Generalized reduction of the Poincaré differential equation to Cauchy matrix form

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Abstract

In this paper the Poincaré differential equation of order n with multiple regular singularities is reduced to the Cauchy matrix form.

1 Introduction

Using the transformation of H.L.Turrittin [1, p. 494] we will prove that the Poincaré differential equation of n-th order with multiple regular singularities, can be reduced to the Cauchy matrix form [2, p. 369]. In this paper the results obtained in [3] are generalized.

2 Generalized reduction

Now we will prove the following result.

Theorem. *The Poincaré differential equation*

$$P_n(x)y^{(n)} = \sum_{i=0}^{n-1} P_i(x)y^{(i)}, \tag{1}$$

where

$$P_n(x) = \prod_{i=1}^{k}(x - d_i)^{r_i}, \tag{2}$$

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$$\sum_{i=1}^{k} r_i = n, \qquad (1 \leq k \leq n),$$

$$1 \leq r_k \leq r_{k-1} \leq \cdots \leq r_1 \leq n,$$

reduces to the Cauchy matrix form

$$(xI - D)\frac{dY}{dx} = QY, \tag{3}$$

where

$$D = diag(d_1, \cdots, d_1, d_2, \cdots, d_2, \cdots, d_k, \cdots, d_k), \qquad (rankD \geq 1) \tag{4}$$

$$Q = \begin{bmatrix} Q_1 & 1 & 0 & \cdots & 0 \\ & Q_2 & 1 & \cdots & 0 \\ & & & & \cdot \\ & q_{ij} & & & \cdot \\ & & & & \cdot \\ & & & & Q_k \end{bmatrix} \tag{5}$$

and

$$Y = (y_1, y_2, \cdots, y_n)^T. \tag{6}$$

Proof. The regular singularities in the equality (1) are $x = d_i$, $(1 \leq i \leq k)$ and the following functions

$$(x - d_j)^i P_{n-i}(x)/P_n(x), \quad (1 \leq i \leq n)$$

are holomorphic for $x = d_j$, i.e. the polynomials $P_{n-i}(x)$ must contain the factor $(x - d_j)^{r_j - i}$, $(1 \leq i \leq r_j)$. Hence it follows

$$P_{n-i}(x) = P^*_{n-i}(x)\prod_{j=1}^{k}(x - d_j)^{r_j - i}, \quad (0 < i \leq r_k)$$

$$P_{n-i}(x) = P^*_{n-i}(x)\prod_{j=1}^{s-1}(x - d_j)^{r_j - i}, \quad (r_s < i \leq r_{s-1};\ k \geq s \geq 2) \tag{7}$$

$$P_{n-i}(x) = P^*_{n-i}(x), \quad (r_1 < i \leq n)$$

such that if $r_s < i \leq r_{s-1}$, $(1 \leq s \leq k+1; r_0 = n, r_{k+1} = 0)$ the polynomials $P^*_{n-i}(x)$ in the best case have degree

$$(n - i) - \sum_{i=1}^{s-1} r_i + i(s-1) = n - N_{s-1} + i(s-2),$$

where

$$N_s = \sum_{i=1}^{s} r_i, \quad (1 \leq s \leq k)$$

$$N_0 = 0, \quad N_k = n.$$

In the block matrix (5), each of the blocks Q_s, $(1 \le s \le k)$ has format $r_s \times r_s$ and its form is

$$Q_s = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \cdots & r_s - 2 & 1 \\ a_{s1} & a_{s2} & a_{s3} & \cdots & a_{s,r_s-1} & a_{s,r_s} + r_s - 1 \end{bmatrix}. \tag{8}$$

If we introduce the following substitutions

$$\begin{gathered} t_s^i = \prod_{j=1}^{n} (x - d_j)^{r_j - i}, \\ t_s^i = t_s^{i+1} \psi_s, \quad \psi_s = \prod_{j=1}^{s} (x - d_j), \\ (t_s^i)' = t_s^{i+1} p_s^i, \quad p_s^i = \sum_{j=1}^{s} (r_j - i) \prod_{m=1}^{s} (x - d_m), \\ (1 \le s \le k) \end{gathered} \tag{9}$$

then the equality (7) takes the form

$$\begin{gathered} P_{n-i}(x) = t_{s-1}^i P_{n-i}^*(x), \quad (r_s < i \le r_{s-1};\ k+1 \ge s \ge 2) \\ P_{n-i}(x) = P_{n-i}^*(x), \quad (r_1 < i \le n) \end{gathered} \tag{10}$$

and in the linear transformation of H.L.Turrittin [1]

