Caps embedded in the Klein quadric

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1 Introduction

Let $PG(N, q)$ be the projective space of dimension N over the finite field $GF(q)$. A *k–cap* K in $PG(N, q)$ is a set of k points, no three of which are collinear [14], and a k–cap is called *complete* if it is maximal with respect to set–theoretic inclusion. The maximum value of k for which there exists a k–cap in $PG(N, q)$ is denoted by $m_2(N, q)$ [14]. This number $m_2(N, q)$ is only known, for arbitrary q, when $N \in \{2, 3\}$. Namely, $m_2(2, q) = q + 1$ if q is odd, $m_2(2, q) = q + 2$ if q is even, and $m_2(3, q) = q^2 + 1$, $q > 2$. With respect to the other values of $m_2(N, q)$, apart from $m_2(N, 2) = 2^N$, $m_2(4, 3) = 20$, $m_2(5, 3) = 56$ and $m_2(4, 4) = 41$ [2], only upper bounds are known. Finding the exact value for $m_2(N, q)$, $N \geq 4$ and constructing an $m_2(N, q)$–cap seems to be a very hard problem. In the last few years there has been a certain interest in caps embedded in the Klein quadric $\mathcal{K}$ of $PG(5, q)$ considered as ambient space, and the main purpose is to find lower and upper bounds for a complete cap embedded in $\mathcal{K}$. In this direction, Blokhuis and Sziklai [3] proved a lower bound for the smallest complete cap of the Klein quadric. Precisely such a cap has size at least $const{\cdot}q^{12/7}$. In 1997, Cossidente, Hirschfeld and Storme [8] constructed a cap of size $2q^2 + q + 1$ of $\mathcal{K}$ obtained by gluing together two suitable Veronese surfaces. If we assume q even, it is always possible to extend such a cap to a complete $2(q^2 + q + 1)$–cap of $\mathcal{K}$ [5]. This seems to be the unique known example of smallest complete cap of $\mathcal{K}$. On the other hand Glynn [12] proved (using the Klein correspondence between lines of $PG(3, q)$ and points of $PG(5, q)$) that any line orbit of a Singer cyclic group of $PG(3, q)$ corresponds to a cap of size

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$q^3 + q^2 + q + 1$ embedded in $\mathcal{K}$. Also, he observed that for q odd the above caps are maximally embedded in $\mathcal{K}$ and left the q even case as an open problem. Note that a standard double counting argument on flags and the value of $m(2, q)$ for q even give that the maximum number of lines in $PG(3, q)$ with no three in a planar pencil is $(q^2 + 1)(q + 2)$. Such a value can be taken as a theoretic upper bound. In a recent paper by Ebert, Metsch and Szőnyi [11] it is shown that actually for q even, caps of size larger than $q^3 + q^2 + q + 1$ can be embedded in $\mathcal{K}$, and the authors constructed caps whose deficiency from the theoretic upper bound is $q + 1$. Also they constructed maximal caps of size $q^3 + q^2 \pm r(q+1) - q + 1$ of $\mathcal{K}$, for each value of q, being $q = 2^h$, $h \geq 3$ and $r = 2^{(h+1)/2}$. These caps are bigger then Glynn's caps. In this paper we describe the geometric structure of such caps which turn out to be set–theoretic unions of $q + 1$ Suzuki–Tits ovoids and one elliptic quadric. Our approach uses Singer cyclic groups and the geometric setting used to prove the isomorphism between the groups $Sp(4, q)$ and $O(5, q)$, q even, as explained in [21].

