On quenching of solutions for some semilinear parabolic equations of second order

Théodore K. Boni

1 Introduction

Let Ω be a bounded domain in R^n with boundary $\partial\Omega$ of class C^2. Consider the following boundary value problems:

$$(I) \qquad \frac{\partial u}{\partial t} = Lu + f(u) \quad \text{in} \quad \Omega \times (0,T), \tag{1.1}$$

$$\mu\frac{\partial u}{\partial N} + (1-\mu)u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_o(x) \quad \text{in} \quad \Omega, \tag{1.3}$$

$$\frac{\partial u}{\partial t} = Lu \quad \text{in} \quad \Omega \times (0,T), \tag{1.4}$$

$$(II) \qquad \frac{\partial u}{\partial N} = g(u) \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.5}$$

$$u(x,0) = u_*(x) \quad \text{in} \quad \Omega, \tag{1.6}$$

where

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}), \qquad \frac{\partial u}{\partial N} = \sum_{i,j=1}^{n} \cos(\nu, x_i)a_{ij}(x)\frac{\partial u}{\partial x_j}.$$

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Here the coefficients $a_{ij}(x) \in C^1(\overline{\Omega})$ satisfy the inequalities

$$\lambda_2|\xi|^2 \geq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda_1|\xi|^2$$

for any $\xi \in R^n$ and $x \in \overline{\Omega}$ with positive constants λ_i $(i = 1, 2)$, ν is the exterior normal unit vector on $\partial\Omega$, $\mu \in [0, 1[$ is a function of class C^1 on $\partial\Omega$. For positive values of u, $f(u)$, $g(u)$ are positive and increasing functions with

$$f(0) > 0, \qquad \lim_{u\to b^-} f(u) = \infty,$$

$$g(0) > 0, \qquad \lim_{u\to b^-} g(u) = \infty,$$

where b is a positive number. $u_o(x)$ and $u_*(x)$ are two nonnegative functions of class $C^1(\Omega)$ such that

$$M = \sup_{x\in\Omega} u_o(x) < b, \qquad M' = \sup_{x\in\Omega} u_*(x) < b,$$

$\mu\frac{\partial u_o}{\partial N} + (1-\mu)u_o = 0$ on $\partial\Omega$ and $\frac{\partial u_*}{\partial N} = g(u_*)$ on $\partial\Omega$. In this note, we study the phenomenon of quenching for the problems $(1.1) - (1.3)$ and $(1.4) - (1.6)$.

Definition 1.1. *We say that the solution u of the problem* $(1.1)-(1.3)$ *or* $(1.4)-(1.6)$ *quenches in a finite time if there exists a finite time T_o such that*

$$\lim_{t\to T_o} \sup_{x\in\Omega} u(x,t) = b.$$

T_o is the quenching time of the solution u. $x \in \overline{\Omega}$ is a quenching point of the solution u if there exists a sequence (x_n, t_n) such that $x_n \to x$, $t_n \uparrow T_o$ and $\lim_{n\to\infty} u(x_n, t_n) = b$. The set

$$E_Q = \{x \in \overline{\Omega} \quad \textit{such that} \quad x \textit{ is a quenching point of the solution } u\}$$

is the quenching set of the solution u.

The problem of quenching has been the subject of study of many authors (see, for instance[1,3,4,6,7,8,9,10] and others). In particular in [1], the authors have considered the problem $(1.1) - (1.3)$ in the case where $\mu = 0$. They have shown that if Ω is small enough, then the solution of the problem $(1.1) - (1.3)$ exists in $\Omega \times (0, \infty)$ whereas if Ω is large enough, the solution quenches in a finite time. In this paper, we give other characterizations of quenching for the problem $(1.1) - (1.3)$ based on the nature of certains stationary solutions. These characterizations will be used to obtain the existence and nonexistence of the solution for the problem $(1.1) - (1.3)$ in the case where Ω is unbounded. Moreover, using some isoperimetric inequalities, we also precise some results of Acker and Walter in [1]. Another subject of investigation of the phenomenon of quenching is the quenching set. For the problem $(1.1) - (1.3)$, some results about quenching set have been given in [4]. More precisely, it is proved that under some conditions, the solution of the problem $(1.1) - (1.3)$ in the case where $\mu = 0$ quenches in a finite time and its quenching set is in a compact subset of Ω. For the problem $(1.4) - (1.6)$, we show that under some hypotheses the solution of $(1.4)-(1.6)$ quenches in a finite time and its quenching set is on the boundary $\partial\Omega$ of the domain Ω. The paper is written in the following manner. In Section 2, we obtain the local existence of the solution for the problem $(1.1)-(1.3)$. In Section 3, we characterize the quenching and global existence of the solution for the problem $(1.1)-(1.3)$ in terms of a certain stationary solution. In Section 4, we apply the results of Section 3 to study the existence and nonexistence of the solution for the problem $(1.1)-(1.3)$ in the case where Ω is unbounded. In Section 5, we show that the existence of the solution for the problem $(1.1)-(1.3)$ depends on the existence of a certain stationary solution of this problem. In Section 6, we get other quenching conditions of the solution for the problem $(1.1)-(1.3)$. We also give the asymptotic behavior near the quenching time of this solution. In Sections 7 and 8, we obtain some conditions under which the solution of the problem $(1.4)-(1.6)$ quenches in a finite time and estimate the quenching time of this solution. We also describe its quenching set.

2 Local existence

In this section, we show that for small T, the solution of the problem $(1.1)-(1.3)$ exists in $\Omega\times(0,T)$.

Theorem 2.1. *There exists a finite time T such that the solution u of the problem* $(1.1)-(1.3)$ *exists in $\Omega\times(0,T)$.*

Theorem 2.2. *If the solution u of the problem* $(1.1)-(1.3)$ *exists in $\Omega\times(0,T)$ with*

$$\sup_{(x,t)\in\Omega\times(0,T)} u(x,t)<b,$$

then there exists $T'>T$ such that u exists in $\Omega\times(0,T')$.

Proof of Theorem 2.1. Let $U(x,y,t)$ defined on $\overline{\Omega}\times\overline{\Omega}\times(0,\infty)$, be the fundamental solution of the equation

$$\frac{\partial v}{\partial t}-Lv=0 \quad \text{in} \quad \Omega\times(0,\infty)$$

with the boundary condition

$$\mu\frac{\partial v}{\partial N}+(1-\mu)v=0 \quad \text{on} \quad \partial\Omega\times(0,\infty).$$

It is well known that

$$U(x,y,t)>0 \quad \text{in} \quad \Omega\times(0,\infty), \qquad \int_{\Omega}U(x,y,t)dy\le 1 \tag{2.1}$$

and u is the solution of the problem $(1.1)-(1.3)$ if and only if

$$u(x,t)=\int_{\Omega}U(x,y,t)u(y,0)dy$$

$$+\int_0^t\int_{\Omega}f(u(y,\tau))U(x,y,t-\tau)dyd\tau \quad \text{in} \quad \Omega\times(0,T). \tag{2.2}$$

Put

$$u_1(x,t)=0,$$

$$u_{n+1}(x,t)=\int_\Omega U(x,y,t)u(y,0)dy+\int_0^t\int_\Omega f(u_n(y,\tau))U(x,y,t-\tau)dyd\tau.$$

Since f is increasing and $U>0$, it follows that $u_n>0$ for all $n>1$. Also by recurrence, we easily show that $u_{n+1}\geq u_n$ in $\overline{\Omega\times(0,T)}$. Let δ be a positive number. Suppose that $u_o(x)\leq b-2\delta$ and $u_n\leq b-\delta$, then $u_{n+1}\leq b-\delta$ also provided T is so small that

$$(b-2\delta)+f(b-\delta)\int_0^T\int_\Omega U(x,y,t-\tau)dyd\tau\leq b-\delta,$$

that is to say T is so small that

$$\int_0^T\int_\Omega U(x,y,t-\tau)dyd\tau\leq\frac{\delta}{f(b-\delta)}. \tag{2.3}$$

Since

$$\lim_{t\to0}\int_0^t\int_\Omega U(x,y,t-\tau)dyd\tau=0,$$

take T so small that (2.3) be satisfied. Thus the sequence $(u_n)_{n\geq1}$ is an increasing sequence of continuous functions defined in $\Omega\times(0,T)$ and bounded above by $b-\delta$. By the monotone convergence theorem, $\lim_{n\to\infty}u_n=u$ exists in $\Omega\times(0,T)$ and satisfies the following equality

$$u(x,t)=\int_\Omega U(x,y,t)u(y,0)dy$$

$$+\int_0^t\int_\Omega f(u(y,\tau))U(x,y,t-\tau)dyd\tau \quad \text{in} \quad \Omega\times(0,T).$$

Then we have the result. ∎

Remark 2.3. Changing slightly the proof of Theorem 2.1, we easily prove Theorem 2.2.

