On cup products in some manifolds[*]

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1   Introduction

Results on cup products (we shall consider them only in $\mathbb{Z}_2$-cohomology) can be useful for instance when studying the Lyusternik-Shnirel'man category (cf. James [7]). Namely, by Eilenberg's cup product theorem, if one finds a nontrivial cup product in $H^*(X; \mathbb{Z}_2)$, then its length gives a lower bound for the Lyusternik-Shnirel'man category of the topological space X. In this context, one is naturally interested in the maximum length of nontrivial cup-products in the $\mathbb{Z}_2$-algebra $H^*(X; \mathbb{Z}_2)$; this length is called the cup-length of X, and will be denoted by $\mathrm{cup}(X)$.

In general, $\mathrm{cup}(X)$ is not easily calculable, and its value remains unknown for many important spaces. In view of this, it is good to know how the knowledge of $\mathrm{cup}(X)$ for some X can be used to obtain at least some piece of information also on the cup-length of other spaces somehow related to X. A standard procedure of this type is the following: if $\mathrm{cup}(X)$ is known, and some space Y fibers over X in such a way that to the fibration $p : Y \to X$ we can apply the Leray-Hirsch theorem, then we have $\mathrm{cup}(Y) \geqq \mathrm{cup}(X)$. The reason is clear: in this situation the induced cohomology homomorphism $p^* : H^*(X; \mathbb{Z}_2) \to H^*(Y; \mathbb{Z}_2)$ is a monomorphism. For instance, this procedure can be applied to manifolds suitably fibered over some of the Grassmann manifolds $G_{n,k}$ of all k-dimensional vector subspaces in $\mathbb{R}^n$: if $k \leqq 4$, then for these Grassmannians their cup-length is known, due to R. E. Stong [14].

Speaking about $G_{n,k}$, we shall suppose that $2k \leq n$, because $G_{n,k}$ and $G_{n,n-k}$ are diffeomorphic.

In this note we mainly shall consider smooth fiber bundles with a Grassmannian as fiber and a closed connected manifold as total space. It turns out that for such fiber bundles the known results on cup products in Grassmann manifolds can often be employed for obtaining information on the cup-length in their total spaces. In particular, for n odd we show that the fiber $G_{n,k}$ of such a smooth fiber bundle is totally non-homologous to zero, and therefore (see e.g. Serre [12]) they obey also the Leray-Hirsch theorem.

The results presented here could also serve as a source of conditions under which manifolds cannot be smoothly fibered if the fiber should be a prescribed Grassmannian.

2   The main result and some corollaries

Let $w_i := w_i(\gamma_{n,k}) \in H^i(G_{n,k}; \mathbb{Z}_2)$ be the i-th Stiefel-Whitney class of the canonical k-plane bundle over $G_{n,k}$ ($2k \leq n$). Let us recall that the height of w_1 in $H^*(G_{n,k}; \mathbb{Z}_2)$ is

$$\text{height}(w_1) := \sup\{m; w_1{}^m \neq 0\},$$

and for $2^s < n \leq 2^{s+1}$ one has (see Stong [14])

$$\text{height}(w_1) = \begin{cases} n - 1 & \text{if } k = 1, \\ 2^{s+1} - 2 & \text{if } k = 2 \text{ or if } k = 3 \text{ and } n = 2^s + 1, \\ 2^{s+1} - 1 & \text{otherwise.} \end{cases}$$

Now our main result is the following.

Theorem Let $p : E \longrightarrow B$ be a smooth fiber bundle with E a closed connected manifold and with fiber the Grassmann manifold $G_{n,k}$ ($2 \leq 2k \leq n$).

(a) If each power of 2 dividing n divides also k, then the induced map $p^* : H^*(B; \mathbb{Z}_2) \longrightarrow H^*(E; \mathbb{Z}_2)$ is a monomorphism and $\text{cup}(E) \geq \text{cup}(B)$.

(b) If n is odd, then the fiber $G_{n,k}$ is totally non-homologous to zero in E (in other words, if $i : G_{n,k} \longrightarrow E$ is the fiber inclusion, then $i^* : H^*(E; \mathbb{Z}_2) \longrightarrow H^*(G_{n,k}; \mathbb{Z}_2)$ is an epimorphism).

(c) If n is odd, then one has $\text{cup}(E) \geq \text{cup}(G_{n,k}) + \text{cup}(B)$.

