Second and higher order boundary value problems of nonsingular type

Ravi P. Agarwal   Donal O'Regan

Abstract

Existence of positive solutions are established for second and higher order boundary value problems even in the case when $y \equiv 0$ may also be a solution.

1 Introduction.

We are concerned with boundary value problems of nonsingular type. In particular in Section 2 we discuss the second order problem

$$\begin{cases} y'' + \phi(t)\, f(t, y, y') = 0, \;\; 0 < t < 1 \\ y(0) = y'(1) = 0, \end{cases} \tag{1.1}$$

and in Section 3 we discuss the n^{th} order focal problem

$$\begin{cases} (-1)^{n-p}\, y^{(n)} = \phi(t)\, f(t, y, y', ..., y^{(p-1)}), \;\; 0 < t < 1 \\ y^{(i)}(0) = 0, \;\; 0 \le i \le p-1 \\ y^{(i)}(1) = 0, \;\; p \le i \le n-1; \end{cases} \tag{1.2}$$

here $1 \le p \le n-1$ is fixed. We are interested in solutions y to (1.1) or (1.2) with $y > 0$ on $(0, 1]$ even if $y \equiv 0$ is a solution of (1.1) or (1.2). This paper provides a new technique for showing that (1.1) or (1.2) has a solution $y > 0$ on $(0, 1]$. The stategy involves using (i). approximating problems, (ii). a Leray–Schauder alternative, (iii). lower type inequalities [2], and (iv). a limiting argument (via

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the Arzela–Ascoli Theorem). This technique will enable us to obtain new and very general existence results for both (1.1) and (1.2).

To conclude this section we gather together some results which will be used throughout this paper. Suppose $y \in C^{n-1}[0,1] \cap C^n(0,1)$ satisfies

$$\begin{cases} (-1)^{n-p}\, y^{(n)} > 0 \quad \text{on} \quad (0,1) \\ y^{(i)}(0) = a \geq 0, \;\; 0 \leq i \leq p-1 \\ y^{(i)}(1) = 0, \;\; p \leq i \leq n-1. \end{cases}$$

In [2] we showed

$$y^{(i)}(t) \geq t^{p-i}\, y^{(i)}(1) = t^{p-i} \sup_{t\in[0,1]} |y^{(i)}(t)| \tag{1.3}$$

for $t \in [0,1]$ and $i \in \{0, ..., p-1\}$.

Next we present an existence principle for

$$\begin{cases} y'' + \phi(t)\, F(t,y,y') = 0, \;\; 0 < t < 1 \\ y(0) = a \geq 0 \\ y'(1) = b \geq 0. \end{cases} \tag{1.4}$$

Theorem 1.1. [8]. *Suppose*

$$\phi \in C(0,1) \;\; \textit{with} \;\; \phi > 0 \;\; \textit{on} \;\; (0,1) \;\; \textit{and} \;\; \phi \in L^1[0,1] \tag{1.5}$$

and

$$F : [0,1] \times \mathbf{R}^2 \to \mathbf{R} \quad \textit{is continuous} \tag{1.6}$$

are satisfied. In addition suppose there is a constant $M > a + b$, independent of λ, with

$$|y|_1 = \max\{|y|_0\,,\; |y'|_0\} \neq M$$

for any solution $y \in C^1[0,1] \cap C^2(0,1)$ to

$$\begin{cases} y'' + \lambda\, \phi(t)\, F(t,y,y') = 0, \;\; 0 < t < 1 \\ y(0) = a \\ y'(1) = b \end{cases} \tag{1.7_λ}$$

for each $\lambda \in (0,1)$; here $|u|_0 = \sup_{[0,1]} |u(t)|$. Then (1.4) has a solution $y \in C^1[0,1] \cap C^2(0,1)$ with $|y|_1 \leq M$.

