A common characterization of ovoids, non-singular quadrics and non-singular Hermitian varieties in PG(d, n)

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Abstract

A property is given which characterizes the ovoids, the non-singular quadrics and the non-singular Hermitian varieties of a projective space $\mathrm{PG}(d,n)$, d odd; whereas, the same property is shown to be typical of the Hermitian arcs and of the non-singular Hermitian varieties if d is even.

1 Introduction

Let $\mathrm{PG}(d,n)$ be the d-dimensional projective space over the Galois field of n elements, $d \geq 2$. The objects of $\mathrm{PG}(d,n)$ we consider throughout this paper are essentially ovoids, non-singular quadrics and non-singular Hermitian varieties, whose main properties we assume known [3,5].

If $\mathcal{O}$ is a non-empty set of points of $\mathrm{PG}(d,n)$, denote by $\mathcal{T}(\mathcal{O})$ the set of lines of $\mathrm{PG}(d,n)$ contained in $\mathcal{O}$ or having exactly one point in common with $\mathcal{O}$. If $p \in \mathcal{O}$, $\mathcal{T}_p(\mathcal{O})$ denotes the set of all elements of $\mathcal{T}(\mathcal{O})$ through p. If $\mathcal{O}$ contains a line, it is *ruled* and every line contained in $\mathcal{O}$ is said a *line of* $\mathcal{O}$.

Now, let S be a subspace of $\mathrm{PG}(d,n)$ containing $\mathcal{O}$ and let $p \in \mathcal{O}$. If the union of the elements of $\mathcal{T}_p(\mathcal{O})$ contained in S is a hyperplane of S, we denote it by $\mathrm{H}_p(\mathcal{O},\mathrm{S})$ and we say that $\mathrm{H}_p(\mathcal{O},\mathrm{S})$ is *the hyperplane (the line* if dim S=2*) of* S *tangent to* $\mathcal{O}$

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at p. In the following we shall omit the indication of the subspace S if it is the whole space $\mathrm{PG}(d,n)$. So, if this is the case, the notation $\mathrm{H}_p(\mathcal{O})$ (or H_p if the context is clear) will be used instead of $\mathrm{H}_p(\mathcal{O},\mathrm{S})$ and we shall say that $\mathrm{H}_p(\mathcal{O})$ is *the tangent hyperplane (the tangent line* if dim S=2*) to* $\mathcal{O}$ *at* p.

Theorem 1.1 ([1,2]). Let $\mathcal{O}$ be a ruled set of points of $\mathrm{PG}(d,n)$, $d \geq 2$. If, for any $p \in \mathcal{O}$, there exists the tangent hyperplane to $\mathcal{O}$ at p, then $\mathcal{O}$ is a (non-singular) quadric or n is a square and $\mathcal{O}$ is a (non-singular) Hermitian variety.

Theorem 1.2. ([4,6]. Let $\mathcal{O}$ be a non-ruled set of points of $\mathrm{PG}(2,n)$. If there exists a tangent line to $\mathcal{O}$ and every line not in $\mathcal{T}(\mathcal{O})$ has exactly s $(s>1)$ distinct points in common with $\mathcal{O}$, then n is a square and $\mathcal{O}$ is a Hermitian arc.

Theorem 1.3. ([7]). Let $\mathcal{O}$ be a non-ruled set of points of $\mathrm{PG}(d,n)$, $d \geq 3$. If there exists an integer s $(s>1)$ such that, for any hyperplane H, $|\mathrm{H} \cap \mathcal{O}| = 1$ or s and there exists a hyperplane sharing with $\mathcal{O}$ exactly one point, then $d=3$ and $\mathcal{O}$ is an ovoid.

A *supertangent* hyperplane to $\mathcal{O}$ is a tangent hyperplane H_p such that, for any $q \in (\mathrm{H}_p \backslash \{p\}) \cap \mathcal{O}$, there exists the tangent hyperplane to $\mathcal{O}$ at q.

The aim of this paper is to prove the following

Theorem 1.4. Let $\mathcal{O}$ be a set of points of $\mathrm{PG}(d,n)$, $d \geq 2$. If there exists a supertangent hyperplane to $\mathcal{O}$ and every non-tangent hyperplane has exactly s $(s > 1)$ points in common with $\mathcal{O}$, then :

(i) if $d=3$, $\mathcal{O}$ is an ovoid or a non-singular hyperbolic quadric or n is a square and $\mathcal{O}$ is a non-singular Hermitian variety;

(ii) if $d>3$ is odd, $\mathcal{O}$ is a non-singular quadric or n is a square and $\mathcal{O}$ is a non-singular Hermitian variety;

(iii) if $d=2$, n is a square and $\mathcal{O}$ is a Hermitian arc.

(iv) if $d>2$ is even, n is a square and $\mathcal{O}$ is a non-singular Hermitian variety.

From now on, if an ovoid or a Hermitian variety or arc is considered, we will not specify that necessarily $d \leq 3$ and n is a square, respectively.

Now, we need some definitions and notations.

If r and t are non-negative integers and $r \geq t$, we put $[r,t] = n^r + n^{r-1} + \cdots + n^t$.

If S and T are two subspaces of $\mathrm{PG}(d,n)$, $\langle \mathrm{S},\mathrm{T}\rangle$ will denote the subspace spanned by S∪T. Identifying each point p with the set $\{p\}$, we shall write $\langle p,\mathrm{T}\rangle$ ($\langle p,q\rangle$) instead of $\langle \{p\},\mathrm{T}\rangle$ ($\langle \{p\},\{q\}\rangle$), for any subspace T.

Again, let $\mathcal{O}$ be a non-empty set of points of $\mathrm{PG}(d,n)$. A *secant* of $\mathcal{O}$ is a line L such that $2 \leq |\mathrm{L} \cap \mathcal{O}| \leq n$ and an i-secant of $\mathcal{O}$ is a secant meeting $\mathcal{O}$ in exactly i points.

Finally, a *generator* of $\mathcal{O}$ is a subspace of maximum dimension lying on $\mathcal{O}$. The dimension of a generator of $\mathcal{O}$ will be denoted by $\delta(\mathcal{O})$.

2 First results and proof of Theorem 1.4

Throughout this section, $\mathcal{O}$ is a set of points of $\mathrm{PG}(d,n)$, $d \geq 2$, satisfying the conditions in Theorem 1.4 and H_a $(= \mathrm{H}_a(\mathcal{O}))$ is a supertangent hyperplane to $\mathcal{O}$. We define $\delta = \delta(\mathcal{O})$ and, for any $p \in \mathcal{O}$, we denote by k_p the number of lines of $\mathcal{O}$ through p.

Proposition 2.1. If H_p is a tangent hyperplane to $\mathcal{O}$, then $|\mathrm{H}_p \cap \mathcal{O}| = 1 + nk_p$. Moreover, if $\mathcal{O}'$ is the section of $\mathcal{O}$ by a $(d-2)$-dimensional subspace not through p contained in H_p, then $k_p = |\mathcal{O}'|$ and $|\mathrm{H}_p \cap \mathcal{O}| = 1 + n|\mathcal{O}'|$.

Proof. It is obvious. ∎

Proposition 2.2. Let H_p be a tangent hyperplane to $\mathcal{O}$. There exists in H_p a generator of $\mathcal{O}$. Moreover, if G is a generator of $\mathcal{O}$ contained in H_p, then $p \in \mathrm{G}$.