(d) If $n \equiv 2 \pmod 4$ and k is odd or if $n \equiv 0 \pmod 4$ and k is even, then one has $\text{cup}(E) \geq [\frac{1}{2} \text{height}(w_1)]$.

For $G_{n,2}$ ($n \geq 4, 2^s < n \leq 2^{s+1}$), H. Hiller [5] proved that $w_2{}^{n-2} \neq 0$ and that

$$w_2{}^{n-2} = w_1{}^{2^{s+1}-2} w_2{}^{\frac{1}{2}(2n-4-2^{s+1}+2)} = w_1{}^{2^{s+1}-2} w_2{}^{n-2^s-1}$$

in the top dimension. Since $\text{height}(w_1) = 2^{s+1} - 2$ in this case, one obtains

$$\text{cup}(G_{n,2}) = n + 2^s - 3.$$

In addition to this, for $2^s < n \leq 2^{s+1}$ and $n \geq 2k$, we recall from Stong's [14]:

$$\mathrm{cup}(G_{n,k}) = \begin{cases} 2^{s+2} - 3 \cdot 2^{p-1} - 2 + t & \text{if } k = 3, n = 2^{s+1} - 2^p + 2 + t, \\ & \quad 0 \leq t \leq 2^{p-1} - 2, p \geq 1; \\ 2^{s+2} - 3 \cdot 2^{p-1} - 4 & \text{if } k = 3, n = 2^{s+1} - 2^p + 1, p \geq 1; \\ 2^{s+1} + 2^s - 7 & \text{if } k = 4, n = 2^s + 1; \\ 2^{s+1} + 2^s + 2^{r+1} - 7 + j & \text{if } k = 4, n = 2^s + 2^r + 1 + j, \\ & \quad s > r \geq 0, 0 \leq j \leq 2^r - 1; \\ 2^{s+1} + 2^s + 2^{s-2} - 9 & \text{if } k = 5, n = 2^s + 1. \end{cases}$$

We can now give several examples of corollaries of Theorem; they readily follow from Theorem (c) combined with the above facts.

Corollary 1 Let $p : E \longrightarrow B$ be a smooth fiber bundle with E a closed connected manifold and with fiber the Grassmann manifold $G_{n,2}$ ($n \geq 4$). Let $2^s < n \leq 2^{s+1}$. If n is odd, then we have

$$\mathrm{cup}(E) \geq n + 2^s - 3 + \mathrm{cup}(B).$$

Corollary 2 Let $p : E \longrightarrow B$ be a smooth fiber bundle with E a closed connected manifold and with fiber the Grassmann manifold $G_{n,3}$ ($n \geq 6$). Let $2^s < n \leq 2^{s+1}$.
(i) If $n = 2^{s+1} - 2^p + 2 + t$, for some odd t, $1 \leq t \leq 2^{p-1} - 3$, and $p \geq 1$, then we have

$$\mathrm{cup}(E) \geqq 2^{s+2} - 3 \cdot 2^{p-1} - 2 + t + \mathrm{cup}(B).$$

(ii) If $n = 2^{s+1} - 2^p + 1$ for some $p \geq 1$, then we have

$$\mathrm{cup}(E) \geq 2^{s+2} - 3 \cdot 2^{p-1} - 4 + \mathrm{cup}(B).$$

Corollary 3 Let $p : E \longrightarrow B$ be a smooth fiber bundle with E a closed connected manifold and with fiber the Grassmann manifold $G_{n,4}$ ($n \geq 8$).
(a) If $n = 2^s + 1$, then we have

$$\mathrm{cup}(E) \geq 2^{s+1} + 2^s - 7 + \mathrm{cup}(B).$$

(b) If $n = 2^s + 2^r + 1 + j$ ($s > r \geq 0, 0 \leq j \leq 2^r - 1$) is odd, then we have

$$\mathrm{cup}(E) \geq 2^{s+1} + 2^s + 2^{r+1} - 7 + j + \mathrm{cup}(B).$$

Corollary 4 Let $p : E \longrightarrow B$ be a smooth fiber bundle with E a closed connected manifold and with fiber the Grassmann manifold $G_{n,5}$ ($n \geq 10$).
If $n = 2^s + 1$, then $\mathrm{cup}(E) \geq 2^{s+1} + 2^s + 2^{s-2} - 9 + \mathrm{cup}(B)$.