Finally we present an existence principle for

$$\begin{cases} (-1)^{n-p}\, y^{(n)} = \phi(t)\, F(t,y,y',...., y^{(p-1)}), \;\; 0 < t < 1 \\ y^{(i)}(0) = a \geq 0, \;\; 0 \leq i \leq p-1 \\ y^{(i)}(1) = 0, \;\; p \leq i \leq n-1. \end{cases} \tag{1.8}$$

Theorem 1.2. [4]. *Suppose* (1.5) *and*

(1.9) $$F : [0,1] \times \mathbf{R}^p \to \mathbf{R} \quad \text{is continuous}$$

hold. In addition suppose there is a constant $M > a \sum_{i=0}^{p-1} \frac{1}{i!}$*, independent of* λ*, with*

$$|y|_{p-1} = \max\left\{|y|_0, \ldots\ldots, |y^{(p-1)}|_0\right\} \neq M$$

for any solution $y \in C^{n-1}[0,1] \cap C^n(0,1)$ *to*

$(1.10)_\lambda$ $$\begin{cases} (-1)^{n-p}\, y^{(n)} = \lambda\, \phi(t)\, F(t, y, y', \ldots, y^{(p-1)}), & 0 < t < 1 \\ y^{(i)}(0) = a, & 0 \le i \le p-1 \\ y^{(i)}(1) = 0, & p \le i \le n-1 \end{cases}$$

for each $\lambda \in (0,1)$*. Then* (1.8) *has a solution* $y \in C^{n-1}[0,1] \cap C^n(0,1)$ *with* $|y|_{p-1} \le M$*.*

2 Second order problems.

In this section we discuss the second order problem

(2.1) $$\begin{cases} y'' + \phi(t)\, f(t, y, y') = 0, & 0 < t < 1 \\ y(0) = y'(1) = 0. \end{cases}$$

Throughout this section we will assume the following conditions hold:

(2.2) $$\phi \in C(0,1) \text{ with } \phi > 0 \text{ on } (0,1) \text{ and } \phi \in L^1[0,1]$$

(2.3) $$\begin{cases} f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty) \text{ is continuous with} \\ f(t,u,p) > 0 \text{ for } (t,u,p) \in [0,1] \times (0,\infty) \times (0,\infty) \end{cases}$$

(2.4) $$\begin{cases} f(t,u,p) \le w(\max\{u,p\}) \text{ on } [0,1] \times (0,\infty) \times (0,\infty) \text{ with} \\ w \ge 0 \text{ continuous and nondecreasing on } [0,\infty) \end{cases}$$

(2.5) $$\sup_{c \in (0,\infty)} \frac{c}{w(c) \int_0^1 \phi(s)\, ds} > 1$$

and

(2.6) $$\begin{cases} \text{for a constant } H > 0 \text{ there exists a function } \psi_H \text{ continuous} \\ \text{on } [0,1] \text{ and positive on } (0,1), \text{ and constants } \alpha \ge 0,\ \beta \ge 0 \\ \text{with } \alpha + \beta < 1 \text{ and with } f(t,u,p) \ge \psi_H(t)\, u^\alpha\, p^\beta \\ \text{on } [0,1] \times [0,H] \times [0,H]. \end{cases}$$

Theorem 2.1. *Suppose* (2.2)−−(2.6) *hold. Then* (2.1) *has a solution* $y \in C^1[0,1] \cap C^2(0,1)$ *with* $y > 0$ *on* $(0,1]$.

Proof : Choose $M > 0$ with

$$\frac{M}{w(M)\int_0^1 \phi(s)\,ds} > 1. \tag{2.7}$$

Next choose $\epsilon > 0$ and $\epsilon < \frac{M}{2}$ with

$$\frac{M}{w(M)\int_0^1 \phi(s)\,ds + 2\,\epsilon} > 1. \tag{2.8}$$

Let $n_0 \in \{1,2,.....\}$ be chosen so that $\frac{1}{n_0} < \epsilon$ and let $N_0 = \{n_0, n_0+1, ...\}$. We first show that

$$(2.9)^m \qquad \begin{cases} y'' + \phi(t)\,f^\star(t,y,y') = 0, & 0 < t < 1 \\ y(0) = y'(1) = \frac{1}{m} \end{cases}$$

has a solution for each $m \in N_0$; here

$$f^\star(t,u,p) = \begin{cases} f(t,u,p), & u \geq \frac{1}{m},\ p \geq \frac{1}{m} \\ f(t,u,\frac{1}{m}), & u \geq \frac{1}{m},\ p < \frac{1}{m} \\ f(t,\frac{1}{m},p), & u < \frac{1}{m},\ p \geq \frac{1}{m} \\ f(t,\frac{1}{m},\frac{1}{m}), & u < \frac{1}{m},\ p < \frac{1}{m}. \end{cases}$$