Proof. If $\delta = 0$, then $\{p\}$ is a generator of $\mathcal{O}$ contained in H_p. Now, let $\delta \geq 1$ and let $\bar{\mathrm{G}}$ be a generator of $\mathcal{O}$. If $\bar{\mathrm{G}} \not\subseteq \mathrm{H}_p$, consider $\bar{\mathrm{G}} \cap \mathrm{H}_p$. Obviously, $\dim(\bar{\mathrm{G}} \cap \mathrm{H}_p) = \delta - 1$. Since H_p is tangent to $\mathcal{O}$ at p, then $p \notin \bar{\mathrm{G}}$ and $\langle p, \bar{\mathrm{G}} \cap \mathrm{H}_p \rangle \subseteq \mathcal{O}$. The dimension of $\langle p, \bar{\mathrm{G}} \cap \mathrm{H}_p \rangle$ is δ, so $\langle p, \bar{\mathrm{G}} \cap \mathrm{H}_p \rangle$ is a generator of $\mathcal{O}$ contained in H_p.

Now, let G be a generator of $\mathcal{O}$ contained in H_p. If $p \notin \mathrm{G}$, then $\langle p, \mathrm{G} \rangle \subseteq \mathcal{O}$, a contradiction as $\dim \langle p, \mathrm{G} \rangle = \delta + 1$. ∎

Proposition 2.3. No hyperplane is contained in $\mathcal{O}$, i.e. $\delta < d-1$.

Proof. Assume $\delta = d-1$. By Proposition 2.2, $\mathrm{H}_a \subseteq \mathcal{O}$. Thus, $\mathrm{H}_p = \mathrm{H}_a$, for any $p \in \mathrm{H}_a$; moreover, every line not in H_a is a secant of $\mathcal{O}$. It follows that no hyperplane exists tangent to $\mathcal{O}$ at a point not in H_a. Therefore, for any hyperplane H distinct from H_a, H is not tangent to $\mathcal{O}$ and so $|\mathrm{H} \cap \mathcal{O}| = s$.

Let S be a $(d-2)$-dimensional subspace in H_a. Each hyperplane through S distinct from H_a has exactly $s - [d-2,0]$ points in common with $\mathcal{O} \backslash \mathrm{H}_a$. It follows that

$$|\mathcal{O}| = [d-1,0] + n(s - [d-2,0]) = 1 + ns. \tag{2.1}$$

We distinguish two cases:
(i) $d = 2$;
(ii) $d \geq 3$.

Case (i). Consider a point $p \in \mathcal{O} \backslash \mathrm{H}_a$. Since every line through p is s-secant, then

$$|\mathcal{O}| = 1 + (n+1)(s-1) = ns - n + s,$$

from which, by (2.1), $s = n+1$ follows, a contradiction.

Case (ii). Consider a line L not in H_a and put $|\mathrm{L} \cap \mathcal{O}| = c$. Count in two ways the point-hyperplane pairs (p, H), where $p \in (\mathcal{O} \backslash \mathrm{L}) \cap \mathrm{H}$ and $\mathrm{L} \subseteq \mathrm{H}$. By (2.1),

$$(1 + ns - c)[d-3,0] = [d-2,0](s-c),$$

from which

$$c = \frac{s - [d-3,0]}{n^{d-2}} \tag{2.2}$$

follows. Thus, c does not depend on the particular line L not in H_a. Then, every line through a point of $\mathcal{O}\backslash\mathrm{H}_a$ is c-secant. Hence,

$$|\mathcal{O}| = 1 + [d-1,0](c-1). \tag{2.3}$$

By (2.1)-(2.3), we obtain $s = [d-1,0]$. Therefore, the hyperplanes distinct from H_a all are contained in $\mathcal{O}$, a contradiction. ■

Proposition 2.4. If $d = 2$, then $\delta = 0$ and $\mathcal{O}$ is a Hermitian arc. If $d \geq 3$ and $\delta = 0$, then $\mathcal{O}$ is an ovoid.

Proof. If $d = 2$, then, by Proposition 2.3, $\delta = 0$. Thus, we can assume $d \geq 2$ and $\delta = 0$. Since $\delta = 0$, every tangent hyperplane meets $\mathcal{O}$ in exactly one point. Consequently, for any hyperplane H such that $|\mathrm{H} \cap \mathcal{O}| \neq 1$, H is not tangent and so $|\mathrm{H} \cap \mathcal{O}| = s$. The statement follows from Theorem 1.2 or 1.3 according as $d = 2$ or $d = 3$, respectively. ■

The previous proposition exhausts the cases $\delta = 0$ and $d = 2$. So, in what follows, we can assume $\delta \geq 1$ and $d \geq 3$.

Proposition 2.5. Let H_p be a tangent hyperplane and let $\mathcal{O}'$ be the section of $\mathcal{O}$ by a $(d-2)$-dimensional subspace S not on p contained in H_p. We have $\mathcal{O}' \neq \emptyset$ and $\delta(\mathcal{O}') = \delta(\mathcal{O}) - 1$. Moreover, a subspace G′ contained in $\mathcal{O}'$ is a generator of $\mathcal{O}'$ if, and only if, $\mathrm{G}' = \mathrm{S} \cap \mathrm{G}$, for some generator G of $\mathcal{O}$.

Proof. By Proposition 2.2, there exists in H_p a generator of $\mathcal{O}$; moreover, each generator of $\mathcal{O}$ contained in H_p meets S in a subspace of dimension $\delta(\mathcal{O}) - 1$. By Proposition 2.3, $\delta(\mathcal{O}) - 1 < d - 2$. Again by Proposition 2.2, no $\delta(\mathcal{O})$-dimensional subspace exists in $\mathcal{O}'$. Finally, if G′ is a $(\delta(\mathcal{O}) - 1)$-dimensional subspace contained in $\mathcal{O}'$, then $\langle p, \mathrm{G}' \rangle$ is a generator of $\mathcal{O}$; so the statement. ■

Proposition 2.6. Let p and q be two distinct points such that the tangent hyperplanes to $\mathcal{O}$ at p and q, respectively, exist. If $p \notin \mathrm{H}_q$, then $k_p = k_q$.

Proof. Since $p \notin \mathrm{H}_q$, then $\mathrm{H}_p \neq \mathrm{H}_q$ and $q \notin \mathrm{H}_p$. Consider the $(d-2)$-dimensional subspace $\mathrm{H}_p \cap \mathrm{H}_q$. By Proposition 2.1, $k_p = |\mathcal{O} \cap \mathrm{H}_p \cap \mathrm{H}_q| = k_q$. ■

Proposition 2.7. There exists in H_a a point $b \in \mathcal{O}$ such that $\mathrm{H}_b \neq \mathrm{H}_a$.

