2^d families of the classical d-OPS defined by the generating function (1.2).

For instance, when $d = 1$, $d = 2$ and $d = 3$, we have, respectively,

- For $d = 1$: $i = 1, 2$ and $k = 0$. Then Eq.(3.18) has two (2^1) solutions:

$\mathcal{S}_{1,i}$:	$\vartheta_n \ \ (n \geq 1)$
$\mathcal{S}_{1,1}$:	1
$\mathcal{S}_{1,2}$:	$\dfrac{n+\lambda_0+1}{n+\lambda_0}$

Table 2.

In this case, we rediscover with the above solutions $\mathcal{S}_{1,1}$ and $\mathcal{S}_{1,2}$, respectively, the two families of Hermite and Gegenbauer polynomials where $\lambda_0 \to \alpha + 1/2$ in the second case. That is the Von Bachhaus result.

- For $d = 2$: $i = 1, \dots, 4$ and $k = 0, 1$. Then Eq.(3.18) has four (2^2) solutions [6]:

$\mathcal{S}_{2,i}$:	$\vartheta_{2n} \ \ (n \geq 1)$	$\vartheta_{2n+1} \ \ (n \geq 0)$
$\mathcal{S}_{2,1}$:	1	1
$\mathcal{S}_{2,2}$:	1	$\frac{n+\lambda_1+1}{n+\lambda_1}$
$\mathcal{S}_{2,3}$:	$\frac{n+\lambda_0+1}{n+\lambda_0}$	1
$\mathcal{S}_{2,4}$:	$\frac{n+\lambda_0+1}{n+\lambda_0}$	$\frac{n+\lambda_1+1}{n+\lambda_1}$

Table 3.

• For $d = 3$: $i = 1, \ldots, 8$ and $k = 0, 1, 2$. Then Eq.(3.18) has eight (2^3) solutions:

$\mathcal{S}_{3,i}$:	ϑ_{3n} $(n \geq 1)$	ϑ_{3n+1} $(n \geq 0)$	ϑ_{3n+2} $(n \geq 0)$
$\mathcal{S}_{3,1}$:	1	1	1
$\mathcal{S}_{3,2}$:	1	1	$\frac{n+\lambda_2+1}{n+\lambda_2}$
$\mathcal{S}_{3,3}$:	1	$\frac{n+\lambda_1+1}{n+\lambda_1}$	1
$\mathcal{S}_{3,4}$:	1	$\frac{n+\lambda_1+1}{n+\lambda_1}$	$\frac{n+\lambda_2+1}{n+\lambda_2}$
$\mathcal{S}_{3,5}$:	$\frac{n+\lambda_0+1}{n+\lambda_0}$	1	1
$\mathcal{S}_{3,6}$:	$\frac{n+\lambda_0+1}{n+\lambda_0}$	1	$\frac{n+\lambda_2+1}{n+\lambda_2}$
$\mathcal{S}_{3,7}$:	$\frac{n+\lambda_0+1}{n+\lambda_0}$	$\frac{n+\lambda_1+1}{n+\lambda_1}$	1
$\mathcal{S}_{3,8}$:	$\frac{n+\lambda_0+1}{n+\lambda_0}$	$\frac{n+\lambda_1+1}{n+\lambda_1}$	$\frac{n+\lambda_2+1}{n+\lambda_2}$

Table 4.

Now, from (3.15) and taking (3.16) into account, we obtain that

$$\begin{cases} \gamma^0_{n+2} & = \gamma^0_1 \binom{n+d+1}{d} \prod_{j=0}^{n} \Theta_j \, , \; n \geq 0, \\ \widetilde{\gamma}^0_{n+1} & = \gamma^0_1 \binom{n+d}{d} \prod_{j=0}^{n+1} \Theta_j \, , \; n \geq 0, \end{cases} \tag{3.20}$$

where

$$\Theta_j = \begin{cases} \frac{j(\vartheta_j - 1)+1}{(j+d+2)(\vartheta_{j+1}-1)+1} \, , & j = 1, 2, \ldots, \\ \frac{1}{(d+2)(\vartheta_1 - 1)+1} \, , & j = 0. \end{cases} \tag{3.21}$$

In general, by virtue of the solutions (3.19), to compute the coefficients γ^0_n and $\widetilde{\gamma}^0_n$, we need shift the indices as follows $n \to nd + k$. Then (3.20) yields

$$\begin{cases} \gamma^0_{dn+k+2} & = \gamma^0_1 \binom{d(n+1)+k+1}{d} \prod_{j=0}^{dn+k} \Theta_j \, , \; k = 0, 1, \ldots, d-1 \, ; \; n \geq 0, \\ \widetilde{\gamma}^0_{dn+k+1} & = \gamma^0_1 \binom{d(n+1)+k}{d} \prod_{j=0}^{dn+k+1} \Theta_j \, , \; k = 0, 1, \ldots, d-1 \, ; \; n \geq 0. \end{cases} \tag{3.22}$$