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_i \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_{20}(x) & \varphi_1 & 0 & \cdots & 0 & 0 \\ c_{30}(x) & c_{31}(x) & \varphi_2 & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ c_{i0}(x) & c_{i1}(x) & c_{i2}(x) & \cdots & \varphi_{i-1} & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ c_{n0}(x) & c_{n1}(x) & c_{n2}(x) & \cdots & c_{n,n-2}(x) & \varphi_{n-1} \end{bmatrix} \cdot \begin{bmatrix} y \\ y' \\ y'' \\ \cdot \\ \cdot \\ \cdot \\ y^{(i-1)} \\ \cdot \\ \cdot \\ \cdot \\ y^{(n-1)} \end{bmatrix}, \tag{11}$$

where $deg c_{ij}(x) \le j$, it will be

$$\begin{gathered} \varphi_i = t_s^0 (x - d_s)^{i - N_s} = t_s^1 \psi_s (x - d_s)^{i - N_s} = t_s^1 \psi_{s-1} (x - d_s)^{i+1-N_s}, \\ \varphi_i' = t_s^1 (x - d_s)^{i - N_s} [(x - d_s) p_{k-1}^0 + (i - N_{s-1}) \psi_{s-1}], \end{gathered} \tag{12}$$

$$(N_{s-1} < i \le N_s; 1 \le s \le k).$$

Applying the previous substitutions, according to [3] q_{ii} and $c_{i,i-2}(x)$ can be calculated. First it determines

$$c_{n,n-2}(x) = (x-d_k)^{-1}[q_{nn}\varphi_{n-1} - P_{n-1}(x)] - \varphi'_{n-1} =$$
$$= (x-d_k)^{-1}[q_{nn}t_k^0(x-d_k)^{-1} - t_k^1 P^*_{n-1}(x)] - t_k^1(x-d_k)^{-1}[(x-d_k)p^0_{k-1} + (r_k-1)\psi_{k-1}] =$$
$$= t_k^1(x-d_k)^{-1}[(q_{nn} - r_k + 1)\psi_{k-1} - P^*_{n-1}(x) - (x-d_k)p^0_{k-1}].$$

If we substitute $x = d_k$ in the last equation we obtain

$$\begin{gathered} q_{nn} = r_{k-1} + P^*_{n-1}(d_k)/\psi_{k-1}(d_k), \\ c_{n,n-2}(x) = t_k^1 c^*_{n,n-2}(x), \end{gathered} \tag{13}$$

where $c^*_{n,n-2}$ is a polynomial of the form

$$c^*_{n,n-2}(x) = (x-d_k)^{-1}[(q_{nn} - r_k + 1)\psi_{k-1} - P^*_{n-1}(x) - (x-d_k)p^0_{k-1}]. \tag{14}$$

Now let be $N_{k-1} < i \le N_k - 1$. According to [3], by substituting

$$c_{i,i-2}(x) = t_k^1(x-d_k)^{i-N_k} c^*_{i,i-2}(x) \tag{15}$$

and by using of (9), we obtain

$$t_k^1(x-d_k)^{i-N_k} c^*_{i,i-2}(x) =$$
$$= t_k^1(x-d_k)^{i-N_k}[c^*_{i+1,i-1}(x) - p^0_{k-1}] + t_k^1(x-d_k)^{i-1-N_k}\psi_{k-1}[q_{ii} - (i - N_{k-1} - 1)],$$

and hence it follows that

$$\begin{gathered} q_{ii} = i - N_{k-1} - 1, \\ c^*_{i,i-2}(x) = c^*_{i+1,i-1}(x) - p^0_{k-1} = c^*_{n,n-2}(x) - (n-i)p^0_{k-1}, \end{gathered} \tag{16}$$

$$(N_{k-1} < i \le N_{k-1}).$$

For $i = N_{k-1}$, $c_{i+1,i-1}(x) = t^1_{k-1} c^*_{i+1,i-1}(x)$ from (15) and

$$\varphi_{i-1} = t^0_{k-1}(x-d_{k-1})^{-1} = t^1_{k-1}\psi_{k-1}(x-d_{k-1})^{-1} = t^1_{k-1}\psi_{k-2},$$
$$\varphi'_{i-1} = t^1_{k-1}(x-d_{k-1})^{-1}[(x-d_{k-1})p^0_{k-1} + (r_{k-1}-1)\psi_{k-2}],$$