Proof. Assume on the contrary $\mathrm{H}_p = \mathrm{H}_a$, for any $p \in \mathcal{O} \cap \mathrm{H}_a$. Since every line of $\mathcal{O}$ is contained in H_a, then, by Proposition 2.2, no hyperplane exists tangent to $\mathcal{O}$ at a point of $\mathcal{O}\backslash\mathrm{H}_a$. Thus, every hyperplane distinct from H_a is not tangent to $\mathcal{O}$. Again by Proposition 2.2, there exists in H_a a generator G of $\mathcal{O}$ through a and $(\mathrm{H}_a\backslash\mathrm{G}) \cap \mathcal{O} = \emptyset$. Then, $\mathrm{H}_a \cap \mathcal{O} = \mathrm{G}$. By Proposition 2.3, a $(d-2)$-dimensional subspace T exists in H_a such that $G \subseteq T$. Obviously, $T \cap \mathcal{O} = G$. It follows that each hyperplane through T distinct from H_a contains exactly $s - [\delta, 0]$ points of $\mathcal{O}$ out of H_a. Thus,

$$|\mathcal{O} \backslash H_a| = n(s - [\delta, 0]). \tag{2.4}$$

Now, observe that every hyperplane not containing G meets $H_a \cap \mathcal{O}$ $(= G)$ in a $(\delta - 1)$-dimensional subspace. Therefore, counting in two ways the point-hyperplane pairs (p, H), where $p \in H \cap (\mathcal{O} \backslash H_a)$, yields

$$|\mathcal{O} \backslash H_a| [d - 1, 0] = [d - \delta - 1, 1](s - [\delta, 0]) + [d, d - \delta](s - [\delta - 1, 0]). \tag{2.5}$$

From (2.4) and (2.5) $[d, d - \delta] = 0$ follows, a contradiction. ■

Lemma 2.8. If b and c are two distinct points of $\mathcal{O}$ in H_a such that H_b, $H_c \neq H_a$, then $k_b = k_c$.

Proof. Consider a line L through b in $H_a \backslash (H_b \cup H_c)$. Since L is a secant of $\mathcal{O}$, there exists a point $p \in (L \backslash \{b\}) \cap \mathcal{O}$. By Proposition 2.6, $k_b = k_p = k_c$ and the statement follows. ■

Proposition 2.9. There exists the tangent hyperplane to $\mathcal{O}$ at some point of $\mathcal{O} \backslash H_a$.

Proof. Assume on the contrary that the tangent hyperplane to $\mathcal{O}$ at p exists if, and only if, $p \in \mathcal{O} \cap H_a$.

By Proposition 2.7, there exists in H_a a point b of $\mathcal{O}$ such that $H_b \neq H_a$ and, by Proposition 2.1 and Lemma 2.8, every tangent hyperplane to $\mathcal{O}$ distinct from H_a has exactly $1 + nk_b$ points in common with $\mathcal{O}$. Now, consider a secant L of $\mathcal{O}$ through b in H_a and a point $q \in \mathcal{O} \cap (L \backslash \{b\})$. Obviously, $H_q \neq H_b$ and $H_q \cap H_b$ is a $(d - 2)$-dimensional subspace not contained in H_a. Let $|\mathcal{O} \cap H_q \cap H_b| = k$. By Proposition 2.1, $k_b = k$. Denote by t the number of all tangent hyperplanes to $\mathcal{O}$ through $H_q \cap H_b$. We have

$$|\mathcal{O}| = k + t(1 + nk - k) + (n + 1 - t)(s - k). \tag{2.6}$$

Now, consider a $(d - 2)$-dimensional subspace S in H_a not through a. Let $r = |S \cap \mathcal{O}|$. By Proposition 2.1,

$$|\mathcal{O} \cap H_a| = 1 + nr. \tag{2.7}$$

Since the tangent hyperplanes to $\mathcal{O}$ all contain a, then each hyperplane H through S distinct from H_a is not tangent; so $|H \cap \mathcal{O}| = s$. It follows that

$$|\mathcal{O} \backslash H_a| = n(s - r). \tag{2.8}$$

By (2.7) and (2.8),

$$|\mathcal{O}| = 1 + ns. \tag{2.9}$$

Hence, by (2.6) and (2.9),

$$t(1 + nk - s) = 1 + nk - s. \tag{2.10}$$

Since H_b and H_q are two distinct tangent hyperplanes to $\mathcal{O}$ through $\mathrm{H}_b \cap \mathrm{H}_q$, then $t \geq 2$; so (2.10) implies that $1 + nk = s$.

Thus, every hyperplane distinct from H_a meets $\mathcal{O}$ in exactly s points. It follows that $|\mathcal{O} \backslash \mathrm{H}_a| = n(s - |\mathcal{O} \cap \mathrm{T}|)$, for any $(d-2)$-dimensional subspace T contained in H_a. This implies that the integer $|\mathcal{O} \cap \mathrm{T}|$ does not depend on the particular subspace T; so $|\mathcal{O} \cap \mathrm{T}| = r$, for any T.

Now, counting in two ways the point-subspace pairs (p,T) where T is a $(d-2)$-dimensional subspace in H_a and $p \in \mathcal{O} \cap \mathrm{T}$, we conclude (see also the Remark in [7]) that every subspace T is contained in $\mathcal{O}$, i.e. $\mathrm{H}_a \subseteq \mathcal{O}$, a contradiction to Proposition 2.3. ■

Corollary 2.10. There exists a generator of $\mathcal{O}$ not contained in H_a.

Proof. It is an obvious consequence of Propositions 2.2 and 2.9. ■

Proposition 2.11. For any $p \in \mathcal{O} \cap (\mathrm{H}_a \backslash \{a\})$, $\mathrm{H}_p \neq \mathrm{H}_a$.

Proof. By Corollary 2.10, we can consider a generator G of $\mathcal{O}$ not in H_a. Obviously, $a \notin \mathrm{G}$. Let $\mathrm{G}' = \mathrm{G} \cap \mathrm{H}_a$ and $\mathrm{G}'' = \langle a, \mathrm{G}' \rangle$. We have $\mathrm{G}'' \subseteq \mathcal{O}$. Since $\dim \mathrm{G}' = \delta - 1$, then $\dim \mathrm{G}'' = \delta$, so G'' is a generator of $\mathcal{O}$. If $p \notin \mathrm{G}''$, the statement follows from Proposition 2.2. Now, let $p \in \mathrm{G}'$. Since $\mathrm{G} \subseteq \mathrm{H}_p$, then $\mathrm{H}_p \neq \mathrm{H}_a$. Finally, assume $p \in \mathrm{G}'' \backslash \mathrm{G}'$. Let $\{q\} = \langle a, p \rangle \cap \mathrm{G}'$. Since $\mathrm{H}_q \neq \mathrm{H}_a$, there exists a secant L of $\mathcal{O}$ in H_a through q. Let $q' \in \mathcal{O} \cap (\mathrm{L} \backslash \{q\})$. We have $\langle a, q' \rangle \subseteq \mathcal{O}$. If $\mathrm{H}_p = \mathrm{H}_a$, then $\langle p, \langle a, q' \rangle \rangle \subseteq \mathcal{O}$. Since L is contained in the plane $\langle p, \langle a, q' \rangle \rangle$, then $\mathrm{L} \subseteq \mathcal{O}$, a contradiction. Thus $\mathrm{H}_p \neq \mathrm{H}_a$ and the statement is completely proved. ■

Proposition 2.12. Let $p \in \mathcal{O}$. If there exists the tangent hyperplane H_p to $\mathcal{O}$ at p, then $k_p = k_a$ and $|\mathrm{H}_p \cap \mathcal{O}| = 1 + nk_a$.

Proof. Let X be the set of points q of $\mathcal{O} \backslash \mathrm{H}_a$ such that the tangent hyperplane to $\mathcal{O}$ at q exists. By Proposition 2.9, $\mathrm{X} \neq \emptyset$ and, by Proposition 2.6, $k_q = k_a$, for any $q \in \mathrm{X}$. Fix a point $\bar{q} \in \mathrm{X}$. Obviously, $a \notin \mathrm{H}_{\bar{q}}$. By Proposition 2.2, a line L of $\mathcal{O}$ through a exists. Consider a point b in $\mathrm{L} \backslash \mathrm{H}_{\bar{q}}$ distinct from a. By Proposition 2.6, $k_b = k_{\bar{q}}$. Now, let p be a point of $\mathcal{O} \cap \mathrm{H}_a$ distinct from a and b. By Proposition 2.11, H_p, $\mathrm{H}_b \neq \mathrm{H}_a$; then, by Lemma 2.8, $k_p = k_b$. Thus, $k_p = k_q = k_a$, for any $p \in \mathcal{O} \cap \mathrm{H}_a$ and for any $q \in \mathrm{X}$.