■

3.1 Examples

Among all the solutions of the system (3.11)-(3.13), we now restrict our attention to the following two typical examples obtained with the two solutions $\mathcal{S}_{d,1}$ and $\mathcal{S}_{d,2^d}$:

Example 1. We consider the evident solution $\mathcal{S}_{d,1}$ of the Riccati equation (3.17), that is to say, $\vartheta_{dn+k} = 1$, for all indices. In this case we simply write $\vartheta_n = 1 \, , \; n \geq 1$. Thus, from (3.21), we easily obtain that $\Theta_n = 1 \, , \; n \geq 0$ and from (3.20) we have

$$\gamma^0_{n+1} = \widetilde{\gamma}^0_{n+1} = \gamma^0_1 \binom{n+d}{d} \, , \; n \geq 0. \tag{3.23}$$

Therefore $Q_n = B_n$, $n \geq 0$, and then $\{B_n\}_{n\geq 0}$ is at the same time d-orthogonal and Appell sequence.

These polynomials are the d-symmetric ones of the family of polynomials recently obtained in the context of d-orthogonality by the second author in [5] and for $d = 2$ in [6]. They are called the d-orthogonal polynomials of Hermite type. Otherwise, these polynomials were studied in different contexts by several authors (see, e.g., [19]).
When $d = 1$, we meet again the Hermite polynomials.

Example 2. Let us now consider the solution $\mathcal{S}_{d,2^d}$ of the Riccati equation (3.17). In this case we have $\vartheta_{dn+k} \neq 1$, for all indices, that is to say,

$$\vartheta_{dn+k} = \frac{n+\lambda_k+1}{n+\lambda_k}\,,\ n \geq 0\ ,\ k = 0,1,\ldots,d-1, \quad \big(n \neq 0 \ \text{ if } k = 0\big)$$

and (3.21) becomes

$$\Theta_{dn+k} = \begin{cases} \frac{\left(n+\lambda_{k+1}+1\right)\left((d+1)n+\lambda_k+k\right)}{\left(n+\lambda_k\right)\left((d+2)n+\lambda_{k+1}+k+3\right)}\,, & k = 0,1,\ldots,d-1,\ n \geq 0,\ \big(n \neq 0 \,\text{if}\, k = 0\big) \\ \frac{\lambda_1}{\lambda_1+d+2}\,, & k = n = 0. \end{cases} \tag{3.24}$$

Here, we are interested by a particular case of the last solution. Indeed, if we put

$$\lambda_k = \frac{(d+1)\lambda+k}{d}\ ,\ k = 0,1,\ldots,d-1,$$

we obtain that

$$\vartheta_{dn+k} = \frac{dn+k+(d+1)\lambda+d}{dn+k+(d+1)\lambda}\,,\ n \geq 0\ ,\ k = 0,1,\ldots,d-1, \quad \big(n \neq 0 \ \text{ if } k = 0\big),$$

which can be written, changing the indices $dn + k \to n$, as

$$\vartheta_n = \frac{n+(d+1)\lambda+d}{n+(d+1)\lambda}\,,\ n \geq 1. \tag{3.25}$$

Whence (3.24) becomes

$$\Theta_j = \begin{cases} \frac{\left((d+1)\lambda+j+1\right)\left(\lambda+j\right)}{\left((d+1)\lambda+j\right)\left(\lambda+d+j+1\right)}\,, & j = 1,2,\ldots, \\ \frac{(d+1)\lambda+1}{\left(d+1\right)\left(\lambda+d+1\right)}\,, & j = 0, \end{cases} \tag{3.26}$$

and then

$$\prod_{j=0}^{n} \Theta_j = \frac{\Big(n+(d+1)\lambda+1\Big)\Big(\lambda+1\Big)_n}{\Big(\lambda+d+1\Big)_{n+1}}\,,\ n \geq 0, \tag{3.27}$$

where $(\mu)_n$ is Pochhammer's symbol defined by

$$(\mu)_n=\frac{\Gamma(\mu+n)}{\Gamma(\mu)}=\begin{cases}\mu(\mu+1)\dots(\mu+n-1)\,, & n=1,2,\dots,\\ 1\,, & n=0.\end{cases}$$