3 Sufficient conditions of quenching and global existence

In this section, we characterize the quenching and global existence of the solution for the problem (1.1) – (1.3) in terms of the stationary solution described in the following proposition:

Proposition 3.1.
There exists a unique w solution of the following problem:

$$Lw+1=0 \quad in \quad \Omega,$$

$$\mu\frac{\partial w}{\partial N}+(1-\mu)w=0 \quad on \quad \partial\Omega.$$

Proof. It is a well known result (see, for instance [5]).

Theorem 3.2. *Let w_o be the maximum of the solution for the following boundary value problem:*

$$Lw + 1 = 0 \quad in \quad \Omega,$$

$$\mu \frac{\partial w}{\partial N} + (1-\mu)w = 0 \quad on \quad \partial\Omega.$$

(α) If $w_o > \int_0^b \frac{ds}{f(s)}$, then the solution u of the problem (1.1) – (1.3) quenches in a finite time.
(β) If $\sup_{0<s<b-M} \frac{s}{f(s+M)} \geq w_o$, then the solution u of the problem (1.1) – (1.3) exists in $\Omega \times (0,\infty)$ and

$$\sup_{(x,t)\in\Omega\times(0,\infty)} u(x,t) \leq s(M) + M < b$$

where $M = \sup_{x\in\Omega} u_o(x)$ and

$$s(M) = \inf\{s \in (0, b-M) \quad such\ that \quad [\sup_{0<s<b-M} \frac{s}{f(s+M)}]f(s+M) = s\}.$$

Proof.
(α) Assume at first that $u_o(x) = 0$. Let $(0, T_{max})$ be the maximum time interval in which the classical solution u of the problem (1.1) – (1.3) exists. From the maximum principle, $u(x,t) \geq 0$ in $\Omega \times (0, T_{max})$. Put

$$v(x,t) = F(u(x,t)) = \int_0^u \frac{ds}{f(s)}. \tag{3.1}$$

We obtain

$$\frac{\partial v}{\partial t} - Lv = \frac{1}{f(u)}(u_t - Lu) + [\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}u_{x_j}]\frac{f'(u)}{f^2(u)}. \tag{3.2}$$

Since $f(u)$ is an increasing function, from (1.1) we have

$$\frac{\partial v}{\partial t} - Lv - 1 \geq 0 \quad \text{in} \quad \Omega \times (0, T_{max}) \tag{3.3}$$

and

$$v(x,t) = \int_0^u \frac{ds}{f(s)} \geq \frac{u}{f(u)}. \tag{3.4}$$

From (3.4) and (1.2), we also have

$$\mu\frac{\partial v}{\partial N} = \frac{1}{f(u)}\mu\frac{\partial u}{\partial N} = \frac{-(1-\mu)u}{f(u)} \geq -(1-\mu)v, \tag{3.5}$$

that is to say

$$\mu\frac{\partial v}{\partial N} + (1-\mu)v \geq 0 \quad \text{on} \quad \partial\Omega \times (0, T_{max}). \tag{3.6}$$

Since $w_o > \int_0^b \frac{ds}{f(s)}$ and $u(x,t) \leq b$ in $\Omega \times (0, T_{max})$, from (3.1) it follows that

$$\sup_{(x,t)\in\Omega\times(0,T_{max})} v(x,t) < w_o. \tag{3.7}$$

Let z be the solution of the following problem:

$$\frac{\partial z}{\partial t} = Lz + 1 \quad \text{in} \quad \Omega \times (0, \infty), \tag{3.8}$$

$$\mu \frac{\partial z}{\partial N} + (1-\mu) z = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty), \tag{3.9}$$

$$z(x, 0) = 0 \quad \text{in} \quad \Omega. \tag{3.10}$$

¿From the maximum principle, we deduce that

$$v(x,t) \geq z(x,t) \quad \text{in} \quad \Omega \times (0, T_{max}). \tag{3.11}$$

We also have

$$\lim_{t\to\infty} z(x,t) = w(x). \tag{3.12}$$

Therefore from (3.7) and (3.12), there exist $x_o \in \Omega$ and a finite t_o such that

$$z(x_o, t_o) > \sup_{(x,t)\in\Omega\times(0,T_{max})} v(x,t), \tag{3.13}$$

which implies that $t_o \geq T_{max}$. In fact, suppose that $t_o < T_{max}$. From (3.11), we have $v(x_o, t_o) \geq z(x_o, t_o)$ which contradicts (3.13). Consequently, T_{max} is finite and u quenches in a finite time.
Now, suppose that $u_o(x) \geq 0$. From the maximum principle

$$u(x,t) \geq u_1(x,t) \quad \text{in} \quad \Omega \times (0, T_1) \tag{3.14}$$

where u_1 is the solution of the problem $(1.1) - (1.2)$ with $u_1(x, 0) = 0$ in Ω and $(0, T_1)$ is the maximum time interval in which the solutions u and u_1 exist. From the above result, we know that u_1 quenches in a finite time because

$$w_o > \int_0^b \frac{ds}{f(s)}. \tag{3.15}$$

Therefore, from (3.14), u also quenches in a finite time which yields the result.

(β) Assume at first that $u_o(x) = 0$. Then $M = 0$. Put $s(M) = s_o$ and show that for any h satisfying the following problem

$$Lh + f(s_o) = 0 \quad \text{in} \quad \Omega, \tag{3.16}$$

$$\mu \frac{\partial h}{\partial N} + (1-\mu) h = 0 \quad \text{on} \quad \partial\Omega, \tag{3.17}$$

we have $h \leq s_o$. In fact put $k(x) = f(s_o)w(x) - h(x)$. We obtain

$$Lk(x) = -f(s_o) - Lh(x) = 0, \tag{3.18}$$

$$\mu \frac{\partial k(x)}{\partial N} + (1-\mu) k(x) = 0. \tag{3.19}$$

From the maximum principle, we deduce that

$$k(x) = f(s_o)w(x) - h(x) \geq 0 \quad \text{in} \quad \Omega,$$

that is to say

$$h(x) \leq f(s_o)w(x) \leq f(s_o)w_o \leq s_o. \tag{3.20}$$

By Theorem 2.1, there exists a time T_2 such that u exists in $\Omega \times (0, T_2)$. Put $z(x,t) = h(x) - u(x,t)$. From the maximum principle, $h(x) \geq 0$ in Ω. It follows that

$$z(x, 0) \geq 0 \quad \text{in} \quad \Omega, \tag{3.21}$$

because $u_o(x) = 0$ in Ω. Since f is an increasing function, from (3.20), we also have

$$z_t - Lz = f(s_o) - f(u(x,t)) \geq f(h) - f(u) = f^{'}(\xi)z \quad \text{in} \quad \Omega \times (0, T_2) \tag{3.22}$$

where $\xi = (1-\theta)h + \theta u < b$ with $0 \leq \theta \leq 1$. Finally we have

$$\mu \frac{\partial z}{\partial N} + (1-\mu)z = 0 \quad \text{on} \quad \partial\Omega \times (0, T_2). \tag{3.23}$$