The second part of the statement follows from Proposition 2.1, so the proof is complete. ■

Proposition 2.13. Let $d \geq 4$ and let $\mathcal{S}$ be the set of all $(d-2)$-dimensional subspaces in H_a not through a.

(i) If $d > 5$ is odd, then one of the following occurs:

(a) for any $\mathrm{S} \in \mathcal{S}$, $\mathcal{O} \cap \mathrm{S}$ is a non-singular elliptic quadric of S;

(b) for any $\mathrm{S} \in \mathcal{S}$, $\mathcal{O}\cap\mathrm{S}$ is a non-singular hyperbolic quadric of S;
(c) for any $\mathrm{S} \in \mathcal{S}$, $\mathcal{O}\cap\mathrm{S}$ is a non-singular Hermitian variety of S.
(ii) If $d = 5$, then (b) or (c) holds or
(a′) for any $\mathrm{S} \in \mathcal{S}$, $\mathcal{O}\cap\mathrm{S}$ is an ovoid of S.
(iii) If $d = 4$, then
(d) for any $\mathrm{S} \in \mathcal{S}$, $\mathcal{O}\cap\mathrm{S}$ is a Hermitian arc of S.
(iii) If $d > 4$ is even, then
(e) for any $\mathrm{S} \in \mathcal{S}$, $\mathcal{O}\cap\mathrm{S}$ is a non-singular Hermitian variety of S.

Moreover, if $\mathrm{S} \in \mathcal{S}$ and S' is a hyperplane of S not tangent to $\mathcal{O}\cap\mathrm{S}$, then every hyperplane of $\mathrm{PG}(d, n)$ through $\langle a, \mathrm{S}'\rangle$ distinct from H_a is not tangent to $\mathcal{O}$.

Proof. Let $\mathrm{S} \in \mathcal{S}$ and let $\mathcal{O}' = \mathcal{O}\cap\mathrm{S}$. By Proposition 2.5, $\mathcal{O}' \neq \emptyset$. Let $p' \in \mathcal{O}'$. Since $a \in \mathrm{H}_{p'}$ and, by Proposition 2.11, $\mathrm{H}_{p'} \neq \mathrm{H}_a$, then $\mathrm{S} \not\subseteq \mathrm{H}_{p'}$. So, $\dim(\mathrm{H}_{p'} \cap \mathrm{S}) = d-3$. Since a line of S is an element of $\mathcal{T}(\mathcal{O}')$ if, and only if, it belongs to $\mathcal{T}(\mathcal{O})$, then the union of the elements of $\mathcal{T}_{p'}(\mathcal{O}')$ contained in S is $\mathrm{H}_{p'}\cap\mathrm{S}$. So, $\mathrm{H}_{p'}\cap\mathrm{S}$ is the hyperplane $\mathrm{H}_{p'}(\mathcal{O}',\mathrm{S})$ of S tangent to $\mathcal{O}'$ at p'. Write $\mathrm{T}_{p'} = \mathrm{H}_{p'}(\mathcal{O}',\mathrm{S})$, for any $p' \in \mathcal{O}'$.

We distinguish two cases:
(1) $\delta(\mathcal{O}) = 1$;
(2) $\delta(\mathcal{O}) \geq 2$.

Case (1). By Proposition 2.5, $\mathcal{O}'$ contains no line. So, $\mathrm{T}_{p'} \cap \mathcal{O}' = \{p'\}$, for any $p' \in \mathcal{O}'$. Now, let S' be a hyperplane of S not tangent to $\mathcal{O}'$ and let $\mathrm{S}'' = \langle a, \mathrm{S}'\rangle$. Consider a hyperplane H of $\mathrm{PG}(d, n)$ through S'' distinct from H_a. We want to prove that H is not tangent to $\mathcal{O}$. Since no line through a not in H_a is a line of $\mathcal{O}$, then H can not be tangent to $\mathcal{O}$ at a point of $\mathcal{O} \cap (\mathrm{H}\backslash\mathrm{H}_a)$. Now, let $p' \in \mathcal{O} \cap \mathrm{S}'$. Since $\mathrm{H}_{p'}\cap\mathrm{S} = \mathrm{T}_{p'} \neq \mathrm{S}' = \mathrm{H}\cap\mathrm{S}$, then $\mathrm{H} \neq \mathrm{H}_{p'}$. Therefore, $\mathrm{H} \neq \mathrm{H}_{p'}$ for any point $p' \in \mathcal{O}\cap\mathrm{S}'$. Finally, assume $\mathrm{H} = \mathrm{H}_b$ for some point $b \in (\mathcal{O}\backslash\{a\}) \cap (\mathrm{S}''\backslash\mathrm{S})$. Let $\{b'\} = \mathrm{S} \cap \langle a, b\rangle$. Since S' is not tangent to $\mathcal{O}'$, then there exists in S' a point b'' of $\mathcal{O}$ distinct from b'. Since $\langle a, b''\rangle$ is a line of $\mathcal{O}$, then the whole plane $\langle b, \langle a, b''\rangle\rangle \subseteq \mathcal{O}$, which contradicts $\delta(\mathcal{O}) = 1$.

Thus, every hyperplane $\mathrm{H} \neq \mathrm{H}_a$ of $\mathrm{PG}(d, n)$ through S'' is not tangent to $\mathcal{O}$; so, $|\mathrm{H} \cap \mathcal{O}| = s$. Consequently,

$$|\mathcal{O}| = |\mathcal{O} \cap \mathrm{H}_a| + n(s - |\mathcal{O} \cap \mathrm{S}''|). \tag{2.11}$$

Since $|\mathcal{O} \cap \mathrm{S}''| = 1 + n|\mathcal{O}' \cap \mathrm{S}'|$, from (2.11) it follows that $|\mathcal{O}' \cap \mathrm{S}'|$ does not depend on the particular hyperplane S' of S not tangent to $\mathcal{O}'$, i.e. there exists an integer s' such that every hyperplane of S which is not tangent to $\mathcal{O}'$ has exactly s' points in common with $\mathcal{O}'$. Again, let $p' \in \mathcal{O}'$ and let L' be a secant of $\mathcal{O}'$ through p'. Consider a hyperplane U' of S through L'. Since $\mathrm{L}' \subseteq \mathrm{U}'$, then $s' = |\mathrm{U}'\cap\mathcal{O}'| \geq 2$. So, by Theorems 1.2 and 1.3, only two cases can occur:
-$d = 4$ and $\mathcal{O}'$ is a Hermitian arc of the plane S;
-$d = 5$ and $\mathcal{O}'$ is an ovoid of S.

So, the statement is proved.

Case (2). By Proposition 2.5, $\mathcal{O}'$ is ruled. So, by Theorem 1.1, $\mathcal{O}'$ is a (non-singular) quadric or a (non-singular) Hermitian variety of S. If S' is a hyperplane of S not tangent to $\mathcal{O}'$, the same arguments as in Case (1) show that a hyperplane $\mathrm{H} \neq \mathrm{H}_a$ of $\mathrm{PG}(d, n)$ through $\mathrm{S}'' = \langle a, \mathrm{S}'\rangle$ is not tangent to $\mathcal{O}$ at p, for any $p \in \mathcal{O} \cap ((\mathrm{H}\backslash\mathrm{H}_a) \cup \mathrm{S}')$. Now, let $\bar{\mathrm{S}} \in \mathcal{S}\backslash\{\mathrm{S}\}$. Define $\bar{\mathcal{O}}' = \mathcal{O} \cap \bar{\mathrm{S}}$ and $\bar{\mathrm{S}}' = \bar{\mathrm{S}} \cap \mathrm{S}''$.