according to (12), we obtain the equation

$$c_{i,i-2}(x) = t^1_{k-1}(x-d_{k-1})^{-1}\{c^*_{i+1,i-1}(x) + [q_{ii} - (r_{k-1}-1)]\psi_{k-2}\} - t^1_{k-1}p^0_{k-2},$$

which yields to

$$\begin{gathered} q_{ii} = r_{k-1} - 1 - c^*_{i+1,i-1}(d_{k-1})/\psi_{k-2}(d_{k-1}), \\ c_{i,i-2}(x) = t^1_{k-1}c^*_{i,i-2}(x), \end{gathered} \tag{17}$$

$$(i = N_{k-1})$$

where

$$c^*_{i,i-2}(x) = (x - d_k)^{-1}\{c^*_{i+1,i-1}(x) + [q_{ii} - (r_{k-1} - 1)]\psi_{k-2}\} - p^0_{k-2}, \tag{18}$$

$$(i = N_{k-1}).$$

Hence we can suppose that for $N_{s-1} < i \leq N_s$, $(k - 1 \geq s \geq 1)$ it holds

$$c_{i,i-2}(x) = t^1_s(x - d_s)^{i-N_s} c^*_{i,i-2}(x). \tag{19}$$

Indeed, for $N_{s-1} < i \leq N_s$, the equation

$$c_{i,i-2}(x) = (x - d_s)^{-1}[c_{i+1,i-1}(x) + q_{ii}\varphi_{i-1}] - \varphi'_{i-1},$$

can be reduced to the form

$$c^*_{i,i-2}(x) = c^*_{i+1,i-1}(x) - p^0_{s-1} + (x - d_s)^{-1}\psi_{s-1}[q_{ii} - (i - N_{s-1} - 1)].$$

Using the substitution

$$q_{ii} = i - N_{s-1} - 1, \tag{20}$$

can be determined the polynomial

$$c^*_{i,i-2}(x) = c^*_{i+1,i-1}(x) - p^0_{s-1} =$$

$$= c^*_{N_s,N_s-2}(x) - (N_s - i)p^0_{s-1}, \quad (N_{s-1} < i \leq N_s). \tag{21}$$

Since it is $c_{i+1,i-1}(x) = t^1_{s-1}c^*_{i+1,i-1}(x)$, for $i = N_{s-1}$ and

$$\varphi_{i-1} = t^1_{s-1}\psi_{s-2},$$

$$\varphi'_{i-1} = t^1_{s-1}(x - d_{s-1})^{-1}[(x - d_s)p^0_{k-2} + (r_s - 1)\psi_{s-2}],$$

we obtain

$$c_{i,i-2}(x) = t^1_{s-1}(x - d_{s-1})^{-1}[c^*_{i+1,i-1}(x) + (q_{ii} - r_{s-1} + 1)\psi_{s-2}] - t^1_{s-1}p^0_{s-2},$$

and hence we can determine

$$q_{ii} = r_{s-1} - 1 - c^*_{i+1,i-1}(d_{s-1})/\psi_{s-2}(d_{s-1}), \tag{22}$$

obtaining

$$c_{i,i-2}(x) = t^1_{s-1}c^*_{i,i-2}(x), \quad (i = N_{s-1}) \tag{23}$$

where

$$c^*_{i,i-2}(x) = (x - d_{s-1})^{-1}[c^*_{i+1,i-1}(x) + (q_{ii} - r_{s-1} + 1)\psi_{s-2}] - p^0_{s-2}. \tag{24}$$

Thus we determined the polynomials $c_{i,i-2}(x)$, $(n \geq i \geq 2)$ which have the form (19) and the constants q_{ii}, $(n \geq i \geq 2)$ together with $q_{11} = -c_{20}(x) = 0$, uniquely from $P_{n-1}(x)$.