From the maximum principle, we obtain $h(x) \geq u$ in $\Omega \times (0, T_2)$. Consequently

$$u(x,t) \leq s_o < b \quad \text{in} \quad \Omega \times (0, T_2). \tag{3.24}$$

Owing to Theorem 2.2, there exists $T_2^{'} > T_2$ such that the solution u of (1.1) – (1.3) exists in $\Omega \times (0, T_2^{'})$. Reasoning as above, we have $u(x,t) \leq s_o < b$ in $\Omega \times (0, T_2^{'})$. Iterating this process, we obtain $u(x,t) \leq s_o < b$ in $\Omega \times (0, \infty)$.
Now, suppose that $u_o(x) \geq 0$ and let w_1 be the solution of the following problem:

$$\frac{\partial w_1}{\partial t} = Lw_1 + f(w_1) \quad \text{in} \quad \Omega \times (0, T), \tag{3.25}$$

$$\mu \frac{\partial w_1}{\partial N} + (1-\mu)w_1 = (1-\mu)M \quad \text{on} \quad \partial\Omega \times (0, T), \tag{3.26}$$

$$w_1(x, 0) = M \quad \text{in} \quad \Omega. \tag{3.27}$$

Put $v_1(x,t) = w_1(x,t) - M$. We have

$$\frac{\partial v_1}{\partial t} = Lv_1 + f_1(v_1) \quad \text{in} \quad \Omega \times (0, T), \tag{3.28}$$

$$\mu \frac{\partial v_1}{\partial N} + (1-\mu)v_1 = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{3.29}$$

$$v_1(x, 0) = 0 \quad \text{in} \quad \Omega, \tag{3.30}$$

where $f_1(v_1) = f(v_1 + M)$. We obtain $f_1(0) = f(M) > 0$ and $\lim_{t \to b-M} f_1(t) = \infty$. From the above result, we know that $v_1(x,t)$ exists in $\Omega \times (0, \infty)$ and $v_1(x,t) \leq s(M)$ in $\Omega \times (0, \infty)$ because

$$w_o \leq \sup_{0<s<b-M} \frac{s}{f_1(s)} = \sup_{0<s<b-M} \frac{s}{f(s+M)}. \tag{3.31}$$

This implies that $w_1(x,t)$ exists in $\Omega \times (0, \infty)$. Therefore from (3.25) – (3.27), u exists in $\Omega \times (0, T_{max})$ and

$$u(x,t) \leq w_1(x,t) \leq s(M) + M < b \quad \text{in} \quad \Omega \times (0, T_{max}), \tag{3.32}$$

where $(0, T_{max})$ is the maximum time interval in which u exists. Consequently from (3.32) and Theorem 2.2, we deduce that $T_{max} = \infty$, which yields the result. ■

Remark 3.3. If $f(s) = (b-s)^{-p}$ with $p > 0$, we have

$$\int_0^b \frac{ds}{f(s)} = \frac{b^{p+1}}{p+1}, \qquad \sup_{0<s<b-M} \frac{s}{f(s+M)} = \frac{(b-M)^{p+1}p^p}{(p+1)^{p+1}} \quad \text{and} \quad s(M) = \frac{b-M}{p+1}.$$

Corollary 3.4. *Suppose that $L = \Delta$ and Ω contains a domain Ω_* with piecewise analytic boundary. For $x \in \Omega_*$, denote its harmonic radius by $R_x(\Omega_*)$. If*

$$\sup_{x\in\Omega_*} R_x^2(\Omega_*) > 2n \int_0^b \frac{ds}{f(s)},$$

then the solution u of the problem (1.1) – (1.3) *quenches in a finite time. If $f(s) = (b-s)^{-p}$, then the result holds when*

$$\sup_{x\in\Omega_*} R_x^2(\Omega_*) > \frac{2nb^{p+1}}{p+1}.$$

Proof. Let v be the solution of the following problem:

$$\frac{\partial v}{\partial t} = \Delta v + f(v) \quad \text{in} \quad \Omega_* \times (0,T), \tag{3.33}$$

$$v = 0 \quad \text{on} \quad \partial\Omega_* \times (0,T), \tag{3.34}$$

$$v(x,0) = u_o(x) \quad \text{in} \quad \Omega_*. \tag{3.35}$$

¿From the maximum principle $u \geq v$ in $\Omega_* \times (0, T_{max})$ where $(0, T_{max})$ is the maximum time interval in which the solutions u and v exist. Let w be the solution of the following problem:

$$\Delta w + 1 = 0 \quad \text{in} \quad \Omega_*, \quad w = 0 \quad \text{on} \quad \partial\Omega_*.$$

From the results in ([2], Theorem 2.9, p.70), $w(x)$ satisfies the inequality

$$w(x) \geq \frac{R_x^2(\Omega_*)}{2n}.$$

By Theorem 3.2 (α), the solution v quenches in a finite time because

$$w_o = \sup_{x\in\Omega_*} w(x) \geq \frac{\sup_{x\in\Omega_*} R_x^2(\Omega_*)}{2n} > \int_0^b \frac{ds}{f(s)}.$$

This implies that u also quenches in a finite time and we have the result. The case where $f(s) = (b-s)^{-p}$ is a direct consequence of Remark 3.3. ∎

Remark 3.5. Let Ω_* be a bounded domain in R^n with piecewise analytic boundary. For $x \in \Omega_*$, denote its harmonic radius by $R_x(\Omega_*)$. Then we have $R_x(\Omega_*) \geq \text{dist}(x, \partial\Omega_*)$ (see, for instance[2]).

Corollary 3.6. *Suppose that Ω contains a ball B of radius R and let $L = \Delta$. Then the solution u of* (1.1) – (1.3) *quenches in a finite time if*

$$R^2 > 2n \int_0^b \frac{ds}{f(s)}.$$

If $f(s) = (b-s)^{-p}$, *then the result holds when*

$$R^2 > \frac{2nb^{p+1}}{p+1}.$$

Proof. For $x \in B$, let $R_x(B)$ be the harmonic radius of the ball B. by Remark 3.5, we get $\sup\limits_{x\in B} R_x^2(B) \geq R^2$. The rest of the proof is a direct consequence of Corollary 3.4. ■

Corollary 3.7. *Let* $L = \Delta$. *Suppose that*

$$|\Omega| \leq \left(2n \sup_{0<s<b-M} \frac{s}{f(s+M)}\right)^{\frac{n}{2}} \omega_n,$$

where ω_n *denote the volume of the unit sphere in* R^n. *Then the solution* u *of the problem* (1.1) – (1.3) *with* $\mu = 0$ *exists in* $\Omega \times (0,\infty)$ *and*

$$u(x,t) \leq s(M) + M \quad in \quad \Omega \times (0,\infty),$$

where $M = \sup\limits_{x\in\Omega} u_o(x)$ *and*

$$s(M) = \inf\{s \in (0, b-M) \quad such\ that \quad [\sup_{0<s<b-M} \frac{s}{f(s+M)}]f(s+M) = s\}.$$

If $f(s) = (b-s)^{-p}$, *then the result holds when*

$$|\Omega| \leq \left(2n \frac{(b-M)^{p+1}p^p}{(p+1)^{p+1}}\right)^{\frac{n}{2}} \omega_n.$$

Proof. From the results in ([2]), we know that

$$w(x) \leq \frac{1}{2n}\left(\frac{|\Omega|}{\omega_n}\right)^{\frac{2}{n}}.$$

Then by Theorem 3.2 (β), we obtain the result. ■

Corollary 3.8. *Let* $L = \Delta$. *Suppose that* $\Omega \subset\subset (0,l) \times D$ *where* $D \subset R^{n-1}$ *is a bounded domain and* $(0,l) \subset R^1$. *Suppose also that*

$$l \leq \sqrt{8 \sup_{0<s<b-M} \frac{s}{f(s+M)}}.$$

Then the solution u *of the problem* (1.1) – (1.3) *with* $\mu = 0$ *exists in* $\Omega \times (0,\infty)$ *and*

$$u(x,t) \leq s(M) + M \quad in \quad \Omega \times (0,\infty),$$

where $M = \sup\limits_{x\in\Omega} u_o(x)$ *and*

$$s(M) = \inf\{s \in (0, b-M) \quad such\ that \quad [\sup_{0<s<b-M} \frac{s}{f(s+M)}]f(s+M) = s\}.$$