To show $(2.9)^m$ has a solution we consider the family of problems

$$(2.10)^m_\lambda \qquad \begin{cases} y'' + \lambda\,\phi(t)\,f^\star(t,y,y') = 0, & 0 < t < 1 \\ y(0) = y'(1) = \frac{1}{m}, & m \in N_0 \end{cases}$$

for $0 < \lambda < 1$. Let $y \in C^1[0,1] \cap C^2(0,1)$ be any solution of $(2.10)^m_\lambda$. Then $y' \geq \frac{1}{m}$ and $y \geq \frac{1}{m}$ on $[0,1]$. Also from (2.4) we have

$$-y''(t) \leq \phi(t)\,w(|y|_1) \quad \text{for} \quad t \in (0,1);$$

here $|y|_1 = \max\{|y|_0, |y'|_0\}$ and $|u|_0 = \sup_{[0,1]} |u(t)|$. Integrate from t to 1 to obtain

$$y'(t) \leq w(|y|_1) \int_t^1 \phi(x)\,dx + \frac{1}{m} \quad \text{for} \quad t \in [0,1]. \tag{2.11}$$

In particular

$$y'(0) \leq w(|y|_1) \int_0^1 \phi(x)\,dx + \epsilon. \tag{2.12}$$

Also

$$y(1) \leq w(|y|_1) \int_0^1 \phi(x)\,dx + 2\,\epsilon. \tag{2.13}$$

Combine (2.12) and (2.13) to obtain

$$\frac{|y|_1}{w(|y|_1)\int_0^1 \phi(x)\,dx + 2\,\epsilon} \leq 1. \tag{2.14}$$

Now (2.8) together with (2.14) implies $|y|_1 \neq M$.

Thus Theorem 1.1 implies $(2.9)^m$ has a solution y_m with $|y_m|_1 \leq M$. In fact

$$\frac{1}{m} \leq y_m(t) \leq M \quad \text{and} \quad \frac{1}{m} \leq y_m'(t) \leq M \quad \text{for} \quad t \in [0,1] \tag{2.15}$$

and y_m satisfies

$$\begin{cases} y'' + \phi(t)\, f(t, y, y') = 0, \ \ 0 < t < 1 \\ y(0) = y'(1) = \frac{1}{m}. \end{cases}$$

Now (2.6) guarantees the existence of a function $\psi_M(t)$ continuous on $[0,1]$ and positive on $(0,1)$, and constants $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta < 1$ and with $f(t, y_m(t), y_m'(t)) \geq \psi_M(t)\,[y_m(t)]^\alpha\,[y_m'(t)]^\beta$ for $(t, y_m(t), y_m'(t)) \in [0,1] \times [0,M]^2$. The differential equation and (1.3) now imply

$$-[y_m'(t)]^{-\beta}\, y_m''(t) \geq \psi_M(t)\,\phi(t)\,t^\alpha\,[y_m(1)]^\alpha \quad \text{for} \quad t \in (0,1).$$

Integrate from t to 1 and then from 0 to 1 to obtain

$$y_m(1) \geq [y_m(1)]^{\frac{\alpha}{1-\beta}} \int_0^1 \left((1-\beta) \int_t^1 \psi_M(s)\,\phi(s)\,s^\alpha\,ds\right)^{\frac{1}{1-\beta}} dt$$

and so

$$y_m(1) \geq \left(\int_0^1 \left((1-\beta) \int_t^1 \psi_M(s)\,\phi(s)\,s^\alpha\,ds\right)^{\frac{1}{1-\beta}} dt\right)^{\frac{1-\beta}{1-(\alpha+\beta)}} \equiv a_0. \tag{2.16}$$

This together with (1.3) gives

$$y_m(t) \geq a_0\, t \quad \text{for} \quad t \in [0,1]. \tag{2.17}$$

Of course it is immediate that

$$\begin{cases} \{y_m^{(j)}\}_{m \in N_0} \ \text{ is a bounded, equicontinuous} \\ \text{family on } [0,1] \text{ for each } j = 0,1. \end{cases} \tag{2.18}$$