Then, from (3.20) we easily obtain

$$\gamma^0_{n+2}=\frac{\gamma^0_1}{d+1}\binom{n+d+1}{d}\frac{\big(n+(d+1)\lambda+1\big)\big(\lambda+1\big)_n}{\big(\lambda+d+1\big)_{n+1}}\,,\ n\geq 0 \tag{3.28}$$

and

$$\tilde{\gamma}^0_{n+1}=\frac{\gamma^0_1}{d+1}\binom{n+d}{d}\frac{\big(n+(d+1)\lambda+d+1\big)\big(\lambda+1\big)_n}{\big(\lambda+d+1\big)_{n+1}}\,,\ n\geq 0. \tag{3.29}$$

Now, if we put

$$\gamma=\frac{\gamma^0_1}{(d+1)!}\frac{\Gamma(\lambda+d+1)}{\Gamma(\lambda+1)}, \tag{3.30}$$

the above identities become, respectively,

$$\gamma^0_{n+1}=\gamma\frac{(n+d)!\,\big(n+(d+1)\lambda\big)\,\Gamma(\lambda+n)}{n!\,\Gamma(\lambda+d+n+1)}\,,\ n\geq 0 \tag{3.31}$$

and

$$\tilde{\gamma}^0_{n+1}=\gamma\frac{(n+d)!\,\big(n+(d+1)(\lambda+1)\big)\,\Gamma(\lambda+n+1)}{n!\,\Gamma(\lambda+d+n+2)}\,,\ n\geq 0. \tag{3.32}$$

From these, by taking into account of the dependence on the parameter λ and putting

$$P_n(x;d)=P_n(x;d,\lambda)\,,\ n\geq 0,$$

we have $Q_n(x;d,\lambda)=P_n(x;d,\lambda+1)\,,\ n\geq 0$, that is,

$$P'_{n+1}(x;d,\lambda)=(n+1)P_n(x;d,\lambda+1)\,,\ n\geq 0. \tag{3.33}$$

Furthermore, the relation (3.10) leads to

$$P_{n+d+1}(x;d,\lambda)=\frac{1}{n+d+2}P'_{n+d+2}(x;d,\lambda)-\frac{\sigma_n}{n+1}P'_{n+1}(x;\lambda),\ n\geq 0, \tag{3.34}$$

or

$$P_{n+d+1}(x;d,\lambda)=P_{n+d+1}(x;d,\lambda+1)-\sigma_nP_n(x;d,\lambda+1),\ n\geq 0, \tag{3.35}$$

where

$$\sigma_n=d\gamma\frac{(n+d+1)!\,\Gamma(\lambda+n+1)}{n!\,\Gamma(\lambda+d+n+2)}\,,\ n\geq 0. \tag{3.36}$$

Note that, these polynomials were first studied by Humbert [14] and later generalized by Gould [11]. When $\lambda=0$ and $\lambda=1$, we obtain, repectively, the d-orthogonal Tchebychev polynomials studied in [7,8].
When $d=1\,,\gamma=1/4$ and the parameter $\lambda\to\alpha+1/2$, we obtain again the Gegenbauer polynomials.

4 Some properties of the polynomials given in the two examples

4.1 Explicit representation for the polynomials $P_n(.;d)$

We have

$$\begin{aligned}\sum_{n\geq 0} c_n P_n(x;d)t^n &= G\Big[(d+1)xt - t^{d+1}\Big]\\ &= \sum_{n\geq 0} a_n\Big((d+1)xt - t^{d+1}\Big)^n\\ &= \sum_{n\geq 0} a_n \sum_{p=0}^{n}(-1)^p \binom{n}{p}\Big((d+1)xt\Big)^{n-p} t^{(d+1)p}\\ &= \sum_{n\geq 0}\sum_{p=0}^{[\frac{n}{d+1}]}(-1)^p a_{n-dp}\binom{n-dp}{p}\Big((d+1)x\Big)^{n-(d+1)p} t^n,\end{aligned}$$

where $[\frac{n}{d+1}]$ denotes the integer part of $\frac{n}{d+1}$.
Now, comparing the coefficients of t^n, we get

$$P_n(x;d) = \frac{1}{c_n}\sum_{p=0}^{[\frac{n}{d+1}]}(-1)^p a_{n-dp}\binom{n-dp}{p}\Big((d+1)x\Big)^{n-(d+1)p}. \tag{4.1}$$

Recall that $c_n = (d+1)^n\, a_n\,,\ n \geq 0$.
Note that for two special values of a_n the resulting polynomials are known:

- for $G[z] = e^z$, then $a_n = 1/n!$, we have the Gould-Hopper polynomials;
- for $G[z] = (1-z)^{-\alpha}$, $\alpha \notin \mathbb{N}$, then $a_n = (\alpha)_n/n!$, we obtain the Humbert polynomials.