Now we can see that the polynomials $c_{i,i-j}(x)$ can be expressed as

$$c_{i,i-j}(x) = t_s^{j-1}(x-d_s)^{i-N_s}c^*_{i,i-j}(x), \tag{25}$$

$$(N_{s-1} < i \le N_s; 1 \le s \le k),$$

where it understands that the factor $(x-d_j)$ up to potention of nonpositive integers is equal to 1, i.e.

$$(x-d_s)^{r_s-N_s+i+1-j} = 1, \quad (r_s - N_s + i + 1 \le j)$$

$$(x-d_j)^{r_j-j+1} = 1, \quad (r_j + 1 \le j).$$

For $j=2$, from the formulas (16) can be obtained the formulas (19), and hence for $i = n+1$ the formulas (10) correspond to the formulas (16), i.e.

$$c_{n+1,n+1-j}(x) = P_{n-j+1}(x) = t_k^{j-1}P^*_{n-j+1}(x).$$

The equality (25) will be proved by induction with respect to the subdiagonal row j. Now we will consider the rows $c_{i,i-j}(x)$ for $N_{s-1} < i \le N_s$. In this case it holds

$$(x-d_s)[c_{i,i-j-1}(x) + c'_{i,i-j}(x)] =$$

$$= c_{i+1,i-j}(x) + \sum_{\nu=0}^{j-2} q_{i,i-\nu}c_{i-\nu,i-j}(x) + q_{i,i-j+1}\varphi_{i-j}, \quad (N_{s-1} < i \le N_s). \tag{26}$$

Let us suppose that equations (22) hold for $i = N_s + 1$, then we can prove by induction of $j, (j = 2, 3, \cdots)$ that the $r_s \times r_s$ matrix $[q_{i,i-j}]$, $(0 \le j \le r_s - 1)$ is a joint matrix. Indeed, for $j = 2$ we have

$$(x-d_s)c_{i,i-3}(x) - c_{i+1,i-2}(x) =$$

$$= t_s^2(x-d_s)^{i-N_s+1}\{[(q_{ii}-i+N_s)\psi_{s-1}+p_s^1]c^*_{i,i-2}(x) - \psi_s c^{*'}_{i,i-2}(x)\} + q_{i,i-1}t_s^2(x-d_s)^{i-N_s}\psi^2_{s-1}. \tag{27}$$

From the assumption $c_{i+1,i-2}(x) = t_s^2 c^*_{i+1,i-2}(x)$ for $i = N_s$, we can substitute

$$q_{i,i-1} = -c^*_{i+1,i-2}(d_s)/\psi^2_{s-1}(d_s), \quad (i = N_s) \tag{28}$$

and we will prove that $c_{i,i-3}(x)$ can be determined in the form

$$c_{i,i-3}(x) = t_s^2 c^*_{i,i-3}(x), \quad (i = N_s). \tag{29}$$

Substituting the equation (29) in (27) for $i = N_s - 1$ we obtain $q_{i,i-1} = 0$ and the equation (25) for $i = N_s - 1$. The equation (27) for $N_s - 1 \ge i \ge N_{s-1} + 2$ reduces to

$$(x-d_s)[c^*_{i,i-3}(x) - c^*_{i+1,i-2}(x)] =$$

$$= (x-d_s)\{[(q_{ii} - i + N_s)\psi_{s-1} + p_k^1]c^*_{i,i-2}(x) - \psi_s c^{*'}_{i,i-2}(x)\} + q_{i,i-1}\psi^2_{s-1},$$

$$(i = N_s - 1, N_s - 2, \cdots, N_{s-1} + 2)$$

where it follows that

$$q_{i,i-1} = 0, \quad (i = N_s - 1, N_s - 2, \cdots, N_{s-1} + 2). \tag{30}$$

For $i = N_{s-1} + 1$ the expressions $c_{i,i-2}(x) = t^1_{s-1} c^*_{i,i-2}(x)$ and $\varphi_{i-2} = t^1_{s-1}\psi_{s-2}$ does not contain the factor $(x - d_s)$. In this case $q_{i,i-1}$ can be determined, if we substitute $x = d_s$ in (26). We will also prove that the polynomials $c_{i,i-3}(x)$ contain the factor t^2_{s-1}.

The previous calculations can be used for all blocks ($N_{s-1} < i \leq N_s;\ k \geq s \geq 1$), which means that the first subdiagonal determines from $P_{n-1}(x)$.

Now we will assume that the polynomials (25) are valid until the first $(j-2)$-nd subdiagonal rows. Then we will prove that (25) holds for the $(j-1)$-st part together with the constants $q_{i,i-j+1}$.