The Arzela–Ascoli Theorem guarantees the existence of a subsequence N of N_0 and a function $y \in C^1[0,1]$ with $y_m^{(j)}$ converging uniformly on $[0,1]$ to $y^{(j)}$ as $m \to \infty$ through N; here $j = 0,1$. Also $y(0) = 0 = y'(1)$ and $y(t) \geq a_0\, t$ for $t \in [0,1]$ (in particular $y > 0$ on $(0,1]$). Now y_m, $m \in N$, satisfies

$$y_m(t) = \frac{1}{m} + \frac{1}{m}t + \int_0^t s\,\phi(s)\,f(s, y_m(s), y_m'(s))\,ds + t \int_t^1 \phi(s)\,f(s, y_m(s), y_m'(s))\,ds \quad \text{for} \quad t \in [0,1].$$

Fix $t \in [0,1]$ and let $m \to \infty$ through N to obtain

$$y(t) = \int_0^t s\,\phi(s)\,f(s, y(s), y'(s))\,ds + t \int_t^1 \phi(s)\,f(s, y(s), y'(s))\,ds.$$

■

Example 2.1. Consider the boundary value problem

$$(2.19)\qquad \begin{cases} y'' + y^{\alpha}\ (y')^{\beta} = 0, & 0 < t < 1 \\ y(0) = y'(1) = 0 \end{cases}$$

with $\alpha \geq 0,\ \beta \geq 0$ and $\alpha+\beta < 1$. Then (2.19) has a solution $y \in C^1[0,1] \cap C^2(0,1)$ with $y > 0$ on $(0,1]$.

Remark 2.1. Notice $y \equiv 0$ is also a solution of (2.19) if $\alpha + \beta \neq 0$.

To see this we will apply Theorem 2.1. Notice (2.2), (2.3), (2.4) (with $w(x) = x^{\alpha+\beta}$), and (2.6) (with $\psi_H = 1,\ \alpha = \alpha$ and $\beta = \beta$) hold. Also

$$\sup_{c\in(0,\infty)} \frac{c}{w(c)\ \int_0^1 \phi(s)\, ds} = \sup_{c\in(0,\infty)} \frac{c}{c^{\alpha+\beta}} = \infty$$

so (2.5) is satisfied. Theorem 2.1 now establishes the result.

Example 2.2. Consider the boundary value problem

$$(2.20)\qquad \begin{cases} y'' + \mu \left(y^{\alpha}\ (y')^{\beta} + \eta_0\, y^{\gamma} + \eta_1 \right) = 0, & 0 < t < 1 \\ y(0) = y'(1) = 0 \end{cases}$$

with $\alpha \geq 0,\ \beta \geq 0,\ \alpha + \beta < 1,\ \gamma > 0,\ \eta_0 \geq 0,\ \eta_1 \geq 0$, and $\mu > 0$. If

$$(2.21)\qquad \mu < \sup_{c\in(0,\infty)} \frac{c}{c^{\alpha+\beta} + \eta_0\, c^{\gamma} + \eta_1}$$

then (2.20) has a solution $y \in C^1[0,1] \cap C^2(0,1)$ with $y > 0$ on $(0,1]$.

Again we apply Theorem 2.1. It is easy to check (2.2), (2.3), (2.4) (with $w(x) = x^{\alpha+\beta} + \eta_0\, x^{\gamma} + \eta_1$), and (2.6) (with $\psi_H = 1,\ \alpha = \alpha$ and $\beta = \beta$) hold. Also

$$\sup_{c\in(0,\infty)} \frac{c}{w(c)\ \int_0^1 \phi(s)\, ds} = \sup_{c\in(0,\infty)} \frac{c}{\mu[c^{\alpha+\beta} + \eta_0\, c^{\gamma} + \eta_1]}$$

so (2.21) guarantees that (2.5) holds. Theorem 2.1 now establishes the result.

3 Higher order problems.

In this section we discuss the n^{th} order focal boundary value problem (here $1 \leq p \leq n-1$ is a fixed integer)

$$(3.1)\qquad \begin{cases} (-1)^{n-p}\, y^{(n)} = \phi(t)\, f(t, y, y', ..., y^{(p-1)}), & 0 < t < 1 \\ y^{(i)}(0) = 0, & 0 \leq i \leq p-1 \\ y^{(i)}(1) = 0, & p \leq i \leq n-1; \end{cases}$$

here $n \geq 2$. Throughout this section we will assume the following conditions hold:

$$(3.2)\qquad \phi \in C(0,1) \ \text{ with } \ \phi > 0 \ \text{ on } \ (0,1) \ \text{ and } \ \phi \in L^1[0,1]$$

$$(3.3)\qquad \begin{cases} f : [0,1] \times [0,\infty)^p \to [0,\infty) \ \text{ is continuous with} \\ f(t, u_0, ..., u_{p-1}) > 0 \ \text{ for } \ (t, u_0, ..., u_{p-1}) \in [0,1] \times (0,\infty)^p \end{cases}$$

$$(3.4)\quad \begin{cases} f(t, u_0, \ldots.., u_{p-1}) \le w(|u|) \text{ on } [0,1] \times (0,\infty)^p \text{ with } w \ge 0 \text{ continuous} \\ \text{and nondecreasing on } [0,\infty); \text{ here } |u| = \max\{u_0, \ldots.., u_{p-1}\} \end{cases}$$

$$(3.5)\quad \begin{cases} \sup_{c\in(0,\infty)} \frac{c}{w(c)} > k_0 \text{ where } k_0 = \max\{r_j : j = 0, \ldots., p-1\} \\ \text{and } r_j = \sup_{t\in[0,1]} \int_0^1 |G^{(j)}(t,s)|\,\phi(s)\,ds \end{cases}$$

and

$$(3.6)\quad \begin{cases} \text{for a constant } H > 0 \text{ there exists a function } \psi_H \text{ continuous on } [0,1] \\ \text{and positive on } (0,1), \text{ and constants } \alpha_i \ge 0 \text{ for } i = 0, \ldots, p-1 \\ \text{with } \sum_{j=0}^{p-1} \alpha_i < 1 \text{ and with } f(t, u_0, \ldots.., u_{p-1}) \ge \psi_H(t) \prod_{i=0}^{p-1} u_i^{\alpha_i} \\ \text{on } [0,1] \times [0,H]^p; \end{cases}$$

here $G(t,s)$ is the Green's function for

$$(3.7)\quad \begin{cases} y^{(n)} = 0 \text{ on } [0,1] \\ y^{(i)}(0) = 0, \ 0 \le i \le p-1 \\ y^{(i)}(1) = 0, \ p \le i \le n-1 \end{cases}$$

and $G^{(j)}(t,s) = \frac{\partial^j}{\partial t^j} G(t,s)$.

Theorem 3.1. *Suppose* (3.2) − −(3.6) *hold. Then* (3.1) *has a solution* $y \in C^{n-1}[0,1] \cap C^n(0,1)$ *with* $y > 0$ *on* $(0,1]$.

Proof : Choose $M > 0$ and then $\epsilon > 0$ and $\epsilon < \frac{m}{\sum_{i=0}^{p-1} \frac{1}{i!}}$ with

$$(3.8)\quad \frac{M}{k_0\,\psi(M) + \epsilon\left(\sum_{i=0}^{p-1} \frac{1}{i!}\right)} > 1.$$

Choose $n_0 \in \{1, 2, \ldots..\}$ with $\frac{1}{n_0} < \epsilon$ and let $N_0 = \{n_0, n_0+1, \ldots.\}$. We first show that

$$(3.9)^m\quad \begin{cases} (-1)^{n-p}\, y^{(n)} = \phi(t)\, f^{\star\star}(t, y, y', \ldots., y^{(p-1)}), \ 0 < t < 1 \\ y^{(i)}(0) = \frac{1}{m}, \ 0 \le i \le p-1 \\ y^{(i)}(1) = 0, \ p \le i \le n-1 \end{cases}$$

has a solution for each $m \in N_0$; here $f^{\star\star} : [0,1] \times \mathbf{R}^p \to [0,\infty)$ is a continuous function with $f^{\star\star}(t, u_0, \ldots.., u_{p-1}) = f(t, u_0, \ldots.., u_{p-1})$ for all $t \in [0,1]$ and all $u_i \ge \frac{1}{m}$, $i = 0, \ldots, p-1$. To show $(3.9)^m$ has a solution we consider the family of problems

$$(3.10)^m_\lambda\quad \begin{cases} (-1)^{n-p}\, y^{(n)} = \lambda\,\phi(t)\, f^{\star\star}(t, y, y', \ldots., y^{(p-1)}), \ 0 < t < 1 \\ y^{(i)}(0) = \frac{1}{m}, \ 0 \le i \le p-1 \\ y^{(i)}(1) = 0, \ p \le i \le n-1 \end{cases}$$