Remark There are plenty of smooth fiber bundles with fiber $G_{n,k}$. For instance, let G be any compact Lie group containing $H := O(n) \times O(k_1) \times \cdots \times O(k_q)$ as its closed subgroup. Of course, $K := O(k) \times O(n - k) \times O(k_1) \times \cdots \times O(k_q)$ is

a closed subgroup of H, and the map $p : G/K \longrightarrow G/H$, $p(aK) = aH$ (where $a \in G$) defines a smooth fiber bundle with fiber $H/K \cong O(n)/(O(k) \times O(n - k))$ which can be identified with the Grassmann manifold $G_{n,k}$. Note that if $G = O(n + k_1 + \cdots + k_q)$, then G/K is the flag manifold of type $(k, n - k, k_1, \ldots, k_q)$ (see e.g. Korbaš, Zvengrowski [9, 3.1]). Another explicit family of fiber bundles with fiber $G_{n,k}$ will be presented in Section 4.

3   Proof of Theorem

For the Grassmann manifold $G_{n,k}$ write $n = \sum_{i \geqq 0} n_i 2^i$, $k = \sum_{i \geqq 0} k_i 2^i$ as the dyadic expansions of n and k. Then the first two Stiefel-Whitney classes of $G_{n,k}$ (that is, of its tangent bundle $TG_{n,k}$) can be expressed by the following formulae (Bartík, Korbaš [1]):

$$
\begin{aligned}
w_1(G_{n,k}) &= n_0 w_1, \\
w_2(G_{n,k}) &= (1 + n_1 + k_0) w_1^2 + n_0 w_2.
\end{aligned}
$$

In addition to this, if n is odd, then (Bartík, Korbaš [1, 3.6.2] or Korbaš [8, 1.1]) for $i \leq k \leq n - k$ the Stiefel-Whitney class $w_i(G_{n,k})$ can uniquely be expressed as

$$
w_i(G_{n,k}) = w_i + P(w_1, \ldots, w_{i-1}), \tag{*}
$$

where P is a $\mathbb{Z}_2$-polynomial.

We also shall need the following.

Fact If $p : E \longrightarrow B$ is a smooth fiber bundle with fiber the Grassmann manifold $G_{n,k}$ and $i : G_{n,k} \longrightarrow E$ is the fiber inclusion, then one has

$$
i^*(w_j(E)) = w_j(G_{n,k}) \tag{**}
$$

for all j.

Indeed, as is well known (see Borel, Hirzebruch [3, 7.4, 7.6])

$$
TE \cong p^*(TB) \oplus \eta,
$$

where η is a real vector bundle over E, whose fibers are the tangent spaces to the fibers $p^{-1}(b) \cong G_{n,k}$ ($b \in B$); η is called the bundle along the fibers. Since the induced vector bundle $i^*p^*(TB) = (p \circ i)^*(TB)$ is isomorphic to the trivial $\dim(B)$-plane bundle $\varepsilon^{\dim(B)}$, we have $i^*(TE) \cong \varepsilon^{\dim(B)} \oplus TG_{n,k}$. That gives (**).

Now we are able to prove Theorem.

Proof of Theorem (a). By P. Sankaran [11] (cf. also Bartík, Korbaš [2]), if each power of 2 dividing n divides also k, then the Grassmann manifold $G_{n,k}$ is not a boundary. Hence by D. Gottlieb [4, Corollary 4], the induced cohomology homomorphism $p^* : H^*(B; \mathbb{Z}_2) \longrightarrow H^*(E; \mathbb{Z}_2)$ is a monomorphism.   ∎

Proof of Theorem (b). Now we have n odd. An easy induction shows that

$$w_i = w_i(G_{n,k}) + P_i(w_1(G_{n,k}), \ldots, w_{i-1}(G_{n,k})), \qquad (***)$$

where P_i is a polynomial, for $i = 1, \ldots, k$. Indeed, we have $w_1 = w_1(G_{n,k})$. Then by (*) for $j \geq 1$ one has

$$w_j = w_j(G_{n,k}) + \text{a polynomial in } w_1, \ldots, w_{j-1}.$$

But the induction hypothesis then implies that

$$w_j = w_j(G_{n,k}) + P_j(w_1(G_{n,k}), \ldots, w_{j-1}(G_{n,k}))$$

for some polynomial P_j.