2 Definitions and preliminary results

Denote by α a linear collineation of $PG(n, q)$, the projective space of dimension n over the Galois field $GF(q)$. Assume that α has a matrix representation $A = (a_{i,j})$, $i, j = 0, 1, \ldots, n$. The *second exterior power* of A, denoted by $\Lambda^{(2)}(A)$ is a matrix of order $\binom{n+1}{2}$ whose rows and columns are denoted by $01, 02, \ldots, 0n; 12, \ldots 1n; n-1n$, and occour in this order, where the element in row ij and column rs is

$$a_{ij,rs} = a_{ir}a_{js} - a_{is}a_{jr},$$

namely, the entries of $\Lambda^{(2)}(A)$ are the 2×2 submatrices of A arranged in lexicographical order. The second exterior power of the collineation α is a linear collineation of the projective space $PG(\binom{n+1}{2}-1, q)$, of the same order of α, which leaves the Grassmannian $\mathcal{G}_{1,n}$ of lines of $PG(n, q)$ invariant, and each collineation of $PG(\binom{n+1}{2}-1, q)$ which leaves the Grassmannian $\mathcal{G}_{1,n}$ of lines of $PG(n, q)$ invariant, comes from a collineation of $PG(n, q)$ [13]. Here we are interested in the second exterior power of a Singer cycle of $PG(3, q)$.

Let ω be a primitive element of $GF(q^4)$ over $GF(q)$ and let $f(x) = x^4 - a_3x^3 - a_2x2 - a_1x - a_0$ be its minimal polynomial over $GF(q)$. The companion matrix $C(f)$ of f given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_0 & a_1 & a_2 & a_3 \end{pmatrix}$$

represents a Singer cycle of $PG(3, q)$. The second exterior power of $C(f)$ is represented by the following matrix

$$\Lambda^{(2)}(C(f)) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -a_0 & 0 & 0 & a_2 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -a_0 & 0 & -a_1 & 0 & a_3 \\ 0 & 0 & -a_0 & 0 & -a_1 & -a_2 \end{pmatrix},$$

and induces a linear collineation $\overline{\Lambda^{(2)}(C(f))}$ of $PG(5,q)$ leaving a Klein quadric $\mathcal{K}=\mathcal{G}_{1,3}$ (the Grassmannian of lines of $PG(3,q)$) invariant. In [6] the canonical form of $\Lambda^{(2)}(C(f))$ was studied. It is the following matrix of $GL(6,q^4)$:

$$\Lambda^{(2)}(D)=diag(\omega^{q+1},\omega^{q^2+1},\omega^{q^3+1},\omega^{q^2+q},\omega^{q^3+q},\omega^{q^3+q^2}),$$

where ω is a primitive element of $GF(q^4)$. In particular, it follows that $\Lambda^{(2)}(C(f))$ has a rational form, say J which is a diagonal block matrix. In particular, it is the direct sum of the companion matrix of a primitive quadratic polynomial over $GF(q)$ and the $(q+1)$–th power of the companion matrix of a primitive quartic polynomial over $GF(q)$. Of course, $\overline{\Lambda^{(2)}(C(f))}$ has order $(q^2+1)(q+1)$. From a geometric point of view, the first block induces a Singer cycle on a projective line l of $PG(5,q)$. The second block fixes (setwise) a solid L of $PG(5,q)$ inducing a partition into elliptic quadrics [10]. So $\overline{\Lambda^{(2)}(C(f))}$ fixes one line l and its conjugate solid L with respect to τ. This is exactly the geometric setting used in [21] to prove the isomorphism between the groups $Sp(4,q)$ and $O(5,q)$, q even.

Remark 1. For completeness and for our future purposes we illustrate the gemetry in which we are moving. In [6] we noted that in $PG(3,q)$ the group fixing setwise a pencil C of linear complexes generated by two linear complexes, say C_1 and C_2 contains a Singer cyclic group of $PG(3,q)$, see also [18]. Note that the base locus of C is an elliptic congruence $\mathcal{E}_3$ of $PG(3,q)$. Using the Klein representation of lines of $PG(3,q)$ as points of $PG(5,q)$, we get that the pencil of linear complexes C is represented by a line l skew to the Klein quadric $\mathcal{K}$. Each point of l is conjugate with respect to $\mathcal{K}$ to a hyperplane H whose intersection with $\mathcal{K}$ (a parabolic quadric) represents the lines of the corresponding linear complex belonging to C. The argument applies to all points of l, giving a pencil of hyperplanes centered in a solid L (conjugate to l) which meets $\mathcal{K}$ in an elliptic quadric since the congruence is elliptic.