Proof. Since D is a bounded domain, there exist numbers l_i $(i = 2, ..., n)$ such that $\Omega \subset\subset (0, l) \times \prod_{i=2}^{n}[0, l_i] = I$. Let $\psi(x_1, x^{'})$ be a function defined in I by

$$\psi(x_1, x^{'}) = \frac{1}{2}x_1(l - x_1),$$

with $x_1 \in (0, l)$ and $x^{'} \in \prod_{i=2}^{n}[0, l_i]$. We obtain

$$\Delta\psi(x_1, x^{'}) + 1 = 0 \quad \text{in} \quad I, \quad \psi(x_1, x^{'}) \geq 0 \quad \text{on} \quad \partial I.$$

Since $\psi(x_1, x^{'}) > 0$ in $\overline{\Omega}$, from the maximum principle, $\psi \geq w$ in Ω, where $w(x)$ is the solution of the following problem

$$\Delta w + 1 = 0 \quad \text{in} \quad \Omega, \quad w = 0 \quad \text{on} \quad \partial\Omega.$$

Since $||\psi||_{L^\infty(I)} = \frac{l^2}{8}$, we also have $w_o = ||w||_{L^\infty(\Omega)} \leq \frac{l^2}{8}$. Then, by Theorem 3.2 (β), u exists in $\Omega \times (0, \infty)$ and

$$u(x, t) \leq s(M) + M \quad \text{in} \quad \Omega \times (0, \infty)$$

because

$$w_o = ||w||_{L^\infty(\Omega)} \leq \frac{l^2}{8} \leq \sup_{0<s<b-M} \frac{s}{f(s + M)}.$$

Therefore, we obtain the result. ∎

4 Application

In this section, we are interested in the existence and nonexistence of the solution for the problem (1.1) – (1.3) in the case where $\Omega = R^m \times \Omega_o$ with $0 \leq m < n$ and $\Omega_o \subset R^{n-m}$ is a bounded domain. Putting $x = (x_m, y)$, we suppose that the coefficients $a_{ij}(x) = a_{ij}(x_m, y)$ and $\mu(x) = \mu(x_m, y)$ are invariant under the translation of x_m for $x_m \in R^m$.

Theorem 4.1. *Let w_{Ω_o} be the maximum of the solution for the following boundary value problem:*

$$L\psi + 1 = 0 \quad \text{in} \quad \Omega_o,$$

$$\mu\frac{\partial\psi}{\partial N} + (1 - \mu)\psi = 0 \quad \text{on} \quad \partial\Omega_o.$$

(α) If $w_{\Omega_o} > \int_0^b \frac{ds}{f(s)}$, then the solution u of the problem (1.1) – (1.3) quenches in a finite time.
(β) If $\sup_{0<s<b-M} \frac{s}{f(s + M)} \geq w_{\Omega_o}$, then the solution u of the problem (1.1) – (1.3) exists in $\Omega \times (0, \infty)$ and

$$\sup_{(x,t)\in\Omega\times(0,\infty)} u(x, t) \leq s(M) + M < b$$

where $M = \sup_{x\in\Omega} u_o(x)$ *and*

$$s(M) = \inf\{s \in (0, b-M) \quad \text{such that} \quad [\sup_{0<s<b-M} \frac{s}{f(s+M)}]f(s+M) = s\}.$$

In the proof of Theorem 4.1, the following lemma will be used.

Lemma 4.2. *Suppose that* $\Omega = R^m \times \Omega_o$*, where* $\Omega_o \subset R^{n-m}$ *is a bounded domain. Then, the problem* (1.1) $-$ (1.3) *has at most one nonnegative classical solution.*

Proof. Let u_1 and u_2 be two nonnegative classical solutions of the problem (1.1) $-$ (1.3). Put $w_2 = u_1 - u_2$. We obtain

$$\frac{\partial w_2}{\partial t} = Lw_2 + f'(\xi)w_2 \quad \text{in} \quad \Omega \times (0,T),$$

$$\mu\frac{\partial w_2}{\partial N} + (1-\mu)w_2 = 0 \quad \text{on} \quad \partial\Omega \times (0,T),$$

$$w_2(x,0) = 0 \quad \text{in} \quad \Omega,$$

where $\xi = (1-\theta)u_1 + \theta u_2$ with $\theta \in [0,1]$. We also have

$$0 \le |w_2(x,t)| < b \quad \text{in} \quad \Omega \times (0,T),$$

because for $i \in \{1,2\}$, $0 \le u_i(x,t) < b$ in $\Omega \times (0,T)$. Since $f'(\xi)$ is bounded for $t < T$, the result follows from the Phragmèn-Lindelöf principle (see, for instance [12]). ■

Proof of Theorem 4.1.

(α) Consider the following problem:

$$\frac{\partial v}{\partial t} = Lv + f(v) \quad \text{in} \quad \Omega \times (0,T), \tag{4.1}$$

$$\mu\frac{\partial v}{\partial N} + (1-\mu)v = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{4.2}$$

$$v(x,0) = 0 \quad \text{in} \quad \Omega. \tag{4.3}$$

Put $x = (x_m, y)$ where $x_m \in R^m$ and $y \in \Omega_o$. Let $v(x,t) = v(x_m, y, t)$ be a nonnegative classical solution of the problem (4.1) $-$ (4.3). Since the operators $\frac{\partial}{\partial t} - L$, $\frac{\partial}{\partial N}$, the domain Ω and the function μ are invariant under the translation of x_m, for any $h \in R^m$, $v_1(x,t) = v(x_m + h, y, t)$ is also a nonnegative solution of the problem (4.1)$-$(4.3). From the uniqueness of the solution, we have $v_1(x,t) \equiv v(x,t)$. Therefore $v(x,t) = v(x_m, y, t)$ depends only on y and t. This implies that the problem (4.1) $-$ (4.3) can be reduced to the following form:

$$\frac{\partial v}{\partial t} = L_{n-m}v + f(v) \quad \text{in} \quad \Omega_o \times (0,T), \tag{4.4}$$

$$\mu\frac{\partial v}{\partial N_{n-m}} + (1-\mu)v = 0 \quad \text{on} \quad \partial\Omega_o \times (0,T), \tag{4.5}$$

$$v(x,0) = 0 \quad \text{in} \quad \Omega_o, \tag{4.6}$$

where

$$L_{n-m}v = \sum_{i,j=m+1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial v}{\partial x_j}), \qquad \frac{\partial v}{\partial N_{n-m}} = \sum_{i,j=m+1}^{n} \cos(\nu, x_i)a_{ij}(x)\frac{\partial v}{\partial x_j}.$$

By Theorem 3.2 (α), we know that v quenches in a finite time because

$$w_{\Omega_o} > \int_0^b \frac{ds}{f(s)}. \tag{4.7}$$

From the maximum principle, we have

$$u(x,t) \geq v(x,t) \quad \text{in} \quad R^m \times \Omega_o \times (0, T_{max})$$

where $(0, T_{max})$ is the maximum time interval in which u and v exist. This implies that u also quenches in a finite time.
(β) Now consider the following problem:

$$\frac{\partial w}{\partial t} = Lw + f(w) \quad \text{in} \quad \Omega \times (0,T), \tag{4.8}$$

$$\mu\frac{\partial w}{\partial N} + (1-\mu)w = (1-\mu)M \quad \text{on} \quad \partial\Omega \times (0,T), \tag{4.9}$$

$$w(x,0) = M \quad \text{in} \quad \Omega, \tag{4.10}$$

where $M = \sup_{x\in\Omega} u_o(x) < b$. Put $w_*(x,t) = w(x,t) - M$. We obtain

$$\frac{\partial w_*}{\partial t} = Lw_* + f_*(w_*) \quad \text{in} \quad \Omega \times (0,T), \tag{4.11}$$