for $0 < \lambda < 1$. Let $y \in C^{n-1}[0,1] \cap C^n(0,1)$ be any solution of $(3.10)^m_\lambda$. Then

$$(3.11)\quad y(t) = \frac{1}{m} \sum_{j=0}^{p-1} \frac{t^j}{j!} + \lambda \int_0^1 (-1)^{n-p}\, G(t,s)\,\phi(s)\, f^{\star\star}(s, y(s), y'(s), \ldots., y^{(p-1)}(s))\,ds$$

for $t \in [0,1]$; here $G(t,s)$ is the Green's function for (3.7). From [2, 9] we know

$$(-1)^{n-p}\, G^{(i)}(t,s) \geq 0, \;\; 0 \leq i \leq p-1 \;\; \text{on} \;\; [0,1]\times[0,1]$$

and

$$(-1)^{n-i}\, G^{(i)}(t,s) \geq 0, \;\; p \leq i \leq n-1 \;\; \text{on} \;\; [0,1]\times[0,1].$$

Consequently

$$y^{(i)}(t) \geq \frac{1}{m} \;\; \text{for} \;\; t \in [0,1] \;\; \text{and} \;\; 0 \leq i \leq p-1 \;\; \text{with} \;\; \sup_{[0,1]} |y^{(i)}(t)| = y^{(i)}(1) \quad \text{for} \;\; 0 \leq i \leq p-1.$$

Also (3.4) and (3.11) imply for $j \in \{0,1,...,p-1\}$ and $t \in [0,1]$ that

$$|y^{(j)}(t)| \leq \frac{1}{m}\sum_{i=j}^{p-1}\frac{1}{(i-j)!} + \int_0^1 |G^{(j)}(t,s)|\, \phi(s)\, \psi(|y|_{p-1})\, ds \leq r_j\, \psi(|y|_{p-1}) + \epsilon \sum_{i=0}^{p-1}\frac{1}{i!};$$

here $|y|_{p-1} = \max\{|y|_0, ..., |y^{(p-1)}|_0\}$ and $|u|_0 = \sup_{[0,1]} |u(t)|$. Consequently for $j = 0,1,...,p-1$ we have

$$|y^{(j)}|_0 \leq k_0\;\psi(|y|_{p-1}) + \epsilon \sum_{i=0}^{p-1}\frac{1}{i!}$$

and so

$$\frac{|y|_{p-1}}{k_0\,\psi(|y|_{p-1}) + \epsilon\left(\sum_{i=0}^{p-1}\frac{1}{i!}\right)} \;\leq\; 1. \tag{3.12}$$

Now (3.8) together with (3.12) implies $|y|_{p-1} \neq M$ and so Theorem 1.2 implies that $(3.9)^m$ has a solution y_m with $|y_m|_{p-1} \leq M$. In fact $\frac{1}{m} \leq y_m^{(i)}(t) \leq M$ for $t \in [0,1]$ and $i = 0,1,...,p-1$. Now (3.6) guarantees the existence of a function $\psi_M(t)$ continuous on $[0,1]$ and positive on $(0,1)$, and constants $\alpha_i \geq 0$, $i = 0,1,...,p-1$ with $\sum_{j=0}^{p-1}\alpha_j < 1$ and with $f(t, y_m(t), ..., y_m^{(p-1)}(t)) \geq \psi_M(t)\prod_{i=0}^{p-1}[y_m^{(i)}(t)]^{\alpha_i}$ for $(t, y_m(t), ..., y_m^{(p-1)}(t)) \in [0,1]\times[0,M]^p$. Thus

$$y_m(t) = \frac{1}{m}\sum_{j=0}^{p-1}\frac{t^j}{j!} + \int_0^1 (-1)^{n-p}\, G(t,s)\, \phi(s)\, f(s, y_m(s), y_m'(s), ..., y_m^{(p-1)}(s))\, ds \tag{3.13}$$

and (1.3) will give

$$y_m^{(j)}(t) \geq \int_0^1 (-1)^{n-p}\, G^{(j)}(t,s)\, \phi(s)\, \psi_M(s)\prod_{i=0}^{p-1}\left[s^{p-i}\, y_m^{(i)}(1)\right]^{\alpha_i}\; ds$$