The subgroup $\langle\overline{\Lambda^{(2)}C(f)}\rangle$ has one orbit of size $q+1$ (the line l), $q+1$ orbits of size q^2+1 (such orbits partition L) and all the other orbits have size $(q+1)(q^2+1)$ and are caps as described in [12]. It is easily seen that L meets $\mathcal{K}$ in an elliptic quadric $\mathcal{E}$, representing the elliptic congruence $\mathcal{E}_3$ (a regular spread of $PG(3,q)$ [13]). Hence the Klein quadric can be partitioned into q caps of size $(q+1)(q^2+1)$ and one elliptic quadric of L. For further details see [6].

Here we are interested in the action of the unique subgroup of $\langle\Lambda^{(2)}(C(f))\rangle$ of order $q+1$. Denote by H such a subgroup. It is easily seen that the linear transformation $(\Lambda^{(2)}(C(f)))^{q^2+1}$ has the following canonical form in $GL(6,q^4)$:

$$diag(\omega^{q^3+q^2+q+1},\omega^{2(q^2+1)},\omega^{q^3+q^2+q+1},\omega^{q^3+q^2+q+1},\omega^{2q(q^2+1)},\omega^{q^3+q^2+q+1}).$$

Since $\omega^{q^3+q^2+q+1}\in GF(q)$ and $\omega^{2(q^2+1)}$ and $\omega^{2q(q^2+1)}$ are distinct elements in $GF(q^2)$, conjugate over $GF(q)$, we have that the induced linear collineation $\overline{(\Lambda^{(2)}(C(f)))^{q^2+1}}$ fixes the line l setwise and the solid L pointwise. All the other orbits have size $q+1$ and are planar conics. These conics lie in planes conjugate (with respect to the polarity of $\mathcal{K}$, say τ) to planes in L meeting $\mathcal{E}_3$ into a conic, see [12]. In particular we have the following

Lemma 2. *The linear collineation* $\overline{(\Lambda^{(2)}(C(f)))^{q^2+1}}$ *fixes the elliptic quadric* $\mathcal{E}$ *pointwise.*

From now on we assume $q = 2^h$, h odd and $h \geq 3$. Set $r = 2^{(h+1)/2}$. Denote by $Sz(q)$ the Suzuki group of $PGL(4, q)$, thought embedded in the symplectic group $Sp(4, q)$ [16]. Associated with the group $Sz(q)$ is an ovoid Ω [20] (the Suzuki–Tits ovoid), in the sense that Ω is invariant under $Sz(q)$. From [16] it follows that $Sz(q)$ contains cyclic subgroups of order $q + 1 + r$ and $q + 1 - r$, actually subgroups of distinct Singer cyclic groups of $PGL(4, q)$. Also the group $Sz(q)$ is associated with a line–spread $\mathcal{S}$ of $PG(3, q)$ [17] (the Lűneburg spread). In particular, the lines of $\mathcal{S}$ are tangent to Ω, see [13, Th. 16.4.12], in the sense that such lines belong to the linear complex $\mathcal{C}$ defined by all the tangents to Ω [19]. Hence, if the group $Sz(q)$ acts on the points of Ω in its natural permutation representation, the second exterior power of $Sz(q)$ acts on the lines of $\mathcal{C}$. The linear complex $\mathcal{C}$ is represented on the Klein quadric $\mathcal{K}$ by a parabolic quadric $\mathcal{P}_4$, obtained as a section of $\mathcal{K}$ by a non–tangent hyperplane, say Π. The Lűneburg spread $\mathcal{S}$ is then represented by an ovoid $\overline{\Omega}$ of $\mathcal{P}_4$.

Lemma 3. *The regular spread* $\mathcal{E}_3$ *and the Lűneburg spread* $\mathcal{S}$ *have in common* $q+1+r$ *(resp.* $q+1-r$*) lines. In particular the lines of* $\mathcal{E}_3$ *are tangent to* Ω.