$$\mu\frac{\partial w_*}{\partial N} + (1-\mu)w_* = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{4.12}$$

$$w_*(x,0) = 0 \quad \text{in} \quad \Omega, \tag{4.13}$$

where $f_*(w_*) = f(w_* + M)$. We also have $f_*(0) = f(M) > 0$ and $\lim_{t\to b-M} f_*(t) = \infty$. As above, we know that $w_*(x,t) = w_*(x_m, y, t)$ depends only on y and t. Moreover $(4.11) - (4.13)$ may be written in the following form

$$\frac{\partial w_*}{\partial t} = L_{n-m}w_* + f_*(w_*) \quad \text{in} \quad \Omega_o \times (0,T), \tag{4.14}$$

$$\mu\frac{\partial w_*}{\partial N_{n-m}} + (1-\mu)w_* = 0 \quad \text{on} \quad \partial\Omega_o \times (0,T), \tag{4.15}$$

$$w_*(x,0) = 0 \quad \text{in} \quad \Omega_o. \tag{4.16}$$

Therefore by Theorem 3.2 (β), $w_*(x,t)$ exists in $R^m \times \Omega_o \times (0,\infty)$ and is bounded above by $s(M) \in]0, b-M[$ because

$$w_{\Omega_o} \leq \sup_{0<s<b-M} \frac{s}{f_*(s)} = \sup_{0<s<b-M} \frac{s}{f(s+M)}. \tag{4.17}$$

Consequently $w(x,t)$ exists in $R^m \times \Omega_o \times (0,\infty)$ and is bounded above by $s(M) + M \in]0,b[$. Therefore from $(4.8) - (4.10)$ and the maximum principle, u exists in $R^m \times \Omega_o \times (0, T_{max})$ and

$$u(x,t) \leq w(x,t) \leq s(M) + M \quad \text{in} \quad R^m \times \Omega_o \times (0, T_{max}) \tag{4.18}$$

where $(0, T_{max})$ is the maximum time interval in which the solution u exists. Since $s(M) + M < b$, from (4.18) and Theorem 2.2, $T_{max} = \infty$ and we have the result. ■

Corollary 4.3. *Let $L = \Delta$ and suppose that $\Omega = R^m \times \Omega_o$ where $0 \le m \le n$ and $\Omega_o \subset R^{n-m}$ is a bounded domain.*
(i) First case: If $m = n$, then the solution u of the problem (1.1) – (1.3) quenches in a finite time.
(ii) Second case: If $0 \le m < n$ and Ω_o contains a domain Ω_* with piecewise analytic boundary such that

$$\sup_{x\in\Omega_*} R_x^2(\Omega_*) > 2n \int_0^b \frac{ds}{f(s)},$$

then the solution u of the problem (1.1) – (1.3) quenches in a finite time.
If $\mu = 0$ and $\Omega_o \subset\subset (0,l) \times D_o$ with

$$l \le \sqrt{8 \sup_{0<s<b-M} \frac{s}{f(s+M)}},$$

then the solution u of the problem (1.1) – (1.3) exists in $\Omega \times (0,\infty)$ and

$$\sup_{(x,t)\in\Omega\times(0,\infty)} u(x,t) \le s(M) + M < b$$

where $M = \sup_{x\in\Omega} u_o(x)$ and

$$s(M) = \inf\{s \in (0, b-M) \quad \text{such that} \quad [\sup_{0<s<b-M} \frac{s}{f(s+M)}]f(s+M) = s\}.$$

Proof (*i*) $n = m$. The proof is an easy consequence of Corollary 3.6. In fact, since the Green's function of the heat equation is positive, we have $u \ge 0$ in $R^n \times (0,T)$. Let B be a ball of radius R such that

$$R^2 > 2n \int_0^b \frac{ds}{f(s)}.$$

From the maximum principle, we have $u \ge v$ in $B \times (0, T_{max})$ where v is a solution of the problem (1.1) – (1.2) with $v(x,0) = 0$ in the case where $\Omega = B$, $\mu = 0$ and $(0, T_{max})$ is the maximum time interval in which the solutions u and v exist. Then, by Corollary 3.6, v quenches in a finite time because

$$R^2 > 2n \int_0^b \frac{ds}{f(s)}.$$

This implies that u also quenches in a finite time, which yields the result.
(ii) Let v be the solution of the following problem:

$$\frac{\partial v}{\partial t} = \Delta v + f(v) \quad \text{in} \quad \Omega_p \times (0,T), \tag{4.19}$$

$$v = 0 \quad \text{on} \quad \partial\Omega_p \times (0,T), \tag{4.20}$$

$$v(x,0) = u_o(x) \quad \text{in} \quad \Omega_p, \tag{4.21}$$

where $\Omega_p = \Omega_* \times (0,T)$. ¿From the maximum principle $u \ge v$ in $R^m \times \Omega_* \times (0, T_{max})$ where $(0, T_{max})$ is the maximum time interval in which the solutions u and v exist. Let w_{Ω_*} be the maximum of the solution for the following problem:

$$\Delta w + 1 = 0 \quad \text{in} \quad \Omega_*, \qquad w = 0 \quad \text{on} \quad \partial\Omega_*.$$

From the results in ([2], Theorem 2.9, p.70), $w(x)$ satisfies the inequality

$$w(x) \geq \frac{R_x^2(\Omega_*)}{2n}.$$

By Theorem 4.1 (α), the solution v quenches in a finite time because

$$w_{\Omega_*} = \sup_{x\in\Omega_*} w(x) \geq \frac{\sup_{x\in\Omega_*} R_x^2(\Omega_*)}{2n} > \int_0^b \frac{ds}{f(s)}.$$

This implies that u also quenches in a finite time and we obtain the first result. Now let w_{Ω_o} be the maximum of the solution for the following problem:

$$\Delta w + 1 = 0 \quad \text{in} \quad \Omega_o, \quad w = 0 \quad \text{on} \quad \partial\Omega_o.$$

To prove the second part of our theorem, by Theorem 4.1 (β), it is sufficient to show that

$$w_{\Omega_o} = ||w||_{L^\infty(\Omega_o)} \leq \frac{l^2}{8} \leq \sup_{0<s<b-M} \frac{s}{f(s+M)}.$$

But, this follows from the proof of Corollary 3.8. ∎

5 Another characterization of quenching

In this section, we show that the global existence of the solution for the problem (1.1)−(1.3) depends on the existence of a certain stationary solution of this problem.

Theorem 5.1.
Consider the following problem:

$$Lv + f(v) = 0 \quad \text{in} \quad \Omega, \tag{P1}$$

$$\mu\frac{\partial v}{\partial N} + (1-\mu)v = 0 \quad \text{on} \quad \partial\Omega. \tag{P2}$$

First case: If the solution v of the problem $(P1)-(P2)$ exists with $v_o = \sup_{x\in\overline{\Omega}} v(x) < b$, then the solution u of the problem $(1.1)-(1.3)$ exists in $\Omega \times (0,\infty)$ for $u_o(x) \leq v(x)$ in Ω. Moreover

$$u(x,t) \leq v_o \quad \text{in} \quad \Omega \times (0,\infty).$$

Second case: If the solution v of the problem $(P1)-(P2)$ does not exist, then the solution u of the problem $(1.1)-(1.3)$ quenches in a finite time.
The following lemma will be used in the proof of Theorem 5.1.
Lemma 5.2. *Suppose that $l(s)$ is a bounded and increasing function in $(0,\infty)$. Then we have*

$$\lim_{t\to\infty} l'(t) = 0.$$

Proof. We get

$$\int_0^t l'(s)ds = l(t) - l(0) \leq C < \infty.$$

It follows that $\int_0^\infty l'(s)ds < \infty$, which leads to the result. ∎

Proof of Theorem 5.1.