4.2 Zeros of the polynomials $P_n(.;d)$

From the recurrence relation (1.13), we easily state the following properties:

P1- $(d+1)$ consecutive polynomials do not vanish simultaneously.

Also, using the recurrence relation (1.13) to construct, for $n > d$, the following system:

$$\begin{cases} -xP_{r-1}(x;d) + P_r(x;d) = 0, & \text{for } 1 \leq r \leq d;\\ \gamma^0_{r-d}P_{r-d-1}(x;d) - xP_{r-1}(x;d) + P_r(x;d) = 0, & \text{for } d+1 \leq r \leq n.\end{cases} \tag{4.2}$$

If x_0 is a zero of $P_n(x;d)$, then the above system can be viewed as an homogeneous linear system which unknowns are $P_r(x_0;d)\,;\ 0 \leq r \leq n-1$. The determinant of this system is null since, otherwise, $P_0(x_0;d) = 0$.
Thus, it is readily verified that the system (4.2) can be also written in the matrix equation as

$$x\boldsymbol{P} = \mathbb{A}\boldsymbol{P}, \tag{4.3}$$

where

$$\boldsymbol{P} = {}^t\Big(P_0,\ P_1,\ P_2,\ \ldots P_{n-1}\Big),$$

and $\mathbb{A} = \Big(a_{ij}\Big)$ is the n-square matrix given by

$$a_{ij} = \begin{cases} 1\,, & \text{for} \quad 1 \le i \le n-1\,, \quad j = i+1; \\ \gamma_j^0\,, & \text{for} \quad 1 \le j \le n-1\,, \quad i = d+j; \\ 0\,, & \text{otherwise.} \end{cases} \tag{4.4}$$

This leads us to state:

P2 - The zeros of $P_n(x;d)\,,\ n > d$, are the eigenvalues of the matrix $\mathbb{A}$.

Numereous papers in the literature dealt with the properties of the eingenvalues of a square matrix (see, e.g., [9,17]). As a consequence of the Gerchgorin's Theorem [17, p.51], for instance, we can locate the zeros of $P_n(x;d)$ in the disk $\mathcal{D}(0,\rho_n)$ where $\rho_n = \sup\limits_{1\le j\le n-d} \Big(1 + |\gamma_j^0|\Big)$.

Other properties of the zeros of $P_n(x;d)$ may be deduced from the explicit expression (4.1) $\Big($or from the d-symmetry property$\Big)$:

P3 - If x_0 is a zero of $P_n(x;d)$, then $\xi_k x_0\,,\ k = 0,1,\ldots,d$, are also zeros of $P_n(x;d)$.

P4 - It is easy verified that 0 is a zero of $P_{(d+1)n+k}(x;d)$ of multiplicity k.

4.3 A $(d+1)$-order differential equation

Theorem 4.1 *Let $\{P_n(.;d)\}_{n\ge0}$ be the polynomials sequence generated by (1.2). If $\{P_n(.;d)\}_{n\ge0}$ is d-OPS, then the polynomial $P_n(.;d)\,,\ n = 0,1,\ldots,$ satisfy the following $(d+1)$-order differential equation:*

$$\Bigg[D^{d+1} - \frac{c_n}{c_{n-d}}\Big(xD - n\Big)\prod_{j=0}^{d-1}\Big(A_{n-1-j}\Big(xD - n + d + 1\Big) + n - j\Big)\Bigg]y = 0\,,\ n > d\,, \tag{4.5}$$

where $y = P_n(x;d)\,,\ D = d/dx$ and

$$A_n = 1 - \frac{c_n}{c_{n-d}}\gamma_{n-d+1}^0 = \frac{(n+1)\Big(\vartheta_{n-d} - 1\Big)}{(n+1)\Big(\vartheta_{n-d} - 1\Big) + 1}\ ,\ n > d. \tag{4.6}$$

Proof. From (3.1) we have

$$c_n[xD - n]P_n(x;d) - c_{n-d}DP_{n-d}(x;d) = 0\,,\quad n > d, \tag{4.7}$$

$$[xD - n]P_n(x;d) = 0\,,\quad 0 \le n \le d. \tag{4.8}$$

The relation (4.7) can be written also

$$DP_{n-d}(x;d) = \frac{c_n}{c_{n-d}}[xD - n]P_n(x;d). \tag{4.9}$$

Otherwise, by shifting $n \to n - d - 1$ in (3.7) we obtain

$$DP_{n+1}(x;d) - [xD + 1]P_n(x;d) = -\gamma_{n-d+1}^0 DP_{n-d}(x;d). \tag{4.10}$$