Proof. An elliptic quadric of $PG(3, q)$ admits a Singer cyclic group T of order q^2+1. Hence it is always possible to choose T in such a way $|T \cap Sz(q)| = q+1+r$ (resp. $q + 1 - r$) [7]. Note that $q^2 + 1 = (q + 1 + r)(q + 1 - r)$. Using the isomorphism $Sp(4, q) \simeq O(5, q)$, we get that the subgroup of $O(5, q)$ fixing $\mathcal{E}$ is $SL(2, q^2)\cdot\mathrm{Gal}(GF(q^2), GF(q))$ [9]. Such a group always contains a Singer cyclic subgroup of order $q^2 + 1$, say $\bar{T}$ and $|\bar{T} \cap \Lambda^{(2)}(Sz(q))| = q + 1 \pm r$. For the second assertion see [7, Lemma 2.6]. ∎

It follows that the elliptic quadric $\mathcal{E}$ and the ovoid $\overline{\Omega}$ of $\mathcal{K}$ meet into a set, say A, of size $q + 1 + r$ (resp. $q + 1 - r$), see also [1].

3 The cap construction

Denote by $\langle\sigma\rangle$ the Singer cyclic group of $PGL(4, q)$ such that $K = Sz(q) \cap \langle\sigma\rangle$ has order $q + 1 \pm r$. From [15] the subgroup K is irreducible and its centraliser in $PGL(4, q)$, say C coincides with $\langle\sigma\rangle$. Consider the unique subgroup of C of order $q + 1$, say J. Then J is not a subgroup of $Sz(q)$ [16] and so $\Lambda^{(2)}(J)$, the second exterior power of J, does not leave $\overline{\Omega}$ invariant (but of course leaves $\mathcal{K}$ invariant). Under the action of J on Ω, we get a partition of the point–set of $PG(3, q)$ into Suzuki–Tits ovoids (as explained in [7]) and so under the action of $\Lambda^{(2)}(J)$ on $\overline{\Omega}$ we obtain $q+1$ 4–dimensional Suzuki–Tits ovoids, say $\overline{\Omega}_1 = \overline{\Omega}, \ldots, \overline{\Omega}_{q+1}$. Since $\Lambda^{(2)}(J)$ fixes A, we have that such ovoids all meet in the set A. We have proved the following

Proposition 1. *The orbit of* $\overline{\Omega}$ *under the group* $\Lambda^{(2)}(J)$*, consists of* $q+1$ *4–dimensional Suzuki–Tits ovoids all intersecting in the set* A.

Set $\mathcal{O} = (\overline{\Omega_1} \cup \overline{\Omega_2}, \cdots \cup \overline{\Omega_{q+1}}) \setminus A$. Then $|\mathcal{O}| = (q + 1)(q^2 - q \mp r)$, according as $|A| = q+1\pm r$. In particular, $\mathcal{O}$, from the above discussion, is made up of $(q^2-q\mp r)$ conics belonging to the same number of planes through the line l.

Lemma 2. *The Suzuki–Tits ovoids $\overline{\Omega}_1, \dots, \overline{\Omega}_{q+1}$ lie in distinct hyperplanes $\Pi_1, \dots, \Pi_{q+1}$ belonging to the pencil centered in the solid L. In particular, the line l meets each hyperplane Π_i in one point, say P_i, $i = 1, \dots q+1$.*

Proof. We know that $\overline{\Omega_1}$ is contained in a parabolic quadric $\mathcal{P}_4$, obtained as section of $\mathcal{K}$ by a non–tangent hyperplane Π containing L. In particular Π is conjugate with respect to $\mathcal{K}$ to a point on the line l; this argument applies to all points of l and since $\Lambda^{(2)}(J)$ acts transitively on the points of l we are done. Moreover, we have seen that the hyperplane $\Pi = \Pi_1$ represents the linear complex $\mathcal{C}$, and such a hyperplane is conjugate with respect to τ to a point P_1 on l. Since q is even, $P_1 \in \Pi_1$ and it turns out to be the nucleus of the parabolic quadric $\mathcal{P}_4$. ∎