First case:

Since u is the solution of the problem (1.1) − (1.3), owing to Theorem 2.1, there is a finite time T such that $u(x,t)$ exists in $\Omega \times (0,T)$. From the maximum principle, it follows that $u(x,t) \leq v(x) < b$ in $\Omega \times (0,T)$ because $u_o(x) \leq v(x)$. By Theorem 2.2, there exists $T^{'} > T$ such that $u(x,t)$ exists in $\Omega \times (0,T^{'})$. Reasoning as above, we have $u(x,t) \leq v_o < b$ in $\Omega \times (0,T^{'})$. Iterating this process, we obtain the result.

Second case:

Suppose that $\sup_{x\in\Omega} u(x,t) < b$ for all $t \geq 0$. Assume at first that $u(x,0) = 0$. Let $G(x,y)$ be the Green's function of $-L$ with the following boundary condition :

$$\mu \frac{\partial G(x,y)}{\partial N_x} + (1-\mu)G(x,y) = 0.$$

Put

$$w(x,t) = \int_\Omega G(x,y)u(y,t)dy. \tag{5.1}$$

We obtain

$$w_t(x,t) = \int_\Omega G(x,y)u_t(y,t)dy. \tag{5.2}$$

From (1.1) and (5.2), we also have

$$w_t(x,t) = -u(x,t) + \int_\Omega G(x,y)f(u(y,t))dy. \tag{5.3}$$

From the maximum principle

$$u_t \geq 0 \quad \text{in} \quad \Omega \times (0,T) \tag{5.4}$$

because $Lu(x,0) + f(u(x,0)) \geq 0$. Therefore

$$\lim_{t\to\infty} u(x,t) := z(x) \tag{5.5}$$

exists because u is a bounded and increasing function. Consequently, from (5.3), (5.4), (5.5) and the monotone convergence theorem, we have

$$\lim_{t\to\infty} w_t(x,t) = -z(x) + \int_\Omega G(x,y)f(z(y))dy. \tag{5.6}$$

Since

$$G(x,y) \geq 0 \quad \text{and} \quad \sup_{x\in\Omega} \int_\Omega G(x,y)dy < \infty,$$

from (5.1), (5.2) and (5.4), it follows that $w_t(x,t) \geq 0$ and w is bounded. Then Lemma 5.2 implies that $\lim_{t\to\infty} w_t(x,t) = 0$ for all $x \in \Omega$. Therefore from (5.6), we obtain

$$z(x) = \int_\Omega G(x,y)f(z(y))dy.$$

Consequently we have

$$Lz + f(z) = 0 \quad \text{in} \quad \Omega,$$

$$\mu\frac{\partial z}{\partial N} + (1-\mu)z = 0 \quad \text{on} \quad \partial\Omega,$$

which is a contradiction to our hypothesis. Therefore, u quenches in a finite time. Now suppose that $u(x,0) = u_o(x) \geq 0$. From the maximum principle,

$$u(x,t) \geq v(x,t) \quad \text{in} \quad \Omega \times (0, T_{max}) \tag{5.7}$$

where v is the solution of the problem $(1.1) - (1.2)$ with $v(x,0) = 0$ and $(0, T_{max})$ is the maximum time interval in which u and v exist. From the above result, we know that v quenches in a finite time. Then from (5.7), u also quenches in a finite time, which yields the result. ∎

6 Asymptotic behavior near the quenching time

In this section, we obtain some conditions under which the solution of the problem $(1.1) - (1.3)$ quenches in a finite time and we describe the asymptotic behavior of this solution near its quenching time.

Theorem 6.1.
Suppose that the function $f(s)$ is positive, increasing, convex for positive values of s, $Lu_o(x) + f(u_o(x)) \geq 0$ in Ω and $\int_0^b \frac{ds}{f(s)} < \infty$. Finally suppose that there is a constant $A < b$ close to b such that

$$sf^{'}(s) \geq f(s) \quad for \quad s \geq A.$$

Then the solution u of the problem $(1.1) - (1.3)$ quenches in a finite time T and there exist two constants c_1 and c_2 such that

$$H_f(c_2(T-t)) \leq \sup_{x \in \overline{\Omega}} u(x,t) \leq H_f(c_1(T-t))$$

where $H_f(s)$ is the inverse function of $F(s) = \int_s^b \frac{ds}{f(s)}$.

Corollary 6.2. *Suppose that $f(u) = (b-u)^{-p}$ with $p > 0$ and $Lu_o(x)+f(u_o(x)) \geq 0$ in Ω. Then the solution u of the problem $(1.1) - (1.3)$ quenches in a finite time T and there exist two constants C_1 and C_2 such that*

$$b - C_2(T-t)^{\frac{1}{p+1}} \leq \sup_{x \in \overline{\Omega}} u(x,t) \leq b - C_1(T-t)^{\frac{1}{p+1}}.$$

Proof of Theorem 6.1.
Let $(0,T)$ be the maximum time interval in which the solution u of the problem $(1.1) - (1.3)$ exists. Since $u_o(x) \geq 0$, from the maximum principle, we have $u \geq 0$ in $\Omega \times (0,T)$. Put $w = \frac{\partial u}{\partial t}$. We obtain

$$\frac{\partial w}{\partial t} = Lw + f^{'}(u)w \quad \text{in} \quad \Omega \times (0,T), \tag{6.1}$$

$$\mu \frac{\partial w}{\partial N} + (1-\mu)w = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{6.2}$$

$$w(x,0) \geq 0 \quad \text{in} \quad \Omega. \tag{6.3}$$

From the maximum principle, it follows that

$$\frac{\partial u}{\partial t} \geq c > 0 \quad \text{in} \quad \Omega \times (\varepsilon_o, T) \tag{6.4}$$

for $\varepsilon_o > 0$. Put $J(x,t) = u_t - \delta f(u)$ where δ is a constant which will be determined later. We have

$$\frac{\partial J}{\partial t} - LJ = \frac{\partial}{\partial t}(u_t - Lu) - \delta f'(u)(u_t - Lu) + f''(u)[\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j}]$$

$$\geq f'(u)u_t - \delta f'(u)f(u) = f'(u)J \quad \text{in} \quad \Omega \times (0,T)$$

because f is convex and $u_t - Lu = f(u)$. We also have

$$J(x, \varepsilon_o) = u_t(x, \varepsilon_o) - \delta f(u(x, \varepsilon_o)) \quad \text{in} \quad \Omega.$$

From (6.4), choose $\delta < \dfrac{c}{f(A)}$ small enough that

$$J(x, \varepsilon_o) > 0 \quad \text{in} \quad \Omega. \tag{6.5}$$

Show that $J(x,t) \geq 0$ in $\overline{\Omega} \times (\varepsilon_o, T)$. In fact suppose that J admits a negative minimum in (x_o, t_o) in $\overline{\Omega} \times (\varepsilon_o, T)$. From the maximum principle, $(x_o, t_o) \in \partial\Omega \times (\varepsilon_o, T)$.
If $u(x_o, t_o) < A < b$, from (6.4), we have

$$J(x_o, t_o) = u_t(x_o, t_o) - \delta f(u(x_o, t_o)) \geq c - \delta f(A)$$

because f is an increasing function. Since $\delta < \dfrac{c}{f(A)}$, we obtain $J(x_o, t_o) > 0$ which is a contradiction.