By the same argument as above, $\bar{\mathcal{O}}'$ is a (non-singular) quadric or a (non-singular) Hermitian variety of $\bar{\mathrm{S}}$. By Proposition 2.1,

$$|\mathcal{O} \cap \mathrm{H}_a| = 1 + n|\mathcal{O}'| = 1 + n|\bar{\mathcal{O}}'|; \tag{2.12}$$

similarly,

$$|\mathcal{O} \cap \mathrm{S}''| = 1 + n|\mathcal{O} \cap \mathrm{S}'| = 1 + n|\mathcal{O} \cap \bar{\mathrm{S}}'|. \tag{2.13}$$

From (2.12) it follows that $|\mathcal{O}'|=|\bar{\mathcal{O}}'|$; so $\mathcal{O}'$ and $\bar{\mathcal{O}}'$ both are (non-singular) elliptic quadric or hyperbolic quadric or Hermitian varieties; moreover, by (2.13), $|\mathcal{O} \cap \mathrm{S}'| = |\mathcal{O} \cap \bar{\mathrm{S}}'|$. Consequently, $\bar{\mathrm{S}}'$ is a hyperplane of $\bar{\mathrm{S}}$ not tangent to $\bar{\mathcal{O}}'$. Again, the same argument as in Case (1) shows that $\mathrm{H} \neq \mathrm{H}_p$, for any $p \in \mathcal{O} \cap \bar{\mathrm{S}}'$. Since $\bar{\mathrm{S}}$ is not a particular element of $\mathcal{S} \backslash \{\mathrm{S}\}$, then we can affirm that $\mathrm{H} \neq \mathrm{H}_p$, for any $p \in (\mathcal{O} \backslash \{a\}) \cap \mathrm{S}''$.

Hence, H is not a tangent hyperplane to $\mathcal{O}$.

In order to conclude the proof, we observe that the same arguments as in Case (1) can be used to prove that, for any $\mathrm{S} \in \mathcal{S}$, the hyperplanes of S not tangent to $\mathcal{O}' = \mathcal{O} \cap \mathrm{S}$ all intersect $\mathcal{O}'$ in the same number of points. Consequently, if d is even, $\mathcal{O}'$ is necessarely a Hermitian variety of S.

The statement is completely proved. ■

Let $d \geq 4$. From now on, we shall say that $\mathcal{O}$ is of hyperbolic (elliptic) type if the case (b) ((a) or (a′)) of Proposition 2.13 occurs. Similarly, $\mathcal{O}$ will be said of Hermitian type if the case (c), (d) or (e) of Proposition 2.13 arises.

Proposition 2.14. We have:

(i) $\delta = \frac{d-1}{2}$, if $d = 3$ or $d \geq 5$ is odd and $\mathcal{O}$ is of hyperbolic or Hermitian type;

(ii) $\delta = \frac{d-3}{2}$, if $d \geq 5$ is odd and $\mathcal{O}$ is of elliptic type;

(iii) $\delta = \frac{d-2}{2}$, if $d \geq 4$ is even (in this case $\mathcal{O}$ is of Hermitian type).

Proof. The statement follows from Proposition 2.3 or 2.5 according as $d = 3$ or $d > 3$. ■

Lemma 2.15. For any $p \in (\mathrm{H}_a \backslash \{a\}) \cap \mathcal{O}$, there exists a generator of $\mathcal{O}$ through a and p. Moreover, if $p, q \in (\mathrm{H}_a \backslash \{a\}) \cap \mathcal{O}$, $p \neq q$ and $a \notin \langle p, q\rangle$, then there exists a generator of $\mathcal{O}$ through p not on q.

Proof. First, let $d = 3$. By Proposition 2.14, $\langle a, p\rangle$ is a generator of $\mathcal{O}$; so the statement.

Now, let $d \geq 4$. Consider a $(d-2)$-dimensional subspace S in H_a such that $p \in \mathrm{S}$ and $a \notin \mathrm{S}$. Let $\mathcal{O}' = \mathcal{O} \cap \mathrm{S}$. By Proposition 2.13, there exists a generator G' of $\mathcal{O}'$ through p. Thus, by Proposition 2.5, $\langle a, \mathrm{G}'\rangle$ is a generator of $\mathcal{O}$.

Let $q \notin \langle a, p\rangle$. Consider a $(d-2)$-dimensional subspace U in H_a not through a such that $p, q \in \mathrm{U}$ and define $\mathcal{O}'' = \mathcal{O} \cap \mathrm{U}$. By Proposition 2.13, there exists a generator G'' of $\mathcal{O}''$ through p not on q. Of course, $\langle a, \mathrm{G}''\rangle$ satisfies the required conditions, so the statement is completely proved.

■

Proposition 2.16. If p and q are distinct points of $\mathrm{H}_a \cap \mathcal{O}$, then $\mathrm{H}_p \neq \mathrm{H}_q$.

Proof. If one of the two points p and q is the point a, then the statement follows from Proposition 2.11.

Now, let $p, q \neq a$. We distinguish two cases:

(i) a, p, q are non-collinear;

(ii) a, p, q are collinear.

Case (i). By Lemma 2.15, there exists a generator G of $\mathcal{O}$ such that $p \in \mathrm{G}$ and $q \notin \mathrm{G}$. By Proposition 2.2, $\mathrm{G} \nsubseteq \mathrm{H}_q$; on the other hand, $\mathrm{G} \subseteq \mathrm{H}_p$. Thus, $\mathrm{H}_p \neq \mathrm{H}_q$.

Case (ii). First of all, observe that $\langle a, p\rangle \subseteq \mathrm{H}_a \cap \mathrm{H}_p$. Since $\mathrm{H}_p \neq \mathrm{H}_a$, a line L through a exists in $\mathrm{H}_p \backslash \mathrm{H}_a$. Obviously, L is a secant of $\mathcal{O}$. Let $y \in \mathcal{O} \cap (\mathrm{L} \backslash \{a\})$. Of course, $\langle p, y\rangle \subseteq \mathrm{H}_p$. If $\mathrm{H}_p = \mathrm{H}_q$, then $\langle q, \langle p, y\rangle\rangle \subseteq \mathcal{O}$, a contradiction as L $\subseteq \langle q, \langle p, y\rangle\rangle$. ■

Lemma 2.17. Let G be a generator of $\mathcal{O}$ through a.

(i) If $\delta = \frac{d-1}{2}$, every hyperplane through G is tangent to $\mathcal{O}$ at a point of G.

(ii) If $\delta = \frac{d-3}{2}$, there exists in H_a a $(\delta+2)$-dimensional subspace U containing G such that every hyperplane through U is tangent to $\mathcal{O}$ at a point of G.

(iii) If $\delta = \frac{d-2}{2}$, there exists in H_a a $(\delta+1)$-dimensional subspace U containing G such that every hyperplane through U is tangent to $\mathcal{O}$ at a point of G.

Proof. If $\delta{=}(d-1)/2$, then the number of points of G is equal to the number of hyperplanes through G. So the statement follows from Proposition 2.16.