For $N_{s-1} < i \leq N_s$ according to the assumption, it follows that

$$\begin{gathered} c'_{i,i-1}(x) = t^j_{s-1}(x-d_s)^{i-j-N_{s-1}}\{[P^{j-1}_{s-1}(x-d_s)+ \\ +\psi_{s-1}(i-j+1-N_{s-1})]c^*_{i,i-j}(x) + \psi_{s-1}(x-d_s)c^{*'}_{i,i-j}(x)\}, \\ c_{i-\nu,i-j}(x) = t^{j-\nu-1}_{s-1}(x-d_s)^{i-j+1-N_{s-1}}c^*_{i-\nu,i-j}(x) = \\ = t^j_{s-1}\psi^{\nu+1}_{s-1}(x-d_s)^{j-i+1-N_{s-1}}c^*_{i-\nu,i-j}(x) \end{gathered} \tag{31}$$

and

$$\varphi_{i-j} = t^0_{s-1}(x-d_s)^{i-j-N_{s-1}} = t^j_{s-1}\psi^j_{s-1}(x-d_s)^{i-j-N_{s-1}}. \tag{32}$$

Let $2 \leq j \leq r_k$. Then for $c_{i+1,i-j}(x) = t^j_{s-1}(x-d_s)^{r_s-j}c^*_{i+1,i-j}(x)$, $(i = N_s)$ from (26), (31) and (32) we obtain

$$\begin{aligned} q_{i,i-j+1} &= -\, c^*_{i+1,i-j}(d_s)/\psi^j_{s-1}(d_s), \\ c_{i,i-j+1}(x) &= t^j_s c^*_{i,i-j-1}(x), \quad (i = N_s). \end{aligned} \tag{33}$$

By substituting (33) in (26), by continuing of this procedure, can be determined $c_{i,i-j-1}(x)$ in the form (25). For $N_s - 1 \geq i \geq N_{s-1} + j$ we obtain

$$(x-d_s)\{c^*_{i,i-j-1}(x)+[p^{j-1}_{s-1}(x-d_s)+\psi_{s-1}(i-j+1-N_{s-1})]c^*_{i,i-j}(x)+\psi_{s-1}(x-d_s)c^{*'}_{i,i-j}(x)\} =$$

$$= (x-d_s)[c^*_{i+1,i-j}(x) + \sum_{\nu=0}^{j-2} q_{i,i-\nu}\psi^{\nu+1}_{s-1}c^*_{i-\nu,i-j}(x)] + q_{i,i-j+1}\psi^j_{s-1},$$

and hence it follows that

$$q_{i,i-j+1} = 0, \quad (i = N_s - 1, N_s - 2, \cdots, N_{s-1} + j) \tag{34}$$

and $c_{i,i-j-1}(x)$ can be determined uniquely.

Thus, we verified that the matrix

$$Q_s = \begin{bmatrix} q_{N_{s-1}+1,N_{s-1}+1} & 1 & 0 & \cdots & 0 \\ 0 & q_{N_{s-1}+2,N_{s-1}+2} & 1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ q_{N_s,N_{s-1}+1} & q_{N_s,N_{s-1}+2} & q_{N_s,N_{s-1}+3} & \cdots & q_{N_s,N_s} \end{bmatrix}$$

is a joint matrix.

From (31) and (32), if $N_{s-1} < i \le N_{s-1} + j - 1$ or if $j > r_s$, the right term of (26) does not contain the factor $(x - d_s)$ longer. From these cases can be determined the constants $q_{i,i-j+1}$, if $x = d_s$ substitutes in (26), and then it obtains $c_{i,i-j-1}(x)$ from the expressions of dividing of the right side of (26) by $(x - d_s)$. We also note that for $j < r_1$, the factor $t^j_{s-1} = t^j_m$, $(j < r_m)$ moves in the next block. The previous calculations can be applied to all blocks $N_{s-1} < i \le N_s$, $(k \ge s \ge 1)$, which means that the $(j-1)$-st subdiagonal parts are determined from $P_{n-j}(x)$. $\|$

Example. For the Poincaré differential equation

$$x^2(x-1)y''' = x(x-1)y'' + (x-1)y' + y$$

where

$$P_3(x) = x^2(x-1), P_2(x) = x(x-1), P_1(x) = x-1, P_0(x) = 1,$$
$$d_1 = d_2 = 0, d_3 = 1, \varphi_1(x) = x, \varphi_2(x) = x^2,$$

the coefficients of the matrix Q, given by the equation (5) have values

$$q_{11} = 0, q_{21} = -2, q_{22} = 4, q_{31} = 1, q_{32} = 0, q_{33} = 0,$$

and the coefficients of the matrix of transformation (11) are

$$c_{31}(x) = -3x, c_{30}(x) = 2, c_{20}(x) = 0.$$

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