If $u(x_o, t_o) > A$, then we have $\mu \dfrac{\partial J}{\partial N}(x_o, t_o) + (1 - \mu) J(x_o, t_o) < 0$, which implies that

$$u(x_o, t_o) f'(u(x_o, t_o)) < f(u(x_o, t_o)).$$

Therefore, we have again a contradiction because by hypothesis $uf'(u) \geq f(u)$ for $u \geq A$. We deduce that $u_t(x,t) \geq f(u)$ in $\Omega \times (\varepsilon_o, T)$ that is

$$-(F(u))_t \geq \delta. \tag{6.6}$$

Integrating (6.6) over (ε_o, T), it follows that

$$\infty > F(u(x, \varepsilon_o)) \geq F(u(x, \varepsilon_o)) - F(u(x,T)) \geq \delta(T - \varepsilon_o). \tag{6.7}$$

Therefore, T is finite and u quenches in a finite time. Integrating again (6.6) over (t, T), we have

$$F(u(x,t)) \geq F(u(x,t)) - F(u(x,T)) \geq \delta(T - t). \tag{6.8}$$

Since H_f is a decreasing function, from (6.8) we obtain

$$\sup_{x\in\overline{\Omega}} u(x,t) \leq H_f(\delta(T-t)).$$

Now put $U(t) = \sup_{x\in\overline{\Omega}} u(x,t)$. Since $\overline{\Omega}$ is compact, there exist $x_i \in \overline{\Omega}$ $(i=1,2)$ such that $U(t_i) = u(x_i,t_i)$ for $t_i \geq 0$. Let $h = t_2 - t_1$. We also have

$$U(t_2) - U(t_1) \leq u(x_2,t_2) - u(x_2,t_1) = hu_t(x_2,t_2) + 0(h).$$

Consequently

$$\frac{U(t_2)-U(t_1)}{t_2-t_1} \leq u_t(x_2,t_2) + 0(1). \tag{6.9}$$

Since $Lu(x_2,t_2) \leq 0$, we obtain

$$u_t(x_2,t_2) \leq f(u(x_2,t_2)) = f(U(t_2)). \tag{6.10}$$

In the fact that

$$\lim_{t_1\to t_2} \frac{U(t_2)-U(t_1)}{t_2-t_1} = U^{'}(t_2),$$

from (6.9) and (6.10), we deduce that $U^{'}(t_2) \leq f(U(t_2))$. Therefore

$$\sup_{x\in\overline{\Omega}} u(x,t) \geq H_f(T-t),$$

which gives the result. ■

7 Quenching time.

In this section, we give some conditions under which the solution of the problem (1.4) – (1.6) quenches in a finite time and estimate its quenching time.

Theorem 7.1. *Suppose that* $\int_0^b \frac{ds}{g(s)} < \infty$ *and for positive values of* s*,* $g(s)$ *is positive and increasing. Then the solution* u *of the problem* (1.4) – (1.6) *quenches in a finite time* T *and*

$$T \leq \frac{|\Omega|}{|\partial\Omega|}\int_m^b \frac{ds}{g(s)},$$

where $m = \inf_{x\in\Omega} u_*(x)$.

Proof. Let $(0,T)$ be the maximum time interval in which the solution u of the problem (1.4) – (1.6) exists. Our aim is to show that T is finite and satisfies the above inequality. Since $u_*(x) \geq 0$ in Ω, from the maximum principle $u(x,t) \geq 0$ in $\Omega \times (0,T)$. Multiplying (1.4) by $\frac{1}{g(u)}$, we have after integration over Ω

$$-\frac{d}{dt}\int_\Omega G(u(x,t))dx = \int_{\partial\Omega} ds + \int_\Omega \frac{g^{'}(u)}{g^2(u)} \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx, \tag{7.1}$$

where $G(s) = \int_s^b \frac{dz}{g(z)}$. Since g is an increasing function, from (7.1) we obtain

$$-\frac{d}{dt}\int_\Omega G(u(x,t))dx \geq |\partial\Omega|. \tag{7.2}$$

Integrating (7.2) over $(0,T)$ and using the fact that $G(u(x,T)) \geq 0$, we have

$$\infty > |\Omega| \int_m^b \frac{ds}{g(s)} \geq \int_\Omega \int_{u_*(x)}^b \frac{ds}{g(s)} dx \geq T|\partial\Omega|. \tag{7.3}$$

Then the solution u of the problem (1.4) – (1.6) quenches in a finite time T and we obtain the result. ∎

8 Quenching set.

In this section, we describe the quenching set of the solution for the problem (1.4) – (1.6). More precisely, we show that under some conditions, the solution of the problem (1.4) – (1.6) quenches in a finite time and its quenching set is on the boundary $\partial\Omega$ of the domain Ω.

Theorem 8.1.
Suppose that $\int_0^b \frac{dz}{g(z)} < +\infty$, $Lu_*(x) \geq 0$ *and for positive values of* s, $g(s)$ *is positive, increasing and convex. Then the solution* u *of the problem* (1.4) – (1.6) *quenches in a finite time* T *and there is a constant* $\delta > 0$ *such that the following estimate holds:*

$$\sup_{x\in\overline{\Omega}} u(x,t) \leq H_g(\delta(T-t))$$

where $H_g(s)$ *is the inverse function of* $G(s) = \int_s^b \frac{dz}{g(z)}$.

Theorem 8.2. *Suppose that the hypotheses of Theorem 8.1 are satisfied. Suppose also that there is a positive constant* C_o *such that*

$$sg^{'}(H_g(s)) \leq C_o \quad \text{for} \quad s > 0.$$

Then the solution u *of the problem* (1.4) – (1.6) *quenches in a finite time* T *and* $E_Q \subset \partial\Omega$, *where* E_Q *is the quenching set of the solution* u.

Remark 8.3. If $g(s) = (b-s)^{-p}$, then we may take $C_o = \frac{p}{p+1}$.

Proof of Theorem 8.1. Let $(0,T)$ be the maximum time interval in which the solution u of the problem (1.4) – (1.6) exists and put $w = u_t$. Since $Lu_*(x) \geq 0$, we have

$$\frac{\partial w}{\partial t} - Lw = 0 \quad \text{in} \quad \Omega \times (0,T),$$

$$\frac{\partial w}{\partial N} = g^{'}(u)w \quad \text{on} \quad \partial\Omega \times (0,T),$$

$$w(x,0) \geq 0 \quad \text{in} \quad \Omega.$$

¿From the maximum principle

$$w(x,t) \geq c > 0 \quad \text{in} \quad \Omega \times (\varepsilon_o, T), \tag{8.1}$$

for $\varepsilon_o > 0$. Consider the following function: $J(x,t) = u_t - \delta g(u)$. From (8.1), take δ small enough that

$$J(x,\varepsilon_o) = u_t(x,\varepsilon_o) - \delta g(u(x,\varepsilon_o)) > 0. \tag{8.2}$$

We also have

$$\frac{\partial J}{\partial N} = g^{'}(u)(u_t - \delta g(u)) = g^{'}(u)J \quad \text{on} \quad \partial\Omega \times (\varepsilon_o, T). \tag{8.3}$$

Finally we have

$$\frac{\partial J}{\partial t} - LJ = \frac{\partial}{\partial t}(u_t - Lu) - \delta g^{'}(u)(u_t - Lu) + g^{''}(u)[\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j}].$$

Since g is a convex function, from (1.4) we obtain

$$J_t - LJ \geq 0 \quad \text{in} \quad \Omega \times (\varepsilon_o, T). \tag{8.4}$$

From the maximum principle, we deduce that

$$J(x,t) \geq 0 \quad \text{in} \quad \Omega \times (\varepsilon_o, T),$$

that is to say

$$u_t \geq \delta g(u) \quad \text{in} \quad \Omega \times (\varepsilon_o, T). \tag{8.5}$$

Since $\int_0^b \frac{dz}{g(z)} < +\infty$, then the function $G(s) = \int_s^b \frac{dz}{g(z)}$ is well defined. Therefore from (8.5), it follows that

$$-(G(u))_t \geq \delta \quad \text{in} \quad \Omega \times (\varepsilon_o, T). \tag{8.6}$$

Integrating (8.6) over (ε_o, T), we have

$$\infty > G(u(x,\varepsilon_o)) \geq G(u(x,\varepsilon_o)) - G(u(x,T)) \geq \delta(T - \varepsilon_o).$$

This implies that T is finite and u quenches in a finite time. On the other hand, integrating (8.6) over (t,T), we also have

$$G(u(x,t)) \geq G(u(x,t)) - G(u(x,T)) \geq \delta(T - t).$$

Since G is a decreasing function, so is H_g and we obtain

$$\sup_{x \in \overline{\Omega}} u(x,t) \leq H_g(\delta(T-t)).$$

Therefore, the theorem is proved. ■

Proof of Theorem 8.2.
By Theorem 8.1, we know that u quenches in a finite time T. Then our aim is to show that $E_Q \subset \partial\Omega$. Let $d(x) = \text{dist}(x, \partial\Omega)$ and $v(x) = d^2(x)$ for $x \in N_\varepsilon(\partial\Omega)$ where

$$N_\varepsilon(\partial\Omega) = \{x \in \Omega \quad \text{such that} \quad d(x) < \varepsilon\}.$$