Now, assume $\delta{=}(d-3)/2$. Since $\delta \geq 1$, then $d \geq 5$. Consider a $(d-2)$-dimensional subspace S in H_a not through a. Let $\mathcal{O}' = \mathrm{S} \cap \mathcal{O}$ and $\mathrm{S}' = \mathrm{S} \cap \mathrm{G}$. By Proposition 2.5, S′ is a generator of $\mathcal{O}'$. We have $\dim \mathrm{S}' = \delta - 1 {=} (d-5)/2$. By Proposition 2.13, $\mathcal{O}'$ is a non-singular elliptic quadric or an ovoid of S. Consequently, there exists in S a $(\delta+1)$-dimensional subspace U′ such that $\mathcal{O}' \cap \mathrm{U}' = \mathrm{S}'$. Let $\langle a, \mathrm{U}'\rangle{=}\mathrm{U}$. Obviously, $\mathrm{U} \cap \mathcal{O} = \mathrm{G}$. Thus, for any $p \in \mathrm{G}$, $\mathrm{H}_p \supseteq \mathrm{U}$. Since the number of points of G is equal to the number of hyperplanes through U, then the statement follows from Proposition 2.16.

A similar argument to that in Case (ii) can be used to prove the statement in Case (iii) where, with the same notations as in Case (ii), $\mathcal{O}'$ is a Hermitian arc or a non-singular Hermitian variety of S according as $d = 4$ or $d > 4$, respectively. ■

Proposition 2.18. Let $p \in \mathcal{O} \backslash \mathrm{H}_a$. For any line L of $\mathcal{O}$ through a, there exists a unique line of $\mathcal{O}$ through p not skew with L. Moreover, $k_p = k_a$.

Proof. We start by proving that two distinct lines of $\mathcal{O}$ though p both having a point in common with L can not exist. Assume on the contrary that two such lines M and M$'$ exist. Let $\{b\}$=L$\cap$M. Since $\mathrm{M}' \subseteq \mathrm{H}_b$, then $\langle b, \mathrm{M}'\rangle \subseteq \mathcal{O}$, a contradiction as the plane $\langle b, \mathrm{M}'\rangle$ contains the line $\langle a, p\rangle$ which is a secant of $\mathcal{O}$.

Thus, there exists at most one line of $\mathcal{O}$ through p having a point in common with L.

By Lemma 2.15, there exists a generator G of $\mathcal{O}$ through L. First, assume d odd. By Proposition 2.14, only two cases can occur:

(i) $\delta = \frac{d-1}{2}$;

(ii) $\delta = \frac{d-3}{2}$.

Case (i). Consider the [δ-1,0] hyperplanes through $\langle p, \mathrm{G}\rangle$. By Lemma 2.17, each of them is tangent to $\mathcal{O}$ at a point of G. Consequently, there exist [δ-1,0] lines of $\mathcal{O}$ through p intersecting H_a in a point of G$\backslash\{a\}$. Since the lines through a which are contained in G are exactly [δ-1,0] and each of them is not skew with at most one line of $\mathcal{O}$ through p, then there exists exactly one line of $\mathcal{O}$ through p having a point in common with L.

So, the first part of the statement is proved. As an easy consequence, $k_p = k_a$ holds.

Case (ii). By Lemma 2.17, there exists in H_a a $(\delta + 2)$-dimensional subspace U through G such that every hyperplane through $\langle p, \mathrm{U}\rangle$ is tangent to $\mathcal{O}$ at a point of G. From now on, the proof runs in the same way as in Case (i).

If d is even, then, by Proposition 2.14, δ=$(d-2)/2$; so (iii) of Lemma 2.17 occurs. A similar argument as above can be used to prove the statement also in this case. ■

By Propositions 2.12 and 2.18, the number k_p of lines of $\mathcal{O}$ through a point $p \in \mathcal{O}$ does not depend on the particular point p. So, in the sequel we define $k = k_p$, for any $p \in \mathcal{O}$.

Proposition 2.19. Let $p \in \mathcal{O} \backslash \mathrm{H}_a$. For any generator G of $\mathcal{O}$ through a, the union of all lines of $\mathcal{O}$ through p intersecting H_a in a point of G is a generator of $\mathcal{O}$.

Proof. By Proposition 2.18, the statement is true if δ (= dim G) = 1.

Now, assume $\delta > 1$. We start by proving that a generator of $\mathcal{O}$ exists through p intersecting G in a subspace of dimension $\delta - 1$. First, observe that, by Proposition 2.18, a line of $\mathcal{O}$ exists through p whose intersection with H_a is a point of G. Now, proceeding by induction, we assume that a h-dimensional subspace S, $1 \leq h < \delta$, exists through p such that $\mathrm{S} \subseteq \mathcal{O}$ and dim (S$\cap$G) $= h - 1$. Consider a line L in G such that $a \in$ L and L$\cap$S=$\emptyset$. By Proposition 2.18, a line M of $\mathcal{O}$ through p exists such that L$\cap$M $\neq \emptyset$. Let $\{q\}$=L$\cap$M. Since $\{p\} \cup \mathrm{G} \subseteq \mathrm{H}_q$, then $\mathrm{S} \subseteq \mathrm{H}_q$, from which $\langle q, \mathrm{S}\rangle \subseteq \mathcal{O}$. Moreover, dim $\langle q, \mathrm{S}\rangle = h + 1$ and dim $(\langle q, \mathrm{S}\rangle \cap \mathrm{G}) = h$.

Thus, we can affirm that a generator G$'$ of $\mathcal{O}$ exists through p such that dim (G$\cap$G$'$) $= \delta - 1$. Since dim (G$\cap$G$'$) $= \delta - 1$, then the number of lines through p in G$'$ which have a point in common with G is equal to the number c of lines through a contained in G. On the other hand, by Proposition 2.18, c is exactly the number of lines through p whose intersection with G is not empty; so, the statement is proved. ∎

Proposition 2.20. If $d = 3$, then $|\mathcal{O}| = 1 + kn + kn^2 - n^2$ and $s = 1 + kn - n$.

Proof. By Proposition 2.2, there exists a line M of $\mathcal{O}$ through a. Let L be a line through a not in H_a. Since, by Proposition 2.14, M is a generator of $\mathcal{O}$, then, by Lemma 2.17, the plane $\langle \mathrm{L}, \mathrm{M}\rangle$ is tangent to $\mathcal{O}$ at a point $p \in \mathrm{M}\backslash\{a\}$. Since the number of lines of $\mathcal{O}$ through p is k, then L is a k-secant of $\mathcal{O}$.

Thus, every line through a not in H_a is a k-secant of $\mathcal{O}$. Since the number of lines through a not contained in H_a is n^2, then $|\mathcal{O}| = |\mathrm{H}_a \cap \mathcal{O}| + n^2(k-1)$, from which, by Proposition 2.1, $|\mathcal{O}| = 1 + kn + kn^2 - n^2$ follows.

By Proposition 2.3, there exists a line T in H_a such that $\mathrm{T}\cap\mathcal{O} = \{a\}$. Let H be a plane through T distinct from H_a. Since the lines through a contained in H and distinct from T all are secant, then H is not a tangent hyperplane to $\mathcal{O}$. Therefore, $s = |\mathrm{H} \cap \mathcal{O}| = 1 + n(k-1) = 1 + nk - n$ and the statement is proved. ∎

Lemma 2.21. Let $d \geq 4$. There exists an integer h such that, for any $p \in \mathcal{O} \cap (\mathrm{H}_a\backslash\{a\})$, the number of lines of $\mathcal{O}$ through p which are not contained in H_a is equal to h. Moreover, we have that

(i) $h = n^{d-3}$, if $\mathcal{O}$ is of hyperbolic or elliptic type;

(ii) $h = \frac{n^{d-3}\sqrt{n}\,+\,n^{d-2}}{\sqrt{n}\,+\,1}$, if $\mathcal{O}$ is of Hermitian type.