As is well-known, the cohomology algebra $H^*(G_{n,k}; \mathbb{Z}_2)$ is generated by the Stiefel-Whitney classes $w_1, \ldots, w_k$. Hence, by what we have just shown, in case of n odd all of this algebra is generated by $w_1(G_{n,k}), \ldots, w_k(G_{n,k})$ as well. But by (**) $w_1(G_{n,k}), \ldots, w_k(G_{n,k})$ lie in the image of the $\mathbb{Z}_2$-algebra homomorphism $i^* : H^*(E; \mathbb{Z}_2) \longrightarrow H^*(G_{n,k}; \mathbb{Z}_2)$. Therefore i^* is an epimorphism. $\blacksquare$

Proof of Theorem (c) In view of Theorem (b), Theorem (c) will be proved if we verify the following lemma.

Lemma Let $p : E \longrightarrow B$ be a smooth fiber bundle with connected base B and connected fiber F. Suppose that the fiber is totally non-homologous to zero. Then one has $\text{cup}(E) \geq \text{cup}(F) + \text{cup}(B)$.

Proof. The cup-length $\text{cup}(F) := r$ is given by some nonzero cup product $a_1 \cdots a_r \in H^*(F; \mathbb{Z}_2)$, and the cup-length $\text{cup}(B) := s$ by some nonzero product $b_1 \cdots b_s \in H^*(B; \mathbb{Z}_2)$. The ring homomorphism $i^* : H^*(E; \mathbb{Z}_2) \longrightarrow H^*(F; \mathbb{Z}_2)$, induced by the fiber inclusion $i : F \longrightarrow E$, is now surjective. Hence there exist $a_1', \ldots, a_r' \in H^*(E; \mathbb{Z}_2)$ with $i^*(a_j') = a_j$, and $i^*(a_1' \cdots a_r') = a_1 \cdots a_r$.

Since the Leray-Hirsch theorem now applies, $H^*(E)$ is a free $H^*(B)$-module, with the action defined by $p^* : H^*(B; \mathbb{Z}_2) \longrightarrow H^*(E; \mathbb{Z}_2)$, and the nonzero product $a_1' \cdots a_r'$ can be taken as an element of its basis. Since p^* is injective, we have $p^*(b_1 \cdots b_s) = p^*(b_1) \cdots p^*(b_s) \neq 0$. Hence the cup product

$$p^*(b_1 \cdots b_s)a_1' \cdots a_r' = p^*(b_1) \cdots p^*(b_s)a_1' \cdots a_r' \in H^*(E; \mathbb{Z}_2)$$

is nonzero, and $\text{cup}(E) \geq s + r$. This closes the proof of Lemma and Theorem (c).$\blacksquare$

Proof of Theorem (d). If $n \equiv 2 \pmod 4$ and k is odd, or if $n \equiv 0 \pmod 4$ and k is even, then $w_2(G_{n,k}) = w_1^2$. Hence (see (**)) we obtain $i^*(w_2^{[\frac{1}{2}\,\text{height}(w_1)]}(E)) = w_2^{[\frac{1}{2}\,\text{height}(w_1)]}(G_{n,k}) \neq 0$, and consequently $w_2^{[\frac{1}{2}\,\text{height}(w_1)]}(E) \neq 0$. The proof of Theorem is complete. $\blacksquare$

4   Remarks

Remark 1 If for some $j < k(n-k) = \dim(G_{n,k})$ some element $a \in H^j(G_{n,k}; \mathbb{Z}_2)$ were a cup-product of the maximum length, then by Poincaré duality (Milnor, Stasheff [10, 11.10]) there would exist a nonzero element $b \in H^{k(n-k)-j}(G_{n,k}; \mathbb{Z}_2)$ such that $ab \neq 0$. But then the length of ab would exceed the length of a, which is a contradiction.

Since, again by Poincaré duality, $H^{k(n-k)}(G_{n,k}; \mathbb{Z}_2) \cong \mathbb{Z}_2$, the above reasoning implies that any cup-product of the maximum length in $H^*(G_{n,k}; \mathbb{Z}_2)$ must be a nonzero monomial $w_1^{t_1} \cdots w_k^{t_k} \in H^{k(n-k)}(G_{n,k}; \mathbb{Z}_2)$. If n is odd, then this $w_1^{t_1} \cdots w_k^{t_k}$ can be written (see the proof of Theorem (b)) as

$$w_1^{t_1}(G_{n,k}) \cdots (w_k(G_{n,k}) + P_k(w_1(G_{n,k}), \ldots, w_{k-1}(G_{n,k})))^{t_k}.$$

The latter is nothing but

$$w_1^{t_1}(G_{n,k}) \cdots w_k^{t_k}(G_{n,k}),$$

because now all cup-products longer than $t_1 + \cdots + t_k$ vanish.