for $t \in [0,1]$ and $j = 0,1,...,p-1$. Consequently

$$y_m^{(j)}(1) \geq \prod_{i=0}^{p-1}\left[y_m^{(i)}(1)\right]^{\alpha_i}\int_0^1 (-1)^{n-p}\, G^{(j)}(1,s)\, \phi(s)\, \psi_M(s)\prod_{i=0}^{p-1} s^{(p-i)\alpha_i}\; ds \tag{3.14}$$

for $j = 0, 1, ..., p-1$. Let

$$\min\{y_m(1), ..., y_m^{(p-1)}(1)\} = y_m^{(j_0)}(1).$$

From (3.14) we have

$$y_m^{(j_0)}(1) \geq \left[y_m^{(j_0)}(1)\right]^{\sum_{i=0}^{p-1} \alpha_i} \int_0^1 (-1)^{n-p}\, G^{(j_0)}(1,s)\, \phi(s)\, \psi_M(s) \prod_{i=0}^{p-1} s^{(p-i)\alpha_i}\, ds$$

and so

$$y_m^{(j_0)}(1) \geq \left(\int_0^1 (-1)^{n-p}\, G^{(j_0)}(1,s)\, \phi(s)\, \psi_M(s) \prod_{i=0}^{p-1} s^{(p-i)\alpha_i}\, ds\right)^{\frac{1}{1-\sum_{i=0}^{p-1} \alpha_i}} \equiv b_0.$$

This together with (1.3) gives

$$y_m^{(j_0)}(t) \geq b_0\, t^{p-j_0} \quad \text{for} \quad t \in [0,1]. \tag{3.15}$$

Consequently

$$y_m(t) \geq a_0\, t^p \quad \text{for} \quad t \in [0,1]; \tag{3.16}$$

here

$$a_0 = \begin{cases} b_0 & \text{if } j_0 = 0 \\ \frac{b_0}{(p-j_0+1)(p-j_0+2).....p} & \text{if } j_0 \in \{1, ..., p-1\}. \end{cases}$$

The Arzela–Ascoli Theorem guarantees the existence of a subsequence N of N_0 and a function $y \in C^{p-1}[0,1]$ with $y_m^{(j)}$ converging uniformly on $[0,1]$ to $y^{(j)}$ as $m \to \infty$ through N; here $j = 0, 1, ..., p-1$. Also $y^{(i)}(0) = 0$ for $0 \leq i \leq p-1$ and $y(t) \geq a_0\, t^p$ for $t \in [0,1]$ (in particular $y > 0$ on $(0,1]$). Fix $t \in [0,1]$ and let $m \to \infty$ through N in (3.13) to obtain

$$y(t) = \int_0^1 (-1)^{n-p}\, G(t,s)\, \phi(s)\, f(s, y(s), y'(s), ..., y^{(p-1)}(s))\, ds.$$

Thus $(-1)^{n-p}\, y^{(n)} = \phi(t)\, f(t, y, y', ..., y^{(p-1)})$ for $t \in (0,1)$ and $y^{(i)}(1) = 0$ for $p \leq i \leq n-1$. ■

Example 3.1. Consider the boundary value problem

$$\begin{cases} (-1)^{n-p}\, y^{(n)} = \prod_{i=0}^{p-1} \left[y^{(i)}\right]^{\alpha_i}, & 0 < t < 1 \\ y^{(i)}(0) = 0, & 0 \leq i \leq p-1 \\ y^{(i)}(1) = 0, & p \leq i \leq n-1. \end{cases} \tag{3.17}$$

with $\alpha_i \geq 0$ for $i = 0, ..., p-1$ and $\sum_{i=0}^{p-1} \alpha_i < 1$. Then Theorem 3.1 guarantees that (3.17) has a solution $y \in C^{n-1}[0,1] \cap C^n(0,1)$ with $y > 0$ on $(0,1]$.

Remark 3.1. Notice $y \equiv 0$ is also a solution of (3.17) if $\sum_{i=0}^{p-1} \alpha_i \neq 0$.

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Ravi P. Agarwal
Department of Mathematics, National University of Singapore,
10 Kent Ridge Crescent,
Singapore 119260

Donal O'Regan
Department of Mathematics, National University of Ireland,
Galway, Ireland.