Proof. Let $p \in \mathcal{O} \cap (\mathrm{H}_a\backslash\{a\})$. If L is a line of $\mathcal{O}$ through p distinct from $\langle a, p\rangle$ and contained in H_a, then, obviously, $\langle a, \mathrm{L}\rangle \subseteq \mathcal{O}$. On the other hand, in every plane containing $\langle a, p\rangle$ and contained in $\mathcal{O}$ (and so contained in H_a) there are exactly n lines of $\mathcal{O}$ through p distinct from $\langle a, p\rangle$. It follows that the number r_p of lines of $\mathcal{O}$ through p distinct from $\langle a, p\rangle$ and contained in H_a is nc, where c denotes the number of planes through $\langle a, p\rangle$ which are contained in $\mathcal{O}$.

Now, let S be a $(d-2)$-dimensional subspace in H_a through p not containing a. Denote by g the number of lines through p contained in $\mathcal{O}\cap\mathrm{S}$. Since a plane Π through $\langle a, p\rangle$ is contained in $\mathcal{O}$ if, and only if, $\Pi=\langle a, \mathrm{M}\rangle$, for some line M of $\mathcal{O}$ through p in S, then $c = g$; so,

$$r_p = n\ g. \tag{2.14}$$

Now, assume $\mathcal{O}$ of hyperbolic type (similar arguments apply in the other cases) and denote by l_p the number of all lines of $\mathcal{O}$ through p contained in H_a. Since $g = [d-5, 0] + n^{\frac{d-5}{2}}$, then, by (2.14),

$$l_p = 1 + r_p = [d-4, 0] + n^{\frac{d-3}{2}}. \tag{2.15}$$

On the other hand, by Proposition 2.1,

$$k = [d-3, 0] + n^{\frac{d-3}{2}}. \tag{2.16}$$

Thus, from (2.15) and (2.16) it follows that $k - l_p = n^{d-3}$; so the assertion. ∎

Proposition 2.22. Let $d \geq 4$. We have:

(i) $s = [d-2, 0]$, if $\mathcal{O}$ is of hyperbolic or elliptic type;

(ii) $s = \frac{n^{d-1}-1}{n-1} + \frac{n^{d-1}-(-\sqrt{n})^{d-1}}{\sqrt{n}+1}$, if $\mathcal{O}$ is of Hermitian type.

Moreover,

(iii) $1 + kn - s = \epsilon\, n^{\frac{d-1}{2}}$, where $\epsilon = 1$ if d is odd and $\mathcal{O}$ is of hyperbolic or Hermitian type, whereas $\epsilon = -1$ if d is odd and $\mathcal{O}$ is of elliptic type or d is even (and $\mathcal{O}$ is of Hermitian type).

Proof. Let S be a $(d-2)$-dimensional subspace in H_a not through a. Define $\mathcal{O}'$= $\mathcal{O}\cap$S and consider a hyperplane S$'$ of S not tangent to $\mathcal{O}'$. Let $\mathrm{S}'' = \langle a, \mathrm{S}'\rangle$. Now, consider a hyperplane H of PG(d, n) through S$''$ distinct from H_a. By Proposition 2.13, H is not tangent to $\mathcal{O}$. So, $s = |\mathrm{H}\cap\mathcal{O}|$. Since the number of lines of $\mathcal{O}$ through a not in S$''$ is equal to $|\mathcal{O}'\backslash \mathrm{S}'|$, then $|\mathcal{O}\cap(\mathrm{H}_a\backslash\mathrm{H})| = n\,|\mathcal{O}'\backslash\mathrm{S}'|$. Consequently, the lines of $\mathcal{O}$ not contained in H_a through the points of $\mathcal{O}\cap(\mathrm{H}_a\backslash\mathrm{H})$ are exactly $hn|\mathcal{O}'\backslash\mathrm{S}'|$, h the integer in Lemma 2.21. On the other hand, by Proposition 2.18, there pass exactly $|\mathcal{O}'\backslash\mathrm{S}'|$ lines of $\mathcal{O}$ not contained in H through every point of $\mathcal{O}\cap(\mathrm{H}\backslash\mathrm{H}_a)$. Thus, $|\mathcal{O}\cap(\mathrm{H}\backslash\mathrm{H}_a)| = hn$. Since $|\mathcal{O}\cap\mathrm{H}\cap\mathrm{H}_a| = |\mathcal{O}\cap\mathrm{S}''| = 1 + n|\mathcal{O}'\cap\mathrm{S}'|$, then $s = |\mathcal{O}\cap\mathrm{H}| = 1 + n|\mathcal{O}'\cap\mathrm{S}'| + hn$.

Now, assume $\mathcal{O}$ of hyperbolic type (the proof runs in a similar way in the other cases). Since S$'$ is not tangent to $\mathcal{O}'$,then $|\mathcal{O}'\cap\mathrm{S}'| = [d-4, 0]$; moreover, by Lemma 2.21(i), $h = n^{d-3}$. Therefore, $s = |\mathcal{O}\cap\mathrm{H}| = [d-2, 0]$. Since, by Proposition 2.1, $k = |\mathcal{O}'|$, then (iii) immediately follows from (i). The statement is completely proved. ∎

Proposition 2.23. Let $d \geq 4$. We have:

(i) $|\mathcal{O}| = [d-1, 0] + n^{\frac{d-1}{2}}$, if $\mathcal{O}$ is of hyperbolic type;

(ii) $|\mathcal{O}| = [d-1, 0] - n^{\frac{d-1}{2}}$, if $\mathcal{O}$ is of elliptic type;.

(iii) $|\mathcal{O}| = \frac{n^d-1}{n-1} + \frac{n^d-(-\sqrt{n})^d}{\sqrt{n}+1}$, if $\mathcal{O}$ is of Hermitian type.

Proof. Let S be a $(d-2)$-dimensional subspace in H_a not through a. Define $\mathcal{O}'$= $\mathcal{O}\cap$S. By Proposition 2.1, $k = |\mathcal{O}'|$ and $|\mathrm{H}_a\cap\mathcal{O}| = 1 + n|\mathcal{O}'|$. Let l be the number of lines of $\mathcal{O}$ not contained in H_a. Since $|(\mathrm{H}_a\backslash\{a\})\cap\mathcal{O}| = n|\mathcal{O}'|$, then

$$l = h\; n\; |\mathcal{O}'|, \tag{2.17}$$

h the integer in Lemma 2.21.

Now, count in two ways the point-line pairs (p, L), where $p \in (\mathcal{O}\backslash\mathrm{H}_a)\cap\mathrm{L}$ and $\mathrm{L}\subseteq\mathcal{O}$. Since $k = |\mathcal{O}'|$, we have

$$|\mathcal{O}\backslash\mathrm{H}_a|\; |\mathcal{O}'| = l\; n. \tag{2.18}$$

From (2.17) and (2.18) it follows that $|\mathcal{O}\backslash\mathrm{H}_a| = hn^2$. This implies that

$$|\mathcal{O}| = |\mathcal{O}\cap\mathrm{H}_a| + |\mathcal{O}\backslash\mathrm{H}_a| = 1 + n\; |\mathcal{O}'| + h\; n^2. \tag{2.19}$$

Now, assume $\mathcal{O}$ of hyperbolic type. Since $|\mathcal{O}'| = [d-3, 0] + n^{\frac{d-3}{2}}$, then (i) follows from (2.19) and Lemma 2.21(i). With the help of Lemma 2.21, it is easy to verify that also (ii) and (iii) are consequence of (2.19). So, the statement is completely proved. ∎

Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. By Proposition 2.3, $0 \leq \delta \leq d-2$.
The statement follows from Proposition 2.4 in the following cases:
-$d = 2$;
-$d \geq 3$ and $\delta = 0$.