Theorem 3. *The set $\mathcal{O}$ is a cap embedded in $\mathcal{K}$.*

Proof. Assume that three points P_1, P_2, P_3 on $\mathcal{O}$ are collinear on the line r. Then r is completely contained in $\mathcal{K}$. Since each $\overline{\Omega_i}$, $i = 1, \dots q+1$ is a cap we suppose that P_1, P_2 lie on $\overline{\Omega_j}$ and P_3 lies on $\overline{\Omega_k}$ for some j and k, $j \neq k$. Using the Klein representation of lines of $PG(3,q)$ as points of $PG(5,q)$ on $\mathcal{K}$, we have that the lines r_1, r_2, r_3 corresponding to the points P_1, P_2 and P_3 belong to the same pencil. This is a contradiction since r_1 and r_2 belong to a Lűneburg spread and so are skew. Assume now that P_1 lies on $\overline{\Omega_j}$, P_2 lies on $\overline{\Omega_k}$ and, P_3 lies on $\overline{\Omega_m}$, $j \neq k \neq m$. In this case we can argue as follows. Consider the projection Q of $\overline{\Omega_1} \subset \mathcal{P}_4 \subset \Pi_1$ from the point P_1 onto L. Remember that P_1 is the nucleus of $\mathcal{P}_4$ and so every line through P_1 contains exactly one point of $\mathcal{P}_4$ [9], [21]. Under this projection, the generating lines of $\mathcal{P}_4$ are mapped bijectively into the lines of a non–singular linear complex which may be taken, by an appropriate choice of coordinates, to be $\mathcal{C}$. Hence Q is a Suzuki–Tits ovoid meeting $\mathcal{E}$ in $q+1 \pm r$ points. If ρ denotes the polarity associated to $\mathcal{E}$, then for each point R belonging to $Q \setminus \mathcal{E}$, R^ρ is a plane meeting $\mathcal{E}$ in a conic and so $(R^\rho)^\tau$ is a plane meeting $\mathcal{K}$ in a conic which is an orbit of a point of $\overline{\Omega} \setminus \mathcal{E}$ under the action of the subgroup $\langle (\Lambda^{(2)}(C(f)))^{q^2+1} \rangle$. Of course such a conic is contained in the set $\mathcal{O}$. We can conclude using the second part of [11, Lemma 3.1]. ∎

Corollary 4. *The set $\mathcal{O} \cup \mathcal{E}$ is a cap of $\mathcal{K}$.*

Proof. The union of a Suzuki–Tits ovoid $\bar{\Omega}_i$ and $\mathcal{E}$ is a cap since both represent line spreads of $PG(3,q)$. Assume that the points $P_1 \in \bar{\Omega}_i$, $P_2 \in \bar{\Omega}_j$, $P_3 \in \mathcal{E}$ are collinear on a line r (which of course belongs to $\mathcal{K}$). We get immediately a contradiction, since, for instance, r is completely contained in the hyperplane Π_i which meets the hyperplane Π_j only in L and $\bar{\Omega}_i$ meets L only in the points of A. ∎

As a by–product we get also infinite families of caps of $PG(4,q)$ embedded in parabolic quadrics. For a similar construction see [7, Sec. 3].

Theorem 5. *There exist caps of size $2q^2 - q + r + 1$ (resp. $2q^2 - q - r + 1$) embedded into parabolic quadrics of $PG(4,q)$, $q = 2^h$, $h \geq 3$.*

Proof. From our previous discussion it is possible to consider the set–theoretic union of a Suzuki–Tits ovoid in a hyperplane Π_i, $i = 1 \dots q+1$, with the elliptic quadric $\mathcal{E}$. ∎

Remark 6. Our method can be applied to any ovoid $\mathcal{D}$ of $PG(3,q)$, q even, assuming that its collineation group does not contain a Singer cyclic subgroup of order $q+1$. In particular, if the line–spread associated to $\mathcal{D}$ and $\mathcal{E}_3$ (regular spread) have no line in common, we get a cap embedded in $\mathcal{K}$ which turns out to be the set–theoretic disjoint union of $q+1$ ovoids of *type* $\mathcal{D}$ and one elliptic quadric.

Remark 7. It would be interesting to have other caps constructions on the Klein quadric with respect to other subgroups of the Suzuki groups, distinct from Singer cyclic subgroups.

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