Since $\partial\Omega$ is of class C^2, then the function $v(x) \in C^2(\overline{N_\varepsilon(\partial\Omega)})$ if ε is sufficiently small. On $\partial\Omega$, we have

$$Lv - \frac{C_o}{v}\sum_{i,j=1}^{n} a_{ij}(x) v_{x_i} v_{x_j}$$

$$= \sum_{i=1}^{n} a_{ii}(x) v_{x_i x_i} + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j}\right) v_{x_i} - \frac{C_o}{v}\sum_{i,j=1}^{n} a_{ij}(x) v_{x_i} v_{x_j}$$

$$= 2\sum_{i=1}^{n} a_{ii}(x) + 2d\sum_{i=1}^{n}\left(\sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j}\right) d_{x_i} - 4C_o\sum_{i,j=1}^{n} a_{ij}(x) d_{x_i} d_{x_j}$$

$$\geq -2\sum_{i=1}^{n} |a_{ii}(x)| - 2d' \sum_{i=1}^{n} \left|\sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j}\right| |\nabla d| - 4C_o\lambda_2 |\nabla d|^2$$

where $d' = \sup_{x\in\overline{\Omega}, y\in\overline{\Omega}} ||x-y||$. Therefore, there exists a positive constant C_1 such that

$$Lv - \frac{C_o}{v}\sum_{i,j=1}^{n} a_{ij}(x) v_{x_i} v_{x_j} \geq -C_1 \quad \text{on} \quad \partial\Omega.$$

Since $v \in C^2(\overline{N_\varepsilon(\partial\Omega)})$ for ε sufficiently small, let ε_o be so small that

$$Lv - \frac{C_o}{v}\sum_{i,j=1}^{n} a_{ij}(x) v_{x_i} v_{x_j} \geq -2C_1 \quad \text{in} \quad \overline{N_{\varepsilon_o}(\partial\Omega)}.$$

We extend v to a function on $\overline{\Omega}$ such that $v \in C^2(\overline{\Omega})$ and $v \geq C_o^* > 0$ in $\overline{\Omega - N_{\varepsilon_o}(\partial\Omega)}$. Then we have

$$Lv - \frac{C_o}{v}\sum_{i,j=1}^{n} a_{ij}(x) v_{x_i} v_{x_j} \geq -C^* \quad \text{in} \quad \overline{\Omega} \tag{8.7}$$

for some $C^* > 0$. Since $H_g(0) = b$, multiplying (8.7) by ϵ small enough, we may assume without loss of generality that C^* and v are sufficiently small so that

$$H_g(\delta(v(x) + C^*(T - \varepsilon_o))) > u(x, \varepsilon_o). \tag{8.8}$$

Put $w(x,t) = H_g(\tau)$ where $\tau = \delta(v(x) + C^*(T-t))$. From (8.8), we obtain

$$w(x, \varepsilon_o) > u(x, \varepsilon_o) \quad \text{in} \quad \Omega.$$

We also have

$$w_t - Lw = -\delta H_g'(\tau)\left[C^* + Lv + \delta\frac{H_g''(\tau)}{H_g'(\tau)}\sum_{i,j=1}^{n} a_{ij}(x) v_{x_i} v_{x_j}\right]. \tag{8.9}$$

Since $H_g(s)$ is the inverse function of $G(s)$, we have $H_g^{'}(s) = -g(H_g(s))$ and $H_g^{''}(s) = -H_g^{'}(s)g^{'}(H_g(s))$. Consequently

$$w_t - Lw = \delta g(H_g(s))[C^* + Lv - \delta g^{'}(H_g(\tau)) \sum_{i,j=1}^{n} a_{ij}(x)v_{x_i}v_{x_j}]. \tag{8.10}$$

Since $sg^{'}(H_g(s)) \leq C_o$ for $s > 0$, using the fact that $g^{'}(H_g(s))$ is a decreasing function ($g^{'}$ is increasing and H_g is decreasing), we have

$$w_t - lw \geq \delta g(H_g(\tau))[C^* + Lv - \frac{C_o}{v} \sum_{i,j=1}^{n} a_{ij}(x)v_{x_i}v_{x_j}]. \tag{8.11}$$

From (8.7) and (8.11), we deduce that

$$w_t - Lw \geq 0 \quad \text{in} \quad \Omega \times (\varepsilon_o, T). \tag{8.12}$$

We also have

$$w(x,t) = H_g(\delta C^*(T-t)) > H_g(\delta(T-t)) \quad \text{on} \quad \partial\Omega \times (\varepsilon_o, T) \tag{8.13}$$

because $C^* < 1$, which implies that

$$w(x,t) > u(x,t) \quad \text{on} \quad \partial\Omega \times (\varepsilon_o, T). \tag{8.14}$$

Consequently, from the maximum principle, it follows that

$$u(x,t) < w(x,t) \quad \text{in} \quad \Omega \times (\varepsilon_o, T). \tag{8.15}$$

Since H_g is decreasing, we obtain

$$u(x,t) \leq H_g(\delta(v(x) + C^*(T-t))) \leq H_g(\delta v(x)). \tag{8.16}$$

Then if $\Omega^{'} \subset\subset \Omega$, from (8.16) we have

$$\sup_{x\in\Omega^{'}, t\in[\varepsilon_o, T)} u(x,t) \leq \sup_{x\in\Omega^{'}} H_g(\delta v(x)) < b,$$

which yields the result. ∎

References

[1] Acker A.,Walter W., *The quenching problem for nonlinear parabolic equations,* Lecture Notes in Mathematics, **564**, Springer-Verlag, New York, 1976.

[2] Bandle C., *Isoperimetric inequalities and applications,* Pitman, London, 1980.

[3] Dai Q., Gu Y., *A short note on quenching Phenomena for semilinear Parabolic equations,* Jour. of Diff. Equat. 137 (1997), pp. 240-250.

[4] Deng K., Levine H. A., *On the blow up of u_t at quenching,* Proc. of Amer. Math. Soc. Vol. 106, 4 (1989), pp. 1049-1056.

[5] Gilbarg D., Trudinger N.S., *Elliptic partial differential equations of second order,* Springer-Verlag 1977.

[6] Kawarada H., *On solutions of initial boundary value problem for $u_t = u_{xx} + \frac{1}{1-u}$,* RIMS Kyoto Univ., **10** (1975), pp. $729 - 736$.

[7] Levine H.A., *The quenching of solutions of linear hyperbolic and parabolic with nonlinear boundary conditions,* SIAM J. Math. Anal., **14** (1983), pp. $1139 - 1153$.

[8] Levine H.A., Lieberman G.M.,*Quenching of solutions of parabolic equations with nonlinear boundary conditions in several dimensions,* J. Reine Ang. Math., **345** (1983), pp. $23 - 38$.

[9] Levine H.A., Montgomery J.T., *Quenching of solutions of some nonlinear parabolic problems,* SIAM J. Math. Anal., **11** (1980), pp. $842 - 847$.

[10] Levine H.A., *Quenching, nonquenching, and beyond quenching for solution of some parabolic equations,* Annali di Matematica pura ed applicata (IV), Vol. CLV (1989), pp. $243 - 260$.

[11] Matano H., *Asymptotic behavior and stability of solutions of semilinear diffusion equations.* Publ. R.I.M.S. Vol. 15, 2 (1979), pp. 401-454.

[12] Protter M.H., Weinberger H.F., *Maximum Principles in Differential Equations,* Prentice Hall, Englewood Cliffs, NJ, 1967.

[13] Walter W., *Differential- und Integral-Ungleichungen,* Springer, Berlin, 1964.

Université Paul Sabatier,
UFR-MIG, MIP, 118 route de Narbonne,
31062 Toulouse, France.
E-mail: boni@mip.ups-tlse.fr