Now, assume $d \geq 3$ and $\delta \geq 1$. Let $q \in \mathcal{O} \backslash \mathrm{H}_a$. By Propositions 2.2 and 2.19, there exists a generator G of $\mathcal{O}$ through q. Denote by r the number of all point-hyperplane pairs (p,H), where $p \in (\mathcal{O}\backslash \mathrm{G}) \cap \mathrm{H}$ and $\mathrm{G} \subseteq \mathrm{H}$. We have

$$r = (|\mathcal{O}| - |\mathrm{G}|)\ [d-2-\delta, 0]. \tag{2.20}$$

Let t be the number of hyperplanes through G tangent to $\mathcal{O}$. By Proposition 2.12, every tangent hyperplane to $\mathcal{O}$ has exactly $1 + nk$ points in common with $\mathcal{O}$. Since the number of points that $\mathcal{O}$ shares with every non-tangent hyperplane is s, then

$$r = t\ (1 + nk - |\mathrm{G}|) + ([d-1-\delta, 0] - t)\ (s - |\mathrm{G}|). \tag{2.21}$$

From (2.20) and (2.21) it follows that

$$|\mathcal{O}|\ [d-2-\delta, 0] + |\mathrm{G}|\ n^{d-1-\delta} = t\ (1 + nk - s) + s\ [d-1-\delta, 0]. \tag{2.22}$$

By Proposition 2.14, only three cases can occur:

(i) $d \geq 3$ odd, $\delta = \frac{d-1}{2}$ and, if $d \geq 5$, $\mathcal{O}$ of hyperbolic or Hermitian type;

(ii) $d \geq 5$ odd, $\delta = \frac{d-3}{2}$ and $\mathcal{O}$ of elliptic type;

(iii) $d \geq 4$ even, $\delta = \frac{d-2}{2}$ and $\mathcal{O}$ of Hermitian type.

If $d = 3$, then, by (2.22) and Proposition 2.20, $t = |\mathrm{G}|$. The same result holds if $d \geq 5$ as a consequence of (2.22) and Propositions 2.22 and 2.23. Thus, the number of the tangent hyperplanes to $\mathcal{O}$ containing G is equal to the number of points of G. On the other hand, if H_p is a hyperplane through G tangent to $\mathcal{O}$, then, by Proposition 2.2, $p \in \mathrm{G}$. Therefore, we can affirm that, for any $p \in \mathrm{G}$, there exists the tangent hyperplane to $\mathcal{O}$ at p. So, there exists the tangent hyperplane to $\mathcal{O}$ at the point q.

Thus, for any $p \in \mathcal{O}$, there exists the tangent hyperplane to $\mathcal{O}$ at p. Then, the statement follows from Theorem 1.1.

Case (ii). Let $\mathrm{G}' = \mathrm{G} \cap \mathrm{H}_a$. We start by proving that there exists a hyperplane through G tangent to $\mathcal{O}$ at a point of $\mathrm{G} \backslash \mathrm{G}'$. Assume on the contrary that such a hyperplane does not exist. Consequently, by Proposition 2.2, if H_p is a tangent hyperplane to $\mathcal{O}$ through G, then $p \in \mathrm{G}'$. So, by Proposition 2.16,

$$t = [\tfrac{d-5}{2}, 0]. \tag{2.23}$$

From (2.22), (2.23) and Propositions 2.22 and 2.23, it follows that $n^{d-2} = 0$, a contradiction.

Thus, there exists through G the tangent hyperplane to $\mathcal{O}$ at $\bar{q}$, for some point $\bar{q} \in \mathrm{G} \backslash \mathrm{G}'$. Define $\mathrm{S}' = \mathrm{H}_{\bar{q}} \cap \mathrm{H}_a$. Obviously, $a \notin \mathrm{S}'$ and $\mathrm{G}' \subseteq \mathrm{S}'$. Let $\mathcal{O}' = \mathcal{O} \cap \mathrm{S}'$. By Proposition 2.5, G' is a generator of $\mathcal{O}'$. Since $\mathcal{O}'$ is a non-singular elliptic quadric of S′, then there exists in S′ a subspace $\bar{\mathrm{S}}$ through G′ such that dim $\bar{\mathrm{S}}$ = 2 + dim G′ = $\delta+1$ and $\bar{\mathrm{S}}\cap\mathcal{O}'$=G′. Let $\mathrm{S}'' = \langle \mathrm{G}, \bar{\mathrm{S}}\rangle$. We have dim S″= $\delta+2$. Since $\bar{\mathrm{S}}\cap\mathcal{O}'$=G′, then every line joining $\bar{q}$ with a point of S″\G intersects $\mathcal{O}$ in the only point $\bar{q}$. It follows that $\mathrm{S}'' \cap \mathcal{O} = \mathrm{G}$.

Now, count in two ways the point-hyperplane pairs (p,H), where $p \in (\mathcal{O}\backslash\mathrm{S}'')\cap\mathrm{H}$ and S″⊆H. If $\bar{t}$ denotes the number of the tangent hyperplane to $\mathcal{O}$ through S″, then, by Proposition 2.12, we have

$$|\mathcal{O}\backslash\mathrm{S}''|\ [\tfrac{d-5}{2}, 0] = \bar{t}\ (1 + kn - |\mathrm{G}|) + ([\tfrac{d-3}{2}, 0] - \bar{t})(s - |\mathrm{G}|). \tag{2.24}$$

Since $|\mathcal{O}\backslash\mathrm{S}''| = |\mathcal{O}| - |\mathrm{G}|$, then (2.24) implies that

$$|\mathcal{O}|[\tfrac{d-5}{2}, 0] + |\mathrm{G}|\ n^{\frac{d-3}{2}} = \bar{t}(1 + kn - s) + s[\tfrac{d-3}{2}, 0],$$

from wich, by Propositions 2.22 and 2.23, $\bar{t} = [\frac{d-3}{2}, 0] = |\mathrm{G}|$ follows.

Thus, the tangent hyperplanes to $\mathcal{O}$ through S″ are so many as the points of G. So, by Proposition 2.2, there exists the tangent hyperplane to $\mathcal{O}$ at p, for any $p \in$ G. In particular, there exists the tangent hyperplane to $\mathcal{O}$ at q.

Thus, there exists the tangent hyperplane to $\mathcal{O}$ at p, for any $p \in \mathcal{O}$. Again, the statement follows from Theorem 1.1.

Case (iii). The proof runs in a similar way as in Case (ii). We just recall that, if $\mathcal{O}'$ is a Hermitian arc or a non-singular Hermitian variety of a projective space S′ of even dimension and G′ is a generator of $\mathcal{O}'$, then there exists in S′ a subspace $\bar{\mathrm{S}}$ through G′ such that $\bar{\mathrm{S}} \cap \mathcal{O}' = \mathrm{G}'$ and dim $\bar{\mathrm{S}}$=1+dim G′.

The statement is completely proved.

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