To substitute (4.9) into (4.10) yields

$$DP_{n+1}(x;d) = \Big[A_n\Big(xD-n\Big)+n+1\Big]P_n(x;d), \tag{4.11}$$

with A_n defined by (4.6).
Now let r be a positive integer. Differentiate (4.11) r times and use the identity $D^r(xD)=(xD+r)D^r$, to obtain

$$D^{r+1}P_{n+1}(x;d) = \Big[A_n\Big(xD-n+r\Big)+n+1\Big]D^rP_n(x;d). \tag{4.12}$$

Now, replace in the last, (r,n) by $(d-j, n-1-j)$; $0\le j\le d-1$, to construct the following system:

$$D^{d+1-j}P_{n-j}(x;d) = \Big[A_{n-1-j}\Big(xD-n+d+1\Big)+n-j\Big]D^{d-j}P_{n-1-j}(x;d)\ ,\ 0\le j\le d-1. \tag{4.13}$$

From which we deduce a relation linking $P_n(x;d)$ and $P_{n-d}(x;d)$, that is

$$D^{d+1}P_n(x;d) = \Big[\prod_{j=0}^{d-1}\Big(A_{n-1-j}\Big(xD-n+d+1\Big)+n-j\Big)\Big]DP_{n-d}(x;d). \tag{4.14}$$

Finally, substitute (4.9) into (4.14) to obtain (4.5), the desired result. ■

Applications.

We conclude this section by giving the differential equations satisfied by the polynomials obtained in the two above examples.

- For Example 1: we have, from (3.9) and (3.23), that

$$\frac{c_n}{c_{n+d}} = \gamma^0_{n+1}\ ,\ n\ge 0,$$

and then, from (4.6) we have $A_n=0\,,\ n\ge 0$.
Thus, taking into account of (4.8), the differential equation (4.5) becomes

$$\Big[D^{d+1}-\frac{d!}{\gamma^0_1}\Big(xD-n\Big)\Big]y=0\,,\ n\ge 0,$$

that is

$$y^{(d+1)}-\frac{d!}{\gamma^0_1}xy'+\frac{d!}{\gamma^0_1}ny=0\,,\ n\ge 0, \tag{4.15}$$

- For Example 2: we have, from (3.9), (3.31) and (3.32), that

$$\frac{c_n}{c_{n+d}} = \frac{(d+1)(n+d+\lambda)}{n+(d+1)\lambda}\gamma_{n+1}^0 = \frac{\gamma(d+1)(n+d)!\Gamma(\lambda+n)}{n!\Gamma(\lambda+d+n)}\,,\ n \geq 0$$

and then, from (4.6) we have $A_n = \dfrac{d(n+1)}{(d+1)(n+\lambda)}\,,\ n \geq 0.$
Thus, taking into account of (4.8), the differential equation (4.5) becomes

$$\Big[D^{d+1} - \gamma^{-1}(d+1)^{-d-1}\Big(xD-n\Big)\prod_{j=0}^{d-1}\Big(d\Big(xD-n\Big)+(d+1)\Big(\lambda+d+n-1-j\Big)\Big)\Big]y = 0\,,$$

$$n \geq 0. \quad (4.16)$$

Remarks 4.1

(a) The differential equation (4.5) is of type (4.6) considered in [1] and generalizes the ones given in [7,8] for $d=2$.

(b) The differential equation (4.15) is a particular case of the one given in [5], which reduces, for $d=1$ and $\gamma_1^0 = 1/2$, to the well-known second-order differential equation satisfied by the Hermite polynomials:

$$y'' - 2xy' + 2ny = 0,$$

where $y = H_n(x)$.

(c) For $d=1$, $\gamma = 1/4$ and $\lambda \to \alpha + 1/2$, in (4.16), we obtain the second-order differential equation satisfied by the Gegenbauer polynomials:

$$(1-x^2)y'' - 2(\alpha+1)xy' + n(n+2\alpha+1)y = 0,$$

where $y = P_n^{(\alpha,\alpha)}(x)$.

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Faculté des Sciences de Monastir
Département de Mathématiques
5019 Monastir,TUNISIE

Université Pierre et Marie Curie
Laboratoire d'Analyse Numérique, Tour 55-65 -5ème étage
4, place Jussieu, 75252 PARIS Cedex 05, FRANCE
e-mail: douak@ann.jussieu.fr