Hence (see (**)) we have

$$w_1^{t_1} \cdots w_k^{t_k} = i^*(w_1^{t_1}(E) \cdots w_k^{t_k}(E)),$$

and of course $w_1^{t_1}(E) \cdots w_k^{t_k}(E)$ is then a nonzero cup-product in $H^*(E; \mathbb{Z}_2)$.

Remark 2 Theorem (b) might appear to be closely related to Theorem C in Shiga, Tezuka [13] (cf. also their note added in proof, p. 106). We can identify

$$G_{n,k} \cong O(n)/(O(k) \times O(n-k)) \cong SO(n)/S(O(k) \times O(n-k)),$$

where $S(O(k) \times O(n-k)) = SO(n) \cap (O(k) \times O(n-k))$. But the order of the Weyl group of $SO(n)$ $(n \geq 3)$ is even (see e.g. Husemoller [6]), and therefore Theorem C of [13] does not say anything in the situations considered in this note.

Remark 3 It seems that possibilities of extending Theorem (b) are quite limited. The following example shows that it does not extend to those $G_{n,k}$ with vanishing Euler-Poincaré characteristic, that is to those with n even and k odd.

Example Let n be even, k odd, $(n, k) \neq (2, 1)$, and let $m \geq 2$. Consider $\mathbb{R}^n$ as $\mathbb{C}^{\frac{n}{2}}$. Then multiplication by i defines a linear automorphism $I : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $I^2 = -\mathrm{id}$, and there is an involution $T : G_{n,k} \longrightarrow G_{n,k}$, $T(D) = I(D)$.

The involution T is free: if $T(D)$ were D for some $D \in G_{n,k}$, then the restriction $I|_D$ would be a linear automorphism of the odd-dimensional vector space D with

$$(\det(I|_D))^2 = \det((I|_D)^2) = \det(-\mathrm{id} : D \longrightarrow D) = -1.$$

Now let $(-1) : S^m \longrightarrow S^m$ denote the antipodal involution on the m-sphere S^m. Let $X := (G_{n,k} \times S^m)/T \times (-1)$ be the obvious quotient manifold. One readily verifies that the map

$$p : X \longrightarrow S^m/(-1) = \mathbb{R}P^m,$$

$$p([(D, x)]) = [x],$$

defines a smooth fiber bundle with fiber $G_{n,k}$.

The fiber $G_{n,k}$ is not totally non-homologous to zero in X. Indeed, if $G_{n,k}$ were totally non-homologous to zero, then by the Leray-Hirsch theorem $H^1(X; \mathbb{Z}_2)$ would contain at least two different nonzero elements, because $H^1(G_{n,k}; \mathbb{Z}_2)$ and $H^1(\mathbb{R}P^m; \mathbb{Z}_2)$ are both isomorphic to $\mathbb{Z}_2$. But in fact we have $H^1(X; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

To see the latter, take another smooth fiber bundle with the same total space X,

$$q : X \longrightarrow G_{n,k}/T,$$

$$q([(D, x)]) = [D].$$

Its fiber is S^m; note that $G_{n,k}/T$ is a smooth manifold, double covered by $G_{n,k}$. Now the homotopy exact sequence of the fibration $q : X \longrightarrow G_{n,k}/T$ gives that the fundamental group $\pi_1(X)$ is isomorphic to $\pi_1(G_{n,k}/T)$. On the other hand, the Grassmann manifold $\tilde{G}_{n,k}$ of oriented k-dimensional vector subspaces in $\mathbb{R}^n$ is a 4-fold universal covering of the manifold $G_{n,k}/T$. The map $\tilde{T} : \tilde{G}_{n,k} \longrightarrow \tilde{G}_{n,k}$ defined by multiplication by i is an automorphism of order four of this universal covering, with $\tilde{T}^2 = -\mathrm{id}$. Hence $\pi_1(G_{n,k}/T)$ contains an element of order four, and we have $\pi_1(G_{n,k}/T) \cong \mathbb{Z}_4 \cong \pi_1(X)$. From this then $H_1(X; \mathbb{Z}_2) \cong \mathbb{Z}_4$, and $H^1(X; \mathbb{Z}_2) \cong \mathrm{Hom}(\mathbb{Z}_4, \mathbb{Z}_2) \cong \mathbb{Z}_2$, as